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MULTIPLE SOLUTIONS FOR $p(x)$-LAPLACIAN EQUATIONS WITH NONLINEARITY SUBLINEAR AT ZERO

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Abstract

We consider the Dirichlet problem for $p(x)$-Laplacian equations of the form

$$-\Delta_{p(x)}u + b(x)|u|^{p(x)-2}u = f(x,u), \quad u \in W_0^{1,p(x)}(\Omega).$$

The odd nonlinearity $f(x,u)$ is $p(x)$-sublinear at $u = 0$ but the related limit need not be uniform for $x \in \Omega$. Except being subcritical, no additional assumption is imposed on $f(x,u)$ for $|u|$ large. By applying Clark's theorem and a truncation method, we obtain a sequence of solutions with negative energy and approaching the zero function $u = 0$.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain, $p : \overline{\Omega} \to \mathbb{R}$ be Lipschitz continuous and

$$1 < p_- := \inf_{x\in\Omega} p(x) \le \sup_{x\in\Omega} p(x) =: p_+ < N. \tag{1.1}$$

We consider the Dirichlet problem for the $p(x)$-Laplacian equation

$$\begin{cases} -\Delta_{p(x)}u + b(x)|u|^{p(x)-2}u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)$-Laplacian of u and $b \in L^{N/p(x)}(\Omega)$. The definition of the space $L^{N/p(x)}(\Omega)$ is given in the next section. Note that b can be sign-changing. Let

$$p^*(x) = \frac{Np(x)}{N - p(x)}.$$

We assume the following conditions on the nonlinearity $f(x, u)$:

(f_1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition and

$$|f(x,t)| \le C_1 + C_2|t|^{q(x)-1} \quad \text{for all } (x,t) \in \Omega \times \mathbb{R},$$

where $q \in C(\overline{\Omega})$ and $1 < q(x) < p^*(x)$ for all $x \in \Omega$;

(f_2) there is a ball $B_r(a) \subset \Omega$ such that

$$\lim_{|t|\to 0} \frac{F(x,t)}{|t|^{p_-}} = +\infty \quad \text{for almost every (a.e.) } x \in B_r(a), \text{ where } F(x,t) = \int_0^t f(x,\cdot). \tag{1.3}$$

When $p(x) \equiv 2$ (thus $p_- = 2$) and $f(x, \cdot)$ is sublinear at zero, then (1.3) holds with $p_- = 2$. For this reason, we say that our problem (1.2) is $p(x)$-sublinear at zero. We emphasise that the limit (1.3) is a pointwise limit, while condition (f_1) means that the nonlinearity $f(x, u)$ is subcritical. Under these mild conditions, we shall prove the following theorem.

THEOREM 1.1. *Suppose that the conditions* (f_1) *and* (f_2) *hold. If* $f(x, \cdot)$ *is odd for all* $x \in \Omega$*, then (1.2) has a sequence of solutions* u_n *such that* $\Phi(u_n) \le 0$, $\Phi(u_n) \to 0$*; where* Φ *is the energy functional given in (3.1).*

This theorem generalises a recent result of He and Wu [5], where the semilinear case $p(x) \equiv 2$, namely

$$\begin{cases} -\Delta u + b(x)u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

is considered assuming $b \in L^{N/2}(\Omega)$ and $f(x, u)$ is subcritical. In particular, He and Wu assumed the pointwise limit

$$\lim_{|t|\to 0} \frac{F(x,t)}{|t|^2} = +\infty \quad \text{for } x \in \Omega. \tag{1.5}$$

However, in their argument, to verify the condition (1.6) in Proposition 1.2 below, they need the inequality

$$F(x,t) \ge c_k^{-2}|t|^2 \qquad \text{for } |t| \le r \text{ and a.e. } x \in \Omega.$$

This could not be true unless the limit (1.5) holds *uniformly*. In the proof of our Theorem 1.1, we fill this gap (see Lemma 3.4) and generalise their result to the quasilinear variable exponent case. Moreover, the verification of the $(PS)_c$ condition, which is crucial for applying variational methods, has been greatly simplified (see Remark 3.3).

Both [5] and our result are based on a new version of Clark's theorem recently proved by Liu and Wang [8]. Our Theorem 1.1 is motivated by [5].

PROPOSITION 1.2 [8, Theorem 1.1]. *Let W be a Banach space and $\Phi \in C^1(W, \mathbb{R})$ be an even coercive functional satisfying the $(PS)_c$ condition for $c \le 0$ and $\Phi(0) = 0$. If for any $k \in \mathbb{N}$ there are a k-dimensional subspace W_k and $\delta_k > 0$ such that*

$$\sup_{W_k \cap S_{\delta_k}} \Phi < 0, \tag{1.6}$$

where $S_r = \{u \in W : \|u\| = r\}$ for $r > 0$, then Φ has a sequence of critical points $u_k \neq 0$ such that $\Phi(u_k) \le 0$, $u_k \to 0$.

Variable exponent variational problems appear in many applications (see [2, 6, 9]). In particular, there has been great interest in elliptic boundary value problems involving the $p(x)$-Laplacian in the last two decades. In [7], a sequence of negative energy solutions of the $p(x)$-Laplacian equation in (1.2) subject to a nonlinear boundary condition is obtained; in addition to (f_1) and (f_2), it is assumed that (1.3) holds uniformly for $x \in \Omega$ and that the nonlinearity is $p(x)$-sublinear at infinity. In [10], the existence of positive solutions of (1.2) with concave and convex nonlinearities is studied via Nehari's method. For other recent results, we refer to [11] for $p(x)$-Laplacian systems and to [1] for $(p(x), q(x))$-Laplacian problems.

2. Variable exponent spaces

To study the problem (1.2), we recall the variable exponent Lebesgue space and Sobolev space (see [4] for more details). For a Lipschitz continuous function $p : \overline{\Omega} \to \mathbb{R}$ satisfying (1.1), let

$$L^{p(x)}(\Omega) = \Big\{u : \Omega \to \mathbb{R} : u \text{ is measurable and } \int |u|^{p(x)} < \infty\Big\}.$$

Here and below, all integrals are taken over Ω. Equipped with the Luxemburg norm,

$$|u|_{p(x)} = \inf\Big\{\lambda > 0 : \int \Big|\frac{u}{\lambda}\Big|^{p(x)} \le 1\Big\},$$

$L^{p(x)}(\Omega)$ becomes a separable uniformly convex Banach space.

The variable exponent Sobolev space $W_0^{1,p(x)}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ under the norm

$$\|u\| = |\nabla u|_{p(x)} = \inf\Big\{\lambda > 0 : \int \Big|\frac{\nabla u}{\lambda}\Big|^{p(x)} \le 1\Big\},$$

which is also a separable uniformly convex Banach space.

From now on, we denote $W = W_0^{1,p(x)}(\Omega)$. The functional $\rho : W \to \mathbb{R}$ defined by

$$\rho(u) = \int \frac{1}{p(x)} |\nabla u|^{p(x)}$$

is crucial for investigating $p(x)$-Laplacian equations like (1.2).

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LEMMA 2.1 [3, Theorem 3.1]. *The functional ρ is of class C^1. Moreover, the functional $\rho' : W \to W^*$ is of type (S_+). Thus, if $u_n \rightharpoonup u$ in W and*

$$\varlimsup_{n\to\infty} \langle \rho'(u_n), u_n - u\rangle \le 0,$$

then $u_n \to u$ in W.

From the definition of the norm $\|\cdot\|$, it is easy to see that:

(1) if $\|u\| \ge 1$, then

$$\|u\|^{p_-} \le \int |\nabla u|^{p(x)} \le \|u\|^{p_+};$$

(2) if $\|u\| \le 1$, then

$$\|u\|^{p_+} \le \int |\nabla u|^{p(x)} \le \|u\|^{p_-}.$$

The following lemma is an easy consequence because $p_- \le p(x) \le p_+$.

LEMMA 2.2

(1) *If $\|u\| \ge 1$, then*

$$\frac{1}{p_+}\|u\|^{p_-} \le \rho(u) \le \frac{1}{p_-}\|u\|^{p_+};$$

(2) *if $\|u\| \le 1$, then*

$$\frac{1}{p_+}\|u\|^{p_+} \le \rho(u) \le \frac{1}{p_-}\|u\|^{p_-}.$$

3. Proof of Theorem 1.1

For the variable exponent Sobolev space $W = W_0^{1,p(x)}(\Omega)$, it is well known that weak solutions of (1.2) are precisely critical points of the C^1-functional $\Phi : W \to \mathbb{R}$,

$$\Phi(u) = \int \Big(\frac{1}{p(x)}(|\nabla u|^{p(x)} + b(x)|u|^{p(x)})\Big) - \int F(x,u). \tag{3.1}$$

At first glance, because b may be sign-changing, the principle part (the first term) of Φ appears to be indefinite. We observe that if we set

$$\tilde f(x,t) = f(x,t) - b(x)|t|^{p(x)-2}t,$$

then the problem (1.2) becomes

$$\begin{cases} -\Delta_{p(x)}u = \tilde f(x,u) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega,\end{cases}$$

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in which the new nonlinearity $\tilde{f}(x,u)$ satisfies the *same* conditions (f_1) and (f_2), and

$$\lim_{|t|\to 0}\frac{\tilde{F}(x,t)}{|t|^{p_-}}=\lim_{|t|\to 0}\Big(\frac{F(x,t)}{|t|^{p_-}}-\frac{b(x)}{p(x)}\frac{|t|^{p(x)}}{|t|^{p_-}}\Big)=\lim_{|t|\to 0}\frac{F(x,t)}{|t|^{p_-}}=+\infty$$

for almost every $x\in B_r(a)$, because $p(x)\geq p_-$. Here, $\tilde{F}(x,t)=\int_0^t\tilde{f}(x,\cdot)$.

In other words, to prove Theorem 1.1, it suffices to consider the case $b(x)=0$. The reason that we state our problem (1.2) with the term $b(x)|u|^{p(x)-2}u$ is to allow comparison with the results of [5, 7, 10].

Therefore, in what follows, we assume $b(x)=0$ so that the functional given in (3.1) becomes $\Phi : W\to\mathbb{R}$,

$$\Phi(u)=\rho(u)-\int F(x,u)=\int\frac{1}{p(x)}|\nabla u|^{p(x)}-\int F(x,u),$$

whose critical points are solutions of (1.2) with $b(x)=0$. To prove Theorem 1.1, we shall apply Proposition 1.2 to find a sequence $\{u_n\}$ of critical points for Φ.

Since we have not assumed any conditions on the nonlinearity $f(x,t)$ for $|t|$ large (except the subcritical growth condition (f_1)), it is not possible to verify the $(PS)_c$ condition for Φ. To overcome this difficulty, we adopt the truncation method of He and Wu [5].

Let $\phi : [0,\infty)\to[0,1]$ be a decreasing C^∞-function such that $|\phi'(t)|\leq 2$,

$$\phi(t)=1 \text{ for } t\in[0,1] \quad\text{and}\quad \phi(t)=0 \text{ for } t\in[2,\infty).$$

We consider the truncated functional $\Psi : W\to\mathbb{R}$,

$$\Psi(u)=\rho(u)-\phi(\rho(u))\int F(x,u).$$

The derivative of Ψ is given by

$$\begin{aligned}\langle\Psi'(u),v\rangle&=\langle\rho'(u),v\rangle-\phi(\rho(u))\int f(x,u)v-\Big(\int F(x,u)\Big)\phi'(\rho(u))\langle\rho'(u),v\rangle\\&=\Big(1-\Big(\int F(x,u)\Big)\phi'(\rho(u))\Big)\langle\rho'(u),v\rangle-\phi(\rho(u))\int f(x,u)v \qquad (3.2)\end{aligned}$$

for $u,v\in W$.

LEMMA 3.1. *The functional Ψ is coercive.*

PROOF. We note that by Lemma 2.2, for $\|u\|\geq 1+(2p_+)^{1/p_-}$,

$$\rho(u)\geq\frac{1}{p_+}\|u\|^{p_-}\geq 2.$$

Hence, $\phi(\rho(u))=0$ and

$$\Psi(u)=\rho(u)\geq\frac{1}{p_+}\|u\|^{p_-}.$$

This implies that Ψ is coercive. □

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LEMMA 3.2. *The functional* Ψ *satisfies* $(PS)_c$ *for* $c \le 0$.

PROOF. Let $\{u_n\}$ be a $(PS)_c$ sequence of Ψ with $c \le 0$, that is, $\Psi(u_n) \to c$, $\Psi'(u_n) \to 0$. Then for n large,

$$-\phi(\rho(u_n)) \int F(x, u_n) = \Psi(u_n) - \rho(u_n) \le \frac{1}{2} - \rho(u_n). \tag{3.3}$$

We claim that

$$1 - \Big(\int F(x, u_n) \Big) \phi'(\rho(u_n)) \ge 1. \tag{3.4}$$

For this purpose, we consider two cases. If $\rho(u_n) < 1$, then $\phi'(\rho(u_n)) = 0$ and (3.4) is an equality. If $\rho(u_n) \ge 1$, then the right-hand side of (3.3) is negative. Noting $\phi(\rho(u_n)) \ge 0$, we have

$$\int F(x, u_n) \ge 0. \tag{3.5}$$

So we also have (3.4) because $\phi'(\rho(u_n)) \le 0$.

The coerciveness of Ψ implies that the $(PS)_c$ sequence $\{u_n\}$ is bounded in W. We may assume that $u_n \rightharpoonup u$ in W. Since f is subcritical (condition (f_1)), by the compact embedding $W \hookrightarrow L^{q(x)}(\Omega)$, Hölder's inequality and the boundedness of the Nemytsky operator

$$\mathcal{N}_f : L^{q(x)}(\Omega) \to L^{q(x)/(q(x)-1)}(\Omega), \quad (\mathcal{N}_f u)(x) = f(x, u(x)),$$

(as shown in [4]), it is well known that up to a subsequence,

$$\Big| \int f(x, u_n)(u_n - u) \Big| \le 2 |f(x, u_n)|_{q(x)/(q(x)-1)} |u_n - u|_{q(x)} \to 0. \tag{3.6}$$

Setting $v = u_n - u$ in (3.2), from $\langle \Psi'(u_n), u_n - u \rangle \to 0$, (3.6) and the boundedness of $\phi(\rho(u_n))$, we obtain

$$\begin{aligned} &\Big(1 - \Big(\int F(x, u_n) \Big) \phi'(\rho(u_n)) \Big) \langle \rho'(u_n), u_n - u \rangle \\ &\quad = \langle \Psi'(u_n), u_n - u \rangle + \phi(\rho(u_n)) \int f(x, u_n)(u_n - u) \to 0. \end{aligned} \tag{3.7}$$

We deduce from this and (3.4) that

$$\langle \rho'(u_n), u_n - u \rangle \to 0.$$

It follows from Lemma 2.1 that $u_n \to u$ in W. □

REMARK 3.3. Although our problem (1.2) is much more general than the problem (1.4) considered in [5], our verification of the $(PS)_c$ condition is much simpler than in [5], where the convergence of $\{u_n\}$ is deduced by estimating $\|u_n - u\|^2$ by the sum of $\langle \Psi'(u_n) - \Psi'(u), u_n - u \rangle$ and four additional complicated terms (see [5, (2.20)]). The key points in our proof are the (S_+) property of ρ' and the observation (3.4).

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We should also point out that the verification of $(PS)_c$ for $c = 0$ in [5] contains a gap. For the $(PS)_0$ sequence $\{u_n\}$, [5, (2.19)] is derived from $2\Psi(u_n) - \|u_n\|^2 \le 0$. However, this may be false because $\Psi(u_n)$ may be positive, even though $\Psi(u_n) \to 0$.

LEMMA 3.4. *For any $k \in \mathbb{N}$, there are a k-dimensional subspace W_k of W and $\delta_k > 0$, such that*

$$\sup_{W_k \cap S_{\delta_k}} \Psi < 0.$$

PROOF. Let $X = \{u \in W : \operatorname{supp} u \subset B_r(a)\}$, W_k be a k-dimensional subspace of X. If the result is not true then, for all $n \in \mathbb{N}$,

$$\sup_{W_k \cap S_{1/n}} \Psi \ge 0.$$

This implies that there is a sequence $\{u_n\} \subset W_k \cap S_{1/n}$, such that

$$\|u_n\| = \frac{1}{n} \to 0, \quad \Psi(u_n) \ge -\frac{1}{n^{p_-}}. \tag{3.8}$$

Since all norms on W_k are equivalent, from $\|u_n\| \to 0$, we deduce $|u_n|_\infty \to 0$.

Let $\eta : \Omega \to [-\infty, \infty]$ be defined by

$$\eta(x) = \varliminf_{n\to\infty} \frac{F(x, u_n(x))}{\|u_n\|^{p_-}}.$$

Then η is measurable. For $x \in B_r(a)$, from the *pointwise* limit (1.3) in (f_2), there is $r_x > 0$ such that $F(x,t) \ge 0$ for $t \in [-r_x, r_x]$. Hence, if $n \gg 1$, then $|u_n|_\infty \le r_x$ and $F(x, u_n(x)) \ge 0$, and so $\eta(x) \ge 0$ for a.e. $x \in B_r(a)$. Consequently, $\eta(x) \ge 0$ for a.e. $x \in \Omega$, because $\operatorname{supp} u_n \subset B_r(a)$.

Let $v_n = \|u_n\|^{-1} u_n$. Since $\dim W_k < \infty$, we have $v_n \to v$ in W_k for some $v \in W_k$, note that $\|v\| = 1$. For $x \in \{v \ne 0\}$, using (1.3) again,

$$\eta(x) = \varliminf_{n\to\infty} \frac{F(x, u_n(x))}{\|u_n\|^{p_-}} = \varliminf_{n\to\infty} \frac{F(x, u_n(x))}{|u_n(x)|^{p_-}} |v_n(x)|^{p_-} = +\infty.$$

By Fatou's lemma, since $\{v \ne 0\}$ has positive Lebesgue measure,

$$\varliminf_{n\to\infty} \int \frac{F(x, u_n)}{\|u_n\|^{p_-}} \ge \int \varliminf_{n\to\infty} \frac{F(x, u_n)}{\|u_n\|^{p_-}} = \int \eta \ge \int_{v\ne 0} \eta = +\infty. \tag{3.9}$$

Because $\|u_n\| \le 1$, we have (see Lemma 2.2)

$$\rho(u_n) \le \frac{1}{p_-} \|u_n\|^{p_-} \le 1.$$

Thus, $\phi(\rho(u_n)) = 1$ and

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$$\begin{aligned}\Psi(u_n) = \Phi(u_n) &= \rho(u_n) - \int F(x, u_n)\\ &\le \frac{1}{p_-}\|u_n\|^{p_-} - \int F(x, u_n)\\ &= \|u_n\|^{p_-}\Big(\frac{1}{p_-} - \int \frac{F(x, u_n)}{\|u_n\|^{p_-}}\Big) = \frac{1}{n^{p_-}}\Big(\frac{1}{p_-} - \int \frac{F(x, u_n)}{\|u_n\|^{p_-}}\Big).\end{aligned}$$

Now, from (3.9), we deduce $n^{p_-}\Psi(u_n) \to -\infty$, contradicting (3.8). □

PROOF OF THEOREM 1.1. Lemmas 3.1, 3.2 and 3.4 permit us to apply Proposition 1.2, and deduce that Ψ has a sequence of critical points $u_k \neq 0$ such that $\Psi(u_k) < 0$ and $u_k \to 0$ in W. For some $K \in \mathbb{N}$, if $k \geq K$,

$$\rho(u_k) \le \frac{1}{p_-}\|u_k\|^{p_-} < 1.$$

Since $\Psi(u) = \Phi(u)$ for $u \in \rho^{-1}[0, 1)$, we see that u_k with $k \geq K$ are critical points of Φ as well, satisfying $\Phi(u_k) < 0$ and $u_k \to 0$ in W. □

REMARK 3.5. Liu and Wang [8, Theorem 3.1] treat the case in which $p(x)$ is a constant $p > 1$. Assuming that $f(x, \cdot)$ is odd *only* in $(-\delta, \delta)$ for some $\delta > 0$, and

$$\lim_{|t|\to 0} \frac{F(x, t)}{|t|^p} = +\infty \tag{3.10}$$

uniformly on some small ball $B_r(x_0) \subset \Omega$, a sequence of negative energy solutions approaching zero is obtained. Liu and Wang truncated the nonlinearity $f(x, t)$ for $|t| > \delta/2$, resulting in a new nonlinearity $\hat{f}(x, t) = 0$ for $|t| > \delta$. Then Proposition 1.2 is applied to get a sequence of solutions u_n for the truncated problem. Since $u_n \to 0$ in $W_0^{1,p}(\Omega)$, a regularity argument then yields $|u_n|_\infty < \delta/2$ for large n. Such u_n are then solutions of the original problem.

To the best of our knowledge, a suitable L^∞-regularity theory is not available for the $p(x)$-Laplacian operator and, at present, we can only deal with the case in which $f(x, \cdot)$ is globally odd and subcritical, as we have done in Theorem 1.1. Our argument in proving Lemma 3.4 can be used to slightly improve [8, Theorem 3.1], requiring only that the limit (3.10) holds pointwise in $B_r(x_0)$.

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