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The Batalin–Vilkovisky Algebra in the String Topology of Classifying Spaces

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Abstract. For almost any compact connected Lie group G and any field $\mathbb{F}_p$, we compute the Batalin–Vilkovisky algebra $H^{*+\dim G}(\mathrm{LBG};\mathbb{F}_p)$ on the loop cohomology of the classifying space introduced by Chataur and the second author. In particular, if p is odd or $p=0$, this Batalin–Vilkovisky algebra is isomorphic to the Hochschild cohomology $HH^*(H_*(G),H_*(G))$. Over $\mathbb{F}_2$, such an isomorphism of Batalin–Vilkovisky algebras does not hold when $G=\mathrm{SO}(3)$ or $G=G_2$. Our elaborate considerations on the signs in string topology of the classifying spaces give rise to a general theorem on graded homological conformal field theory.

1 Introduction

Let M be a closed oriented smooth manifold and let LM denote the space of free loops on M. Chas and Sullivan [4] have defined a product on the homology of LM, called the *loop product*, $H_*(LM)\otimes H_*(LM)\to H_{*-\dim M}(LM)$. They showed that this loop product, together with the homological Batalin–Vilkovisky operator $\Delta\colon H_*(LM)\to H_{*+1}(LM)$, make the shifted free loop space homology $\mathbb{H}_*(LM):=H_{*+\dim M}(LM)$ into a Batalin–Vilkovisky algebra, or BV-algebra. Over $\mathbb{Q}$, when M is simply connected, this BV-algebra can be computed using Hochschild cohomology [11]. In particular, if M is formal over $\mathbb{Q}$, there is an isomorphism of BV-algebras between $\mathbb{H}_*(LM)$ and

$$HH^*(H^*(M;\mathbb{Q}),H^*(M;\mathbb{Q})),$$

the Hochschild cohomology of the symmetric Frobenius algebra $H^*(M;\mathbb{Q})$. Over a field $\mathbb{F}_p$, if $p\neq 0$, this BV-algebra $\mathbb{H}_*(LM)$ is hard to compute. It has been computed only for complex Stiefel manifolds [41], spheres [34], compact Lie groups [19,35], and complex projective spaces [5,18].

Let G be a connected compact Lie group of dimension d and let BG be its classifying space. Motivated by Freed, Hopkins, and Teleman twisted K-theory [13] and by a structure of symmetric Frobenius algebra on $H_*(G)$, Chataur and the second author [6] proved that the homology of LBG, the free loop space with coefficients in a field $\mathbb{K}$, admits the structure of a d-dimensional homological conformal field theory. (More generally, if G acts smoothly on M, Behrend, Ginot, Noohi, and Xu [1, Theorem 14.2] proved that $H_*(L(EG\times_G M))$ is a $(d-\dim M)$-homological conformal field theory.) In particular, the operation associated with a cobordism connecting one-dimensional manifolds called the pair of pants, defines a product on the cohomology

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of LBG, called the *dual of the loop coproduct*, $H^*(\mathrm{LBG}) \otimes H^*(\mathrm{LBG}) \to H^{*-d}(\mathrm{LBG})$. Chataur and the second author showed that the dual of the loop coproduct, together with the cohomological BV-operator $\Delta\colon H^*(\mathrm{LBG}) \to H^{*-1}(\mathrm{LBG})$, make the shifted free loop space cohomology $\mathbb{H}^*(\mathrm{LBG}) := H^{*+d}(\mathrm{LBG})$ into a BV-algebra *up to signs*. Over $\mathbb{F}_2$, Hepworth and Lahtinen [20] extended this result to non-connected compact Lie groups and more difficult, they showed that this d-dimensional homological conformal field theory, in particular this algebra $\mathbb{H}^*(\mathrm{LBG})$, has a unit. One of our results aims to solve the sign issues and to show that, indeed, $\mathbb{H}^*(\mathrm{LBG})$ is a BV-algebra (Corollary C.3).

In fact, one of the highlights in this manuscript is to show that more generally, the dual of a d-homological field theory has, after a d degree shift, the structure of a BV-algebra (Theorems B.1 and C.1). Our elaborate considerations on the signs give many explicit computations on $\mathbb{H}^*(\mathrm{LBG})$ as mentioned below. Surprisingly, these computations enable us to determine the signs on the product of the prop in Theorem B.1; that is, *such local computations in string topology of BG give rise to a general theorem on graded homological conformal field theory*.

Lahtinen [30] computed some non-trivial higher operations in the structure of this d-dimensional homological conformal field theory on the cohomology of BG for some compact Lie groups G. In this paper, we compute the most important part of this d-dimensional homological conformal field theory, namely the BV-algebra $\mathbb{H}^*(\mathrm{LBG}; \mathbb{F}_p)$ for almost any connected compact Lie group G and any field $\mathbb{F}_p$. According to our knowledge, this BV-algebra $\mathbb{H}^*(\mathrm{LBG}; \mathbb{F}_p)$ has never been computed on any example.

Very recently, Grodal and Lahtinen [15] showed that the mod p cohomology of a finite Chevalley group admits a module structure over this algebra $\mathbb{H}^*(\mathrm{LBG}; \mathbb{F}_p)$, where G is the p-compact group of $\mathbb{C}$-rational points associated with the finite group. This result appears in the program to attack Tezuka's question [45] about an isomorphism compatible with the cup products between this group cohomology and this free loop space cohomology of BG. Thus our explicit computations are also strongly relevant to the program.

Our method is completely different from the methods used to compute the BV-algebra $\mathbb{H}_*(LM)$ in the known cases recalled above. Suppose that the cohomology algebra of BG over $\mathbb{F}_p$, $H^*(BG; \mathbb{F}_p)$, is a polynomial algebra $\mathbb{F}_p[y_1, \dots, y_N]$ (few connected compact Lie groups do not satisfy this hypothesis). Then the cup product on $H^*(\mathrm{LBG}; \mathbb{F}_p)$ was first computed by the first author [28](see [24] for a quick calculation). Tamanoi [42] explained the relation between the cap product and the loop product on $H_*(LM)$. Dually, in Theorem 2.2 we give the relation between the cup product on $H^*(\mathrm{LBG})$ and the BV-algebra $\mathbb{H}^*(\mathrm{LBG})$. Knowing the cup product on $H^*(\mathrm{LBG})$, this relation gives the dual of the loop coproduct on $\mathbb{H}^*(\mathrm{LBG})$ (Theorem 3.1). But now, since the cohomological BV-operator Δ (see Appendix E) is a derivation with respect to the cup product, Δ is easy to compute. So finally, on $H^*(\mathrm{LBG})$ we have computed the cup product and the BV-algebra structure at the same time. This has never been done for the BV-algebra $\mathbb{H}_*(LM)$.

If there is no top degree Steenrod operation Sq_1 on $H^*(BG; \mathbb{F}_2)$ or if p is odd or $p = 0$, applying Theorem 3.1, we give an explicit formula for the dual of the loop coproduct $\odot$ in Theorem 4.1 and we show in Theorem 6.2 that there is an isomorphism of BV-algebras between $\mathbb{H}^*(\mathrm{LBG};\mathbb{F}_p)$ and $HH^*(H_*(G;\mathbb{F}_p),H_*(G;\mathbb{F}_p))$, the Hochschild cohomology of the symmetric Frobenius algebra $H_*(G;\mathbb{F}_p)$.

The case $p = 2$ is more intriguing. When $p = 2$, in general we do not give an explicit formula for the dual of the loop coproduct $\odot$ (however, see Theorem 5.4 for a general equation satisfied by $\odot$). But for a given compact Lie group G, applying Theorem 3.1, we are able to give an explicit formula. As examples, we compute the dual of the loop coproduct when $G = \mathrm{SO}(3)$ (Theorem 5.7) or $G = G_2$ (Theorem 5.1). We show (Theorem 6.3) that the BV-algebras $\mathbb{H}^*(\mathrm{LBSO}(3);\mathbb{F}_2)$ and $HH^*(H_*(\mathrm{SO}(3);\mathbb{F}_2),H_*(\mathrm{SO}(3);\mathbb{F}_2))$, the Hochschild cohomology of the symmetric Frobenius algebra $H_*(\mathrm{SO}(3);\mathbb{F}_2)$, are not isomorphic, although the underlying Gerstenhaber algebras are isomorphic. Such a curious result was observed [34] for the Chas–Sullivan BV-algebras $\mathbb{H}_*(LS^2;\mathbb{F}_2)$.

However, for any connected compact Lie group such that $H^*(BG;\mathbb{F}_p)$, is a polynomial algebra, we show (Corollary 4.3 and Theorem 5.8) that as graded algebras

$$\mathbb{H}^*(\mathrm{LBG};\mathbb{F}_p)\cong H_*(G;\mathbb{F}_p)\otimes H^*(BG;\mathbb{F}_p)\cong HH^*\big(H_*(G;\mathbb{F}_p),H_*(G;\mathbb{F}_p)\big).$$

Such isomorphisms of Gerstenhaber algebras should exist (Conjecture 6.1).

We now give the plan of the paper

Section 2: We carefully recall the definition of the loop product and of the loop coproduct, insisting on orientation (Theorem 2.1), and we prove Theorem 2.2.

Section 3: When $H^*(X)$ is a polynomial algebra, following [24, 28], we give the cup product on $H^*(LX)$. Therefore, (Theorem 3.1) the dual of the loop coproduct is completely given by Theorems 2.1 and 2.2.

Section 4 is devoted to the simple case when the characteristic of the field is different from two or when there is no top degree Steenrod operation.

Section 5: The field is $\mathbb{F}_2$. We give some general properties of the dual of the loop coproduct (Lemma 5.3, Theorem 5.4). In particular, we show that it has a unit (Theorem 5.5). As examples, we compute the dual of the loop coproduct on

$$\mathbb{H}^*(\mathrm{LBSO}(3);\mathbb{F}_2) \quad \text{(Theorem 5.7)},$$
$$\mathbb{H}^*(\mathrm{LBG}_2;\mathbb{F}_2) \quad \text{(Theorem 5.1)}.$$

Up to an isomorphism of graded algebras, $\mathbb{H}^*(LX;\mathbb{F}_2)$ is just the tensor product of algebras

$$H^*(X;\mathbb{F}_2)\otimes H_{-*}(\Omega X;\mathbb{F}_2)=\mathbb{F}_2[V]\otimes\Lambda(sV)^\vee \quad \text{(Theorem 5.8)}.$$

As examples, we compute the BV-algebra

$$H^{*+3}(\mathrm{LBSO}(3);\mathbb{F}_2)\cong\Lambda(u_{-1},u_{-2})\otimes\mathbb{F}_2[v_2,v_3] \quad \text{(Theorem 5.13)}$$

and the BV-algebra

$$H^{*+14}(\mathrm{LBG}_2;\mathbb{F}_2)\cong\Lambda(u_{-3},u_{-5},u_{-6})\otimes\mathbb{F}_2[v_4,v_6,v_7] \quad \text{(Theorem 5.14)}.$$

Section 6: After studying the formality and the coformality of BG, we compare the associative algebras, the Gerstenhaber algebras, the BV-algebras $\mathbb{H}^*(\mathrm{LBG})$ and $HH^*(H_*(G),H_*(G))$ under various hypothesis.

Section 7: Independently of the rest of the paper, we show that the loop product on $H_*(\mathrm{LBG};\mathbb{F}_p)$ is trivial if and only if the inclusion of the fibre $\iota\colon \Omega\,\mathrm{BG}\hookrightarrow \mathrm{LBG}$ induces

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a surjective map in cohomology, if and only if $H^*(BG;\mathbb{F}_p)$ is a polynomial algebra, if and only if BG is $\mathbb{F}_p$-formal (when p is odd).

Appendix A: We solve some sign problems in the results [6]. In particular, we correct the definition of integration along the fibre and the main dual theorem concerning the prop structure on $H^*(LX)$.

Appendix B: $\mathbb{H}^*(LX)$ is equipped with a graded associative and graded commutative product $\odot$.

Appendix C: In fact, $\mathbb{H}^*(LX)$ equipped with $\odot$ and the BV-operator Δ is a BV-algebra since the BV identity arises from the lantern relation.

Appendix D: This BV identity comes from seven equalities involving Dehn twists and the prop structure on the mapping class group.

Appendix E: We compare different definitions of the BV-operator $\Delta\colon H^*(LX) \to H^{*-1}(LX)$.

Appendix F: We compute the Gerstenhaber algebra structure on the Hochschild cohomology $HH^*(S(V),S(V))$ of a free commutative graded algebra $S(V)$ (Theorem F.3). In particular, we give the BV-algebra structure on the Hochschild cohomology $HH^*(\Lambda(V),\Lambda(V))$ of a graded exterior algebra $\Lambda(V)$.

2 The Dual of the Loop Coproduct

In this paper, for simplicity, all the results are stated for a connected compact Lie group G. But they are also valid for an exotic p-compact group. Indeed, following [6], we only require that G is a connected topological group (or a pointed loop space) with finite-dimensional cohomology $H^*(G;\mathbb{F}_p)$. This is the main difference from [20], where Hepworth and Lahtinen required the smoothness of G.

Let $\mathbb{K}$ be a field. Let X be a simply-connected space satisfying the condition that $H^*(\Omega X;\mathbb{K})$ is of finite dimension. Then there exists a unique integer d such that $H^i(\Omega X;\mathbb{K}) = 0$ for $i > d$ and $H^d(\Omega X;\mathbb{K}) \cong \mathbb{K}$. In order to describe our results, we first recall the definitions of the product Dlcop on $H^{*+d}(LX;\mathbb{K})$ and of the loop product on $H_{*-d}(LX;\mathbb{K})$ in [6].

Let F be the pair of pants regarded as a cobordism between one ingoing circle and two outgoing circles. The ingoing map $\mathrm{in}\colon S^1 \hookrightarrow F$ and outgoing map $\mathrm{out}\colon S^1 \coprod S^1 \hookrightarrow F$ give the correspondence

$$LX \xleftarrow{\mathrm{map}(\mathrm{in},X)} \mathrm{map}(F,X) \xrightarrow{\mathrm{map}(\mathrm{out},X)} LX \times LX$$

where $\mathrm{map}(\mathrm{in},X)$ and $\mathrm{map}(\mathrm{out},X)$ are orientable fibrations. After orienting them, the integration along the fibre induces a map in cohomology

$$\mathrm{map}(\mathrm{in},X)^!\colon H^{*+d}(\mathrm{map}(F,X)) \longrightarrow H^*(LX)$$

and a map in homology

$$\mathrm{map}(\mathrm{out},X)_!\colon H_*(LX)^{\otimes 2} \longrightarrow H_{*+d}(\mathrm{map}(F,X)).$$

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See Appendix A for the definition of the integration along the fibre. By definition, the loop product is the composite

$$H_*(\mathrm{map}(\mathrm{in}, X)) \circ \mathrm{map}(\mathrm{out}, X)_! \colon H_{p-d}(LX) \otimes H_{q-d}(LX) \longrightarrow H_{p+q-d}(\mathrm{map}(F, X)) \longrightarrow H_{p+q-d}(LX).$$

By definition, the dual of the loop coproduct, denoted Dlcop, is the composite

$$\mathrm{map}(in, X)^! \circ H^*(\mathrm{map}(\mathrm{out}, X)) \colon H^{p+d}(LX) \otimes H^{q+d}(LX) \longrightarrow H^{p+q+2d}(\mathrm{map}(F, X)) \longrightarrow H^{p+q+d}(LX).$$

The pair of pants F is the mapping cylinder of $c \coprod \pi \colon S^1 \coprod (S^1 \coprod S^1) \to S^1 \vee S^1$ where $c\colon S^1 \to S^1 \vee S^1$ is the pinch map and $\pi\colon S^1 \coprod S^1 \to S^1 \vee S^1$ is the quotient map. Therefore the wedge of circles $S^1 \vee S^1$ is a strong deformation retract of the pair of pants F. The retract $r\colon F \overset{\approx}{\twoheadrightarrow} S^1 \vee S^1$ corresponds to lowering his pants and tucking up his trouser legs at the same time:

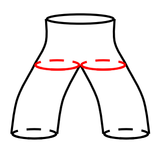

Figure 1: The homotopy between the pairs of pants and the figure eight.

Thus we have the commutative diagram

$$\begin{array}{ccccc} LX & \xleftarrow{\mathrm{map}(\mathrm{in},X)} & \mathrm{map}(F, X) & \xrightarrow{\mathrm{map}(\mathrm{out},X)} & LX^{\times 2} \\ & \nwarrow^{\mathrm{Comp}} & \approx \uparrow \mathrm{map}(r,X) & \nearrow_{q} & \\ & & LX \times_X LX & & \end{array}$$

where Comp is the composition of loops and q is the inclusion. If X were a closed manifold M of dimension d, Comp and q would be embeddings. And the Chas–Sullivan loop product is the composite

$$H_*(\mathrm{Comp}) \circ q_! \colon H_{p+d}(LM) \otimes H_{q+d}(LM) \longrightarrow H_{p+q+d}(LM \times_M LM)) \longrightarrow H_{p+q+d}(LM).$$

while the dual of the loop coproduct is the composite

$$\mathrm{Comp}^! \circ H^*(q) \colon H^{p-d}(LM) \otimes H^{q-d}(LM) \longrightarrow H^{p+q-2d}(LM \times_M LM) \longrightarrow H^{p+q-d}(LM).$$

Therefore, although Comp and q are not fibrations, by an abuse of notation, we will sometimes say that in the case of string topology of classifying spaces [6], the loop product on $H_{*-d}(LX)$ is still $H_*(\mathrm{Comp}) \circ q_!$, while Dlcop is $\mathrm{Comp}^! \circ H^*(q)$.

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The shifted cohomology $\mathbb{H}^*(LX) := H^{*+d}(LX)$ together with the dual of the loop coproduct Dlcop defined in [6] is a BV-algebra, in particular a graded commutative associative algebra, only up to signs, for two reasons.

- First, the integration along the fibre defined in [6] usually does not satisfy the usual property with respect to the product. We have corrected this sign mistake in Appendix A.
- Second, as explained in Appendix A, this is also due to the non-triviality of the prop $\det H_1(F, \partial_{\text{out}}; \mathbb{Z})^{\otimes d}$ (if d is odd).

Nevertheless, we have Theorem C.1. In particular, we have that $\mathbb{H}^*(LX)$ equipped with the operator Δ induced by the action of the circle on LX (see our definition in Appendix E) is a BV-algebra with respect to the product $\odot$ defined by $a \odot b = (-1)^{d(d-|a|)}\,\text{Dlcop}(a \otimes b)$ for $a \otimes b \in H^*(LX) \otimes H^*(LX)$; see Corollary C.3.

In order to investigate Dlcop more precisely, we need to know how the fibration $\text{map}(\text{in}, X)$ is oriented. As explained in [6, §11.5], we must choose a pointed homotopy equivalence $f\colon F/\partial_{\text{in}} \overset{\approx}{\to} S^1$. Then the fibre $\text{map}_*(F/\partial_{\text{in}}, X)$ of $\text{map}(\text{in}, X)$ is oriented by the composite

$$\tau \circ H^d(\text{map}_*(f, X))\colon H^d(\text{map}_*(F/\partial_{\text{in}}, X)) \longrightarrow H^d(\Omega X) \longrightarrow \mathbb{K},$$

where τ is the chosen orientation on ΩX. In this paper, we choose f such that we have the following homotopy commutative diagram

$$\begin{array}{ccc} \text{map}_*(F/\partial_{\text{in}}, X) & \xrightarrow{\text{incl}} & \text{map}(F, X) \\ {\scriptstyle \text{map}_*(f,X)}\uparrow{\scriptstyle\approx} & & {\scriptstyle\approx}\uparrow{\scriptstyle \text{map}(r,X)} \\ \Omega X & \xrightarrow[j]{} & LX \times_X LX \end{array}$$

where incl is the inclusion of the fibre of $\text{map}(\text{in}, X)$ and j is the map defined by $j(\omega) = (\omega, \omega^{-1})$.

Theorem 2.1 *Let $\iota\colon \Omega X \hookrightarrow LX$ be the inclusion of pointed loops into free loops. Let S be the antipode of the Hopf algebra $H^*(\Omega X)$. Let $\tau\colon H^d(\Omega X) \to \mathbb{K}$ be the chosen orientation on ΩX. Let $a \in H^p(LX)$ and $b \in H^q(LX)$ such that $p + q = d$. Then with the above choice of pointed homotopy equivalence $f\colon F/\partial_{\text{in}} \overset{\approx}{\to} S^1$,*

$$a \odot b = (-1)^{d(d-p)}\tau\big(H^p(\iota)(a) \cup S \circ H^q(\iota)(b)\big)1_{H^*(LX)}.$$

Proof Let $F \overset{\text{incl}}{\hookrightarrow} E \overset{\text{proj}}{\twoheadrightarrow} B$ be an oriented fibration with orientation $\tau\colon H^d(F) \to \mathbb{K}$. By definition or by naturality with respect to pull-backs, the integration along the fibre $\text{proj}^!$ is in degree d the composite

$$H^d(E) \xrightarrow{H^d(\text{incl})} H^d(F) \xrightarrow{\tau} \mathbb{K} \xrightarrow{\eta} H^0(B)$$

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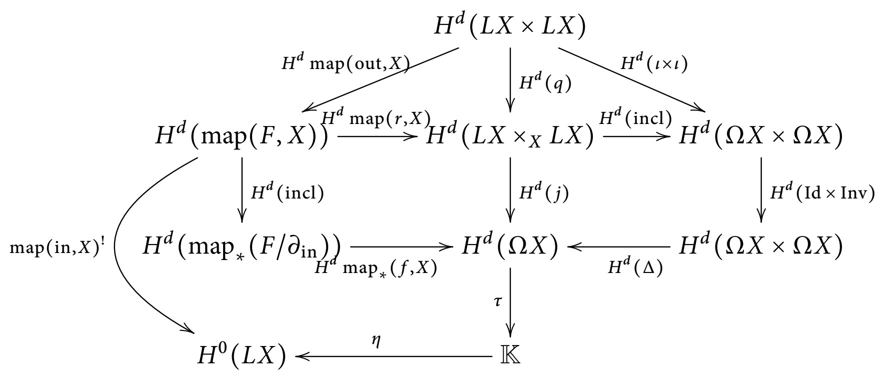
where η is the unit of $H^*(B)$. Therefore Dlcop is given by the commutative diagram



where incl: $\Omega X \times \Omega X \hookrightarrow LX \times_X LX$ is the inclusion and Inv: $\Omega X \to \Omega X$ maps a loop ω to its inverse ω^{-1}. Therefore,

$$\mathrm{Dlcop}(a \otimes b) = \tau\big(H^p(\iota)(a) \cup S \circ H^q(\iota)(b)\big)1_{H^*(LX)}. \qquad \blacksquare$$

We define a bracket $\{\cdot,\cdot\}$ on $H^*(LX)$ with the product $\odot$ and the BV-operator $\Delta\colon H^*(LX) \to H^{*-1}(LX)$ by

$$\{a, b\} = (-1)^{|a|}\Delta(a \odot b) - (-1)^{|a|}\Delta(a) \odot b - a \odot \Delta(b)$$

for a, b in $H^*(LX)$. By Theorem C.3, this bracket is exactly a Lie bracket. The following theorem is an analogue for the string topology of classifying spaces [6] to the theorems of Tamanoi [42] for Chas–Sullivan string topology [4]. This analogy is quite surprising and complete. For our calculations, in the rest of the paper, we use only parts (i)–(iii) of this theorem. Let ev: $LX \twoheadrightarrow X$ be the evaluation map defined by $\mathrm{ev}(\gamma) = \gamma(0)$ for $\gamma \in LX$.

Theorem 2.2 *(Cup products in string topology of classifying spaces) Let X be a simply-connected space such that $H_*(\Omega X; \mathbb{K})$ is finite-dimensional. Let P, $Q \in H^*(X)$, and a and $b \in H^*(LX)$.*

(i) (Cf. [42, Theorem A (1.2)]) *The dual of the loop coproduct*

$$\odot\colon \mathbb{H}^*(LX) \otimes \mathbb{H}^*(LX) \longrightarrow \mathbb{H}^*(LX)$$

is a morphism of left $H^*(X) \otimes H^*(X)$-modules:

$$\begin{aligned}(H^*(\mathrm{ev})(P) \cup a) \odot (H^*(\mathrm{ev})(Q) \cup b)\\ = (-1)^{(|a|-d)|Q|} H^*(\mathrm{ev})(P) \cup H^*(\mathrm{ev})(Q) \cup (a \odot b).\end{aligned}$$

(ii) (See [42, Theorem A (1.3)]) *The cup product with $\Delta \circ H^*(\mathrm{ev})(P)$ is a derivation with respect to the algebra $(\mathbb{H}^*(LX), \odot)$:*

$$\begin{aligned}\Delta \circ H^*(\mathrm{ev})(P) \cup (a \odot b) = (\Delta \circ H^*(\mathrm{ev})(P) \cup a) \odot b\\ + (-1)^{(|P|-1)(|a|-d)} a \odot (\Delta \circ H^*(\mathrm{ev})(P) \cup b).\end{aligned}$$

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(iii) *Let* $r \geq q0$. *Let* $P_1, \ldots, P_r$ *be r elements of* $H^*(X)$. *Denote by* $X_i := \Delta \circ H^*(\mathrm{ev})(P_i)$. *Then*

$$(H^*(\mathrm{ev})(P) \cup a) \odot \big(H^*(\mathrm{ev})(Q) \cup X_1 \cup \cdots \cup X_r \cup b \big) = (-1)^{(|a|-d)(|Q|+|X_1|+\cdots+|X_r|)}$$
$$\times \sum_{0 \leq j_1, \ldots, j_r \leq 1} \pm H^*(\mathrm{ev})(P) \cup H^*(\mathrm{ev})(Q) \cup X_1^{1-j_1} \cup \cdots \cup X_r^{1-j_r} \cup \big((X_1^{j_1} \cup \cdots \cup X_r^{j_r} \cup a) \odot b \big),$$

where $\pm$ *is the sign* $(-1)^{j_1+\cdots+j_r+\sum_{k=1}^r (1-j_k)|X_k|(j_1|X_1|+\cdots+j_{k-1}|X_{k-1}|)}$.

(iv) (See [42, Theorem A(1.4)]) *The cup product with* $\Delta \circ H^*(\mathrm{ev})(P)$ *is a derivation with respect to the bracket*

$$\Delta \circ H^*(\mathrm{ev})(P) \cup \{a, b\}$$
$$= \big\{ \Delta \circ H^*(\mathrm{ev})(P) \cup a, b \big\} + (-1)^{(|P|-1)(|a|-d-1)} \big\{ a, \Delta \circ H^*(\mathrm{ev})(P) \cup b \big\}.$$

(v) (See [42, formula p. 1220, line -9]) *The following formula gives a relation for the cup product of* $H^*(\mathrm{ev})(P)$ *with the bracket*

$$\{H^*(\mathrm{ev})(P) \cup a, b\}$$
$$= H^*(\mathrm{ev})(P) \cup \{a, b\} + (-1)^{|P|(|a|-d-1)} a \odot (\Delta \circ H^*(\mathrm{ev})(P) \cup b).$$

(vi) (See [42, Theorem B]) *The direct sum* $H^*(X) \oplus \mathbb{H}^*(LX)$ *is a BV-algebra where the dual of the loop coproduct* $\odot$, *the bracket, and the* Δ *operator are extended by*

$$P \odot a := H^*(\mathrm{ev})(P) \cup a, \qquad P \odot Q := P \cup Q$$
$$\{P, a\} := (-1)^{|P|} \Delta \circ H^*(\mathrm{ev})(P) \cup a, \qquad \{P, Q\} := 0,$$
$$\Delta(P) := 0.$$

(vii) (See [42, Theorem C]) *Suppose that the algebra* $(\mathbb{H}^*(LX), \odot)$ *has a unit* $\mathbb{I}$. *Let* $s^!: H^*(X) \to H^{*+d}(LX)$ *be the map sending P to* $H^*(\mathrm{ev})(P) \cup \mathbb{I}$. *Then* $s^!$ *is a morphism of BV-algebras with respect to the trivial BV-operator on* $H^*(X)$ *and*

$$H^*(\mathrm{ev})(P) \cup a = s^!(P) \odot a \quad \text{and} \quad (-1)^{|P|} \Delta \circ H^*(\mathrm{ev})(P) \cup a = \{s^!(P), a\}.$$

To prove parts (i) and (ii), it is shorter to use the following lemma. This lemma is just the cohomological version of [4, Theorem 8.2] when we replace the correspondence $LM \times LM \overset{q}{\hookleftarrow} LM \times_M LM \xrightarrow{\mathrm{Comp}} LM$ by its opposite

$$LX \xleftarrow{\mathrm{Comp}} LX \times_X LX \overset{q}{\hookrightarrow} LX \times LX.$$

Similarly, it would have been shorter for Tamanoi to prove [42, Theorem A (1.2), (1.3)] using [4, Theorem 8.2].

Lemma 2.3 *Let* $a = \sum a_1 \otimes a_2 \in H^*(LX \times LX)$ *and* $A \in H^*(LX)$ *such that* $H^*(\mathrm{Comp})(A) = H^*(q)(a)$. *Then for any* $z_1, z_2 \in H^*(LX)$,

$$A \cup (z_1 \odot z_2) = \sum (-1)^{(|z_1|-d)|a_2|} (a_1 \cup z_1) \odot (a_2 \cup z_2).$$

Proof Integration along the fibre, $\mathrm{Comp}^!$, is a morphism of left $H^*(LX)$-modules with the correct signs (see our definition of integration along the fibre in cohomology in Appendix A). Therefore

$$\mathrm{Comp}^!(H^*(\mathrm{Comp})(A)\cup y) = (-1)^{d|A|}A\cup \mathrm{Comp}^!(y).$$

Let $z := z_1\otimes z_2 \in H^*(LX\times LX)$. Since $H^*(q)$ is a morphism of algebras,

$$\begin{aligned}(-1)^{d|A|}\,\mathrm{Dlcop}(a\cup z) &= (-1)^{d|A|}\,\mathrm{Comp}^!\circ H^*(q)(a\cup z)\\ &= (-1)^{d|A|}\,\mathrm{Comp}^!(H^*(\mathrm{Comp})(A)\cup H^*(q)(z))\\ &= A\cup \mathrm{Comp}^!\circ H^*(q)(z) = A\cup \mathrm{Dlcop}(z).\end{aligned}$$

Then the previous equation is

$$\begin{aligned}&A\cup(-1)^{d(|z_1|-d)}z_1\odot z_2\\ &\qquad=\sum(-1)^{d(|a_1|+|a_2|)}(-1)^{d(|a_1|+|z_1|-d)}(-1)^{|a_2||z_1|}(a_1\cup z_1)\odot(a_2\cup z_2).\end{aligned}$$

■

Proof of Theorem 2.2 (i) We have the commutative diagram

$$\begin{array}{ccccc} LX & \xleftarrow{\ \mathrm{Comp}\ } & LX\times_X LX & \xrightarrow{\ q\ } & LX\times LX\\ & \searrow^{\mathrm{ev}} & \downarrow & & \downarrow{\scriptstyle \mathrm{ev}\times\mathrm{ev}}\\ & & X & \xrightarrow[\ \delta\ ]{} & X\times X \end{array}$$

Therefore by applying Lemma 2.3 to $a := H^*(\mathrm{ev}\times\mathrm{ev})(P\otimes Q)$, $A := H^*(\delta\circ\mathrm{ev})(P\otimes Q)$, $z_1 := a$, and $z_2 := b$, we obtain (i).

(ii) By [42, Proof of Theorem 4.2 (4.5)]

$$\mathrm{Comp}^*(\Delta\circ H^*(\mathrm{ev})(P)) = H^*(q)(\Delta\circ H^*(\mathrm{ev})(P)\times 1 + 1\times\Delta\circ H^*(\mathrm{ev})(P)).$$

So we can apply Lemma 2.3 to $a := \Delta\circ H^*(\mathrm{ev})(P)\times 1 + 1\times\Delta\circ H^*(\mathrm{ev})(P)$ and $A := \Delta\circ H^*(\mathrm{ev})(P)$.

(iii) The case $r=0$ is just (i). Now, by induction on r,

$$\begin{aligned}&(H^*(\mathrm{ev})(P)\cup a)\odot(H^*(\mathrm{ev})(Q)\cup X_1\cup\cdots\cup X_{r-1}\cup(X_r\cup b))\\ &\quad= (-1)^{(|a|-d)(|Q|+|X_1|+\cdots+|X_{r-1}|)}\sum_{0\le j_1,\dots,j_{r-1}\le 1}\pm H^*(\mathrm{ev})(P)\cup H^*(\mathrm{ev})(Q)\\ &\qquad\cup X_1^{1-j_1}\cup\cdots\cup X_{r-1}^{1-j_{r-1}}\cup\big((X_1^{j_1}\cup\cdots\cup X_{r-1}^{j_{r-1}}\cup a)\odot(X_r\cup b)\big)\end{aligned}$$

But by (ii),

$$\begin{aligned}&(X_1^{j_1}\cup\cdots\cup X_{r-1}^{j_{r-1}}\cup a)\odot(X_r\cup b)\\ &\quad=\sum_{j_r=0}^{1}(-1)^{|X_r|(|a|-d)+j_r+(1-j_r)|X_r|\sum_{l=1}^{r-1}j_l|X_l|}X_r^{1-j_r}\cup((X_1^{j_1}\cup\cdots\cup X_r^{j_r}\cup a)\odot b).\end{aligned}$$

(iv) By using Theorem 2.2 (ii), the same argument as in [42, Proof of Theorem 4.5] deduces the derivation formula on the bracket.

(v) Again, the arguments are identical to those given by Tamanoi [42, end of proof of Theorem 4.7].

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(vi) As explained by Tamanoi [42, proof of Theorem 4.7], (ii), (iv), and (v) are equivalent to the Poisson and Jacobi identities in the Gerstenhaber algebra

$$H^*(X) \oplus \mathbb{H}^*(LX).$$

By definition of the bracket, this Gerstenhaber algebra is a BV-algebra [42, proof of Theorem 4.8].

(vii) Since $H^{*+d}(LX)$ is an $H^*(X)$-algebra, (Theorem 2.2 (i)), the map

$$s^! \colon H^*(X) \to H^{*+d}(LX), \quad P \mapsto H^*(\mathrm{ev})(P) \cup \mathbb{I},$$

is a morphism of unital commutative graded algebras (we denote this map $s^!$ because this map should coincide with some Gysin map of the trivial section $s\colon X \hookrightarrow LX$ [6]. Indeed, by $H^*(LX)$-linearity, $s^!(P) = s^! \circ H^*(s) \circ H^*(ev)(P) = (-1)^{d|P|} H^*(\mathrm{ev})(P) \cup s^!(1)$.

Since the cup product with $\Delta \circ H^*(\mathrm{ev})(P)$ is a derivation with respect to the dual of the loop coproduct, $\Delta \circ H^*(\mathrm{ev})(P) \cup \mathbb{I} = 0$. Since $\mathbb{H}^*(LX)$ is a BV-algebra, $\Delta(\mathbb{I}) = 0$. Therefore, since Δ is a derivation with respect to the cup product,

$$\Delta(s^!(P)) = \Delta \circ H^*(\mathrm{ev})(P) \cup \mathbb{I} + (-1)^{|P|} H^*(\mathrm{ev})(P) \cup \Delta(\mathbb{I}) = 0 + 0.$$

Now we can conclude using the same arguments as in [42, proof of Theorem 5.1]. ∎

Remark 2.4. Suppose that the algebra $H^*(LX)$ is generated by $H^*(X)$ and $\Delta(H^*(X))$. Then by Theorem 2.2 (iii) when $b = 1$, we see that the dual of the loop coproduct $\odot$ is completely given by the cup product, by the Δ operator, and by its restriction on $\mathbb{H}^*(LX) \otimes 1$. In the following section, we show that this is the case when $H^*(X)$ is a polynomial (see Remark 3.2).

3 The Cup Product on Free Loops and the Main Theorem

Let X be a simply-connected space with polynomial cohomology: $H^*(X)$ is a polynomial algebra $\mathbb{K}[y_1, \dots, y_N]$. The cup product on the free loop space cohomology $H^*(LX; \mathbb{K})$ was first computed by the first author [28, Theorem 1.6]. We now explain how to recover simply this computation following [24, p. 648].

Let $\sigma\colon H^*(X) \to H^{*-1}(\Omega X)$ be the suspension homomorphism and $\sigma(y_i)$ be the suspension image of y_i. By Borel's theorem [38, Chapter VII. Corollary 2.8(2)], which can be easily proved using the Eilenberg–Moore spectral sequence associated with the path fibration $\Omega X \hookrightarrow PX \twoheadrightarrow X$ since $E_2^{*,*} \cong \Lambda(\sigma(y_1), \dots, \sigma(y_N))$,

$$H^*(\Omega X; \mathbb{K}) = \underline{\wedge}(\sigma(y_1), \dots, \sigma(y_N)),$$

where $\underline{\wedge}\sigma(y_i)$ denotes an algebra with a simple system of generators $\sigma(y_i)$ (Here an algebra with a simple system of generators x_i is a graded commutative algebra, denoted $\underline{\wedge}x_i$, such that the products of the form $x_{i_1}x_{i_2}\cdots x_{i_r}$ with $1 \le i_1 < i_2 < \cdots < i_r \le N$ and $r \ge 0$ form a linear basis of the algebra [38, Definition p. 367]). If $\mathrm{ch}(\mathbb{K}) \neq 2$, $\underline{\wedge}\sigma(y_i)$ is just the exterior algebra $\Lambda\sigma(y_i)$.

Let $\Delta\colon H^*(LX) \to H^{*-1}(LX)$ be the operator induced by the action of the circle on LX (Appendix E). Let $\mathcal{D} := \Delta \circ H^*(\mathrm{ev})$ denote the module derivation in [28].

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Since Δ is a derivation with respect to the cup product, $\mathcal{D}$ is a $(H^*(\mathrm{ev}), H^*(\mathrm{ev}))$-derivation [28, Proposition 3.3]. Since Δ and $H^*(\mathrm{ev})$ commutes with the Steenrod operations, $\mathcal{D}$ also commutes with them [28, Proposition 3.3]. Since the composite $H^*(\iota)\circ\mathcal{D}$ is the suspension homomorphism σ [24, Proposition 2(1)], $H^*(\iota)$ is surjective and so by the Leray–Hirsch theorem,

$$H^*(LX;\mathbb{K}) = H^*(X)\otimes\underline{\wedge}(\mathcal{D}(y_1),\dots,\mathcal{D}(y_N))$$

as $H^*(X)$-algebra. Modulo 2, it follows from above that $H^*(LX;\mathbb{F}_2)$ is the polynomial algebra $\mathbb{F}_2[H^*(\mathrm{ev})(y_i), \mathcal{D}y_i]$ quotiented by the relations

$$(\mathcal{D}y_i)^2 = \mathcal{D}(\mathrm{Sq}^{|y_i|-1}y_i).$$

In particular, we have $\Delta(H^*(\mathrm{ev})(y_i)) = \mathcal{D}y_i$ and $\Delta(\mathcal{D}y_i) = 0$, since $\Delta\circ\Delta = 0$. Therefore, we know the cup product and the Δ operator on $H^*(LX;\mathbb{K})$. The following theorem shows that we also know the dual of the loop coproduct.

Theorem 3.1 *Let X be a simply-connected space such that $H^*(X;\mathbb{K})$ is the polynomial algebra $\mathbb{K}[y_1,\dots,y_N]$. Denote again by y_i, the element of $H^*(LX)$, $H^*(\mathrm{ev})(y_i)$, and by x_i, $\Delta\circ H^*(\mathrm{ev})(y_i)$. Often, the cup product $a\cup b$ on $H^*(LX)$ is now simply denoted ab. With respect to this cup product, as algebras we have*

$$H^*(LX) = \mathbb{K}[y_1,\dots,y_N]\otimes\underline{\wedge}(x_1,\dots,x_N).$$

Let d be the degree of $x_1\cdots x_N$. Then the dual of the loop coproduct

$$\odot\colon H^i(LX)\otimes H^j(LX)\longrightarrow H^{i+j-d}(LX)$$

is given inductively (Remark 3.2) by the following four formulas.

(i) *For any a and $b\in H^*(LX)$, for all $1\le i\le N$,*

$$a\odot x_i b = (-1)^{|x_i|(|a|-d)}x_i(a\odot b) - (-1)^{d|x_i|}ax_i\odot b$$

(ii) *Let $\{i_1,\dots,i_l\}$ and $\{j_1,\dots,j_m\}$ be two disjoint subsets of $\{1,\dots,N\}$ such that $\{i_1,\dots,i_l\}\cup\{j_1,\dots,j_m\} = \{1,\dots,N\}$. If we orient $\tau\colon H^d(\Omega X)\overset{\cong}{\to}\mathbb{K}$ by*

$$\tau\circ H^*(\iota)(x_1\dots x_N) = 1,$$

then $x_{i_1}\cdots x_{i_l}\odot x_{j_1}\cdots x_{j_m} = (-1)^{Nm+m}\varepsilon$, where ε is the signature of the permutation

$$\begin{pmatrix} 1 & & \cdots & & & l+m \\ i_1 & \cdots & i_l & j_1 & \cdots & j_m \end{pmatrix}.$$

(iii) *Let $\{i_1,\dots,i_l\}$ and $\{j_1,\dots,j_m\}$ be two disjoint subsets of $\{1,\dots,N\}$ such that $\{i_1,\dots,i_l\}\cup\{j_1,\dots,j_m\}\neq\{1,\dots,N\}$. Then $x_{i_1}\cdots x_{i_l}\odot x_{j_1}\cdots x_{j_m} = 0$.*

(iv) *$\odot$ is a morphism of left $H^*(X)\otimes H^*(X)$-modules: for $P,Q\in H^*(X)$ and $a,b\in H^*(LX)$, one has $(-1)^{|Q|(|a|-d)}Pa\odot Qb = PQ(a\odot b)$.*

Proof Note that if y_i is of odd degree, then $2=0$ in $\mathbb{K}$. (i) and (iv) are particular cases of Theorem 2.2 (i) and (ii). Since $x_{i_1}\cdots x_{i_l}\otimes x_{j_1}\cdots x_{j_m}$ is of degree less than d, for degree reasons, we have (iii).

(ii) Since $H^*(\iota)(x_i) = H^*(\iota)\circ\Delta\circ H^*(\mathrm{ev})(y_i)$ is the suspension of y_i, denoted $\sigma(y_i)$, by Theorem 2.1,

$$x_{i_1}\cdots x_{i_l}\odot x_{j_1}\cdots x_{j_m} = (-1)^{Nm}\tau\big(\sigma(y_{i_1})\cdots\sigma(y_{i_l})\cup S(\sigma(y_{j_1})\cdots\sigma(y_{j_m})\big)1.$$

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Since $\sigma(y_i)$ is a primitive element, $S(\sigma(y_i)) = -\sigma(y_i)$. Since the antipode

$$S\colon H^*(\Omega X) \to H^*(\Omega X)$$

is also a morphism of commutative graded algebras,

$$x_{i_1}\cdots x_{i_l} \odot x_{j_1}\cdots x_{j_m} = (-1)^{Nm+m}\varepsilon\tau(\sigma(y_1)\cdots\sigma(y_N)). \qquad \blacksquare$$

Remark 3.2. We now explain why the four formulas of Theorem 3.1 determine inductively the dual of the loop coproduct $\odot$. For $P \in H^*(X)$ and $\{i_1,\dots,i_l\}$ a strict subset of $\{1,\dots,N\}$, by (ii), (iii), and (iv), $Px_{i_1}\cdots x_{i_l}\odot 1 = 0$ and $Px_1\cdots x_N \odot 1 = P$. Therefore, we know the restriction of $\odot$ on $\mathbb{H}^*(LX)\otimes 1$. Since the algebra $H^*(LX)$ is generated by $H^*(X)$ and $\Delta(H^*(X))$, the product $\odot$ is now given inductively by (i) and (iv) (see Remark 2.4).

The restriction of $\odot\colon \mathbb{H}^*(LX)\otimes 1 \to H^*(X)$ looks similar to the intersection morphism $\iota_!\colon \mathbb{H}_*(LM) \to H_*(\Omega M)$ for a manifold M given by the loop product with the constant pointed loop.

4 Case p Odd or No Sq_1

Let Sq_1 be the operator $H^*(\mathrm{BG};\mathbb{F}_2) \to H^*(\mathrm{BG};\mathbb{F}_2)$ defined by $\mathrm{Sq}_1(x) = \mathrm{Sq}^{\deg x-1}x$ for $x \in H^*(\mathrm{BG};\mathbb{F}_2)$.

Suppose that $H^*(\mathrm{BG};\mathbb{K})$ is a polynomial algebra $\mathbb{K}[y_1,\dots,y_N]$ and that

(H) : $Sq_1 \equiv 0$ on $H^*(\mathrm{BG})$ if $\mathbb{K} = \mathbb{F}_2$ or the characteristic of $\mathbb{K}$ is different from 2.

(Since $\mathrm{Sq}_1(PQ) = P^2\,\mathrm{Sq}_1(Q) + \mathrm{Sq}_1(P)Q^2$, it suffices to check that $\mathrm{Sq}_1(y_i) = 0$ for all i.) Then it follows from Section 3 (or [26, Remark 3.4]) that

$$H^*(\mathrm{LBG};\mathbb{K}) = \wedge(x_1,\dots,x_N)\otimes\mathbb{K}[y_1,\dots,y_N]$$

as an algebra where $x_i := \Delta\circ H^*(ev)(y_i)$. Then we have the following.

Theorem 4.1 *Under hypothesis* (H)*, an explicit form of the dual of the loop coproduct* $\odot\colon H^*(\mathrm{LBG};\mathbb{K})\otimes H^*(\mathrm{LBG};\mathbb{K}) \to H^{*-\dim G}(\mathrm{LBG};\mathbb{K})$ *is given by*

$$x_{i_1}\cdots x_{i_l}a \odot x_{j_1}\cdots x_{j_m}b = (-1)^{\varepsilon'+\varepsilon+m+u+lu+Nm}x_{k_1}\cdots x_{k_u}ab$$

if $\{i_1,\dots,i_l\}\cup\{j_1,\dots,j_m\} = \{1,\dots,N\}$ *and* $x_{i_1}\cdots x_{i_l}a\odot x_{j_1}\cdots x_{j_m}b = 0$ *otherwise, where* $\{i_1,\dots,i_l\}\cap\{j_1,\dots,j_m\} = \{k_1,\dots,k_u\}$*,* $a, b \in H^*(\mathrm{BG})$*,*

$$(-1)^{\varepsilon} = \mathrm{sgn}\begin{pmatrix} j_1 & \cdots & \cdots & \cdots & \cdots & j_m \\ k_1\cdots k_u & j_1\cdots & \widehat{k_1} & \cdots & \widehat{k_u} & \cdots j_m\end{pmatrix},$$

$$(-1)^{\varepsilon'} = \mathrm{sgn}\begin{pmatrix} i_1\cdots i_l & j_1\cdots & \widehat{k_1} & \cdots & \widehat{k_u} & \cdots j_m \\ 1 & \cdots & \cdots & \cdots & \cdots & N\end{pmatrix}.$$

Here $\widehat{x}$ means that the element x disappears from the presentation.

Over $\mathbb{R}$, Behrend, Ginot, Noohi, and Xu [1, 17.23] had the same formula without any signs for their dual hidden loop product $\star$ on $H^*([G/G])$. With our signs, $\odot$ is graded associative and graded commutative (Corollary B.3). In [1, 17.23], $\star$ is commutative, but not graded commutative. For example, by [1, 17.23],

$$x_1\cdots x_{N-1}\star x_2\cdots x_N = x_2\cdots x_N = x_2\cdots x_N\star x_1\cdots x_{N-1},$$

although $x_1\cdots x_{N-1}$ and $x_2\cdots x_N$ are of odd degree in $H^{*+d}(\mathrm{LBG})$.

Proof of Theorem 4.1 To prove Theorem 4.1, by Theorem 3.1 (iv) it suffices to show the formula for the element $x_{i_1}\cdots x_{i_l}\otimes x_{j_1}\cdots x_{j_m}$, namely where $a=b=1$.

Since $x_{k_1}^2=0$, $x_{i_1}\cdots x_{i_l}x_{k_1}\odot x_{j_1}\cdots\widehat{x_{k_1}}\cdots x_{j_m}=0$. So by Theorem 3.1 (i),

$$x_{i_1}\cdots x_{i_l}\odot x_{j_1}\cdots x_{j_m}=(-1)^{|x_{k_1}|(|x_{i_1}\cdots x_{i_l}x_{j_1}\cdots\widehat{x_{k_1}}|-d)}x_{k_1}(x_{i_1}\cdots x_{i_l}\odot x_{j_1}\cdots\widehat{x_{k_1}}\cdots x_{j_m}).$$

By induction on u,

$$x_{i_1}\cdots x_{i_l}\odot x_{j_1}\cdots x_{j_m}=(-1)^{u(l-d)+\varepsilon}x_{k_1}\cdots x_{k_u}(x_{i_1}\cdots x_{i_l}\odot x_{j_1}\cdots\widehat{x_{k_1}}\cdots\widehat{x_{k_u}}\cdots x_{j_m}).$$

By Theorem 3.1 (ii) and (iii),

$$x_{i_1}\cdots x_{i_l}\odot x_{j_1}\cdots\widehat{x_{k_1}}\cdots\widehat{x_{k_u}}\cdots x_{j_m} = \begin{cases}(-1)^{N(m-u)+m-u+\varepsilon'} & \text{if } \{i_1,\dots,i_l\}\cup\{j_1,\dots,j_m\}=\{1,\dots,N\},\\ 0 & \text{otherwise.}\end{cases}$$

■

Corollary 4.2 *Under hypothesis* (H)*, the graded associative commutative algebra* $(\mathbb{H}^*(\mathrm{LBG}),\odot)$ *of Corollary B.3 is unital.*

Proof We see that $x_1\cdots x_N$ is the unit. Theorem 4.1 yields that

$$x_1\cdots x_N\odot x_{j_1}\cdots x_{j_m}b = \operatorname{sgn}\begin{pmatrix} j_1\cdots j_m\\ j_1\cdots j_m\end{pmatrix}\operatorname{sgn}\begin{pmatrix}1\cdots N\\ 1\cdots N\end{pmatrix}(-1)^{m+m+mN+Nm}x_{j_1}\cdots x_{j_m}b.$$

$$x_{i_1}\cdots x_{i_l}a\odot x_1\cdots x_N = \operatorname{sgn}\begin{pmatrix}1 & \cdots & \cdots & \cdots & \cdots & N\\ i_1\cdots i_l & 1\cdots & \widehat{i_1} & \cdots & \widehat{i_l} & \cdots N\end{pmatrix} \operatorname{sgn}\begin{pmatrix} i_1\cdots i_l & 1\cdots & \widehat{i_1} & \cdots & \widehat{i_l} & \cdots N\\ 1 & \cdots & \cdots & \cdots & \cdots & N\end{pmatrix}(-1)^{N+l+l^2+N^2}x_{i_1}\cdots x_{i_l}a.$$

■

Theorem 4.3 *Under hypothesis* (H), $\mathbb{H}^*(\mathrm{LBG})=H^{*+\dim G}(\mathrm{LBG};\mathbb{K})$ *is isomorphic as* BV *algebras to the tensor product of algebras*

$$H^*(\mathrm{BG};\mathbb{K})\otimes H_{-*}(G;\mathbb{K})\cong\mathbb{K}[y_1,\dots,y_N]\otimes\wedge(x_1^\vee,\dots,x_N^\vee)$$

equipped with the BV*-operator* Δ *given by* $\Delta(x_i^\vee\wedge x_j^\vee)=\Delta(y_iy_j)=\Delta(x_j^\vee)=\Delta(y_i)=0$ *for any* i, j *and*

$$\Delta(y_i\otimes x_j^\vee)=\begin{cases}0 & \text{if } i\neq j,\\ 1 & \text{if } i=j.\end{cases}$$

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Proof Since $H^*(G)$ is the Hopf algebra Λx_i with $x_i = \sigma(y_i)$ primitive, its dual is the Hopf algebra $\Lambda x_i^\vee$. By Corollary B.3 and Corollary 4.2, we see that the shifted cohomology $\mathbb{H}^*(\mathrm{LBG})$ is a graded commutative algebra with unit $x_1 \cdots x_N$. This enables us to define a morphism of algebras Θ from

$$H^*(BG;\mathbb{K}) \otimes H_{-*}(G;\mathbb{K}) = \mathbb{K}[y_1,\dots,y_n] \otimes \Lambda(x_1^\vee,\dots,x_N^\vee)$$

to

$$\mathbb{H}^*(\mathrm{LBG}) = \mathbb{K}[y_1,\dots,y_n] \otimes \Lambda(x_1,\dots,x_N)$$

by

$$\Theta(1 \otimes x_j^\vee) = (-1)^{j-1} 1 \otimes (x_1 \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_N),$$
$$\Theta(a \otimes 1) = a \otimes (x_1 \wedge \cdots \wedge x_N)$$

for any a in $\mathbb{K}[V]$. By induction on p, using Theorem 4.1, we have

$$\Theta(a \otimes (x_{j_1}^\vee \wedge \cdots \wedge x_{j_p}^\vee)) = \pm a \otimes (x_1 \wedge \cdots \wedge \widehat{x_{j_1}} \wedge \cdots \wedge \widehat{x_{j_p}} \wedge \cdots \wedge x_N)$$

for any $a \in \mathbb{K}[V]$. Therefore the map Θ is an isomorphism.

The isomorphism Θ sends $1 \otimes \Lambda(x_1^\vee,\dots,x_N^\vee)$ to $1 \otimes \Lambda(x_1,\dots,x_N)$ and sends $\mathbb{K}[y_1,\dots,y_N] \otimes 1$ to $\mathbb{K}[y_1,\dots,y_N] \otimes x_1 \cdots x_N$. Since Δ is null on $1 \otimes \Lambda(x_1,\dots,x_N)$ and $\mathbb{K}[y_1,\dots,y_N] \otimes x_1 \cdots x_N$, Δ is null on $1 \otimes \Lambda(x_1^\vee,\dots,x_N^\vee)$ and $\mathbb{K}[y_1,\dots,y_N] \otimes 1$; we have the first equalities. Moreover, we see that $\Theta(y_i \otimes x_j^\vee) = (-1)^{j-1} y_i x_1 \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_N$ and hence $\Delta\Theta(y_i \otimes x_j^\vee) = 0$ if $i \neq j$. The equalities $\Delta((-1)^{i-1} y_i x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_N) = x_1 \wedge \cdots \wedge x_N = \Theta(1)$ enable us to obtain the second formula. ∎

5 Mod 2 Case

In the case where the operation Sq_1 is non-trivial on $H^*(\mathrm{BG};\mathbb{F}_2)$, the loop coproduct structure on $H^*(\mathrm{LBG};\mathbb{F}_2)$ is more complicated in general. For example, we compute the dual of the loop coproduct on $H^*(\mathrm{LBG}_2;\mathbb{F}_2)$, where G_2 is the simply-connected compact exceptional Lie group of rank 2. Recall that

$$\begin{aligned} H^*(\mathrm{LBG}_2;\mathbb{F}_2) &\cong \underline{\wedge}(x_3,x_5,x_6) \otimes \mathbb{F}_2[y_4,y_6,y_7] \\ &\cong \mathbb{F}_2[x_3,x_5] \otimes \mathbb{F}_2[y_4,y_6,y_7] \Big/ \begin{pmatrix} x_3^4 + x_5 y_7 + x_3^2 y_6 \\ x_5^2 + x_3 y_7 + x_3^2 y_4 \end{pmatrix} \end{aligned}$$

as algebras over $H^*(BG_2;\mathbb{F}_2) \cong \mathbb{F}_2[y_4,y_6,y_7]$, where $\deg x_i = i$ and $\deg y_j = j$; see [28, Theorem 1.7].

Theorem 5.1 *The dual to the loop coproduct*

$$\mathrm{Dlcop}\colon H^*(\mathrm{LBG}_2;\mathbb{F}_2) \otimes H^*(\mathrm{LBG}_2;\mathbb{F}_2) \to H^{*-14}(\mathrm{LBG}_2;\mathbb{F}_2)$$

is commutative and the only non-trivial forms restricted to the submodule

$$\underline{\wedge}(x_3,x_5,x_6) \otimes \underline{\wedge}(x_3,x_5,x_6)$$

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are given by

$$\begin{aligned}
\mathrm{Dlcop}(x_3x_5x_6\otimes 1) &= \mathrm{Dlcop}(x_3x_5\otimes x_6) = \mathrm{Dlcop}(x_3x_6\otimes x_5)\\
&= \mathrm{Dlcop}(x_5x_6\otimes x_3) = 1,\\
\mathrm{Dlcop}(x_3x_5x_6\otimes x_3) &= \mathrm{Dlcop}(x_3x_5\otimes x_3x_6) = x_3,\\
\mathrm{Dlcop}(x_3x_5x_6\otimes x_5) &= \mathrm{Dlcop}(x_3x_5\otimes x_5x_6) = x_5,\\
\mathrm{Dlcop}(x_3x_5x_6\otimes x_6) &= \mathrm{Dlcop}(x_3x_6\otimes x_5x_6) = x_6+y_6,\\
\mathrm{Dlcop}(x_3x_5x_6\otimes x_3x_5) &= x_3x_5,\\
\mathrm{Dlcop}(x_3x_5x_6\otimes x_3x_6) &= x_3x_6+x_3y_6,\\
\mathrm{Dlcop}(x_3x_5x_6\otimes x_5x_6) &= x_5x_6+x_5y_6+y_4y_7,\\
\mathrm{Dlcop}(x_3x_5x_6\otimes x_3x_5x_6) &= x_3x_5x_6+x_3x_5y_6+x_3y_4y_7+y_7^2.
\end{aligned}$$

The proof of Theorem 5.1 will be given after the proof of Theorem 5.7.

Lemma 5.2 *Let $k\colon\{1,\dots,q\}\to\{1,\dots,N\}$, $j\mapsto k_j$ be a map such that for $1\le i\le N$, the cardinality of the inverse image $k^{-1}(\{i\})$ is less than or equal to 2. In $H^*(LX;\mathbb{F}_2)=\mathbb{F}_2[y_1,\dots,y_N]\otimes\triangle(x_1,\dots,x_N)$, the cup product satisfies the equality*

$$x_{k_1}\cdots x_{k_q}=\sum_{\substack{0\le l\le \mathit{cardinal\ of}\ \{k_1,\dots,k_q\},\\ 1\le i_1<\cdots<i_l\le N}} P_{i_1,\dots,i_l}x_{i_1}\cdots x_{i_l},$$

where $P_{i_1,\dots,i_l}$ are elements of $\mathbb{F}_2[y_1,\dots,y_N]$.

Proof Suppose by induction that the lemma is true for $q-1$. If the elements $k_1,\dots,k_q$ are pairwise distinct, take $\{i_1,\dots,i_l\}=\{k_1,\dots,k_q\}$. Otherwise by permuting the elements $x_{k_1},\dots,x_{k_q}$, suppose that $k_{q-1}=k_q$.

$$x_{k_q}^2=\Delta\circ H^*(\mathrm{ev})\circ\mathrm{Sq}^{|y_{k_q}|-1}(y_{k_q})=\sum_{i=1}^N x_iP_i,$$

where $P_1,\dots,P_N$ are elements of $\mathbb{F}_2[y_1,\dots,y_N]$. So $x_{k_1}\cdots x_{k_q}=\sum_{i=1}^N x_{k_1}\cdots x_{k_{q-2}}x_iP_i$. Since $k_q=k_{q-1}$, by hypothesis, k_q in $\{k_1,\dots,k_{q-2}\}$. Therefore the cardinal of

$$\{k_1,\dots,k_{q-2},i\}$$

is less or equal to the cardinal of $\{k_1,\dots,k_q\}$. By our induction hypothesis,

$$x_{k_1}\cdots x_{k_{q-2}}x_i=\sum_{\substack{0\le l\le \mathrm{cardinal\ of}\ \{k_1,\dots,k_{q-2},i\},\\ 1\le i_1<\cdots<i_l\le N}} P_{i_1,\dots,i_l}x_{i_1}\cdots x_{i_l}.$$ ■

Lemma 5.3 *Let $k\colon\{1,\dots,q+r\}\to\{1,\dots,N\}$, $j\mapsto k_j$ be a non-surjective map such that for all $1\le i\le N$, the cardinality of the inverse image $k^{-1}(\{i\})$ is less than 2. Then*

$$\mathrm{Dlcop}(x_{k_1}\cdots x_{k_q}\otimes x_{k_{q+1}}\cdots x_{k_{q+r}})=0.$$

Proof We do an induction on $r\ge 0$.

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Case $r = 0$: By Lemma 5.2, since the cardinal of $\{k_1, \dots, k_q\}$is less than N,

$$\mathrm{Dlcop}(x_{k_1} \cdots x_{k_q} \otimes 1) = \sum_{\substack{0 \leq l < N, \\ 1 \leq i_1 < \cdots < i_l \leq N}} \mathrm{Dlcop}(P_{i_1,\dots,i_l} x_{i_1} \cdots x_{i_l} \otimes 1),$$

where $P_{i_1,\dots,i_l}$ are elements of $\mathbb{F}_2[y_1, \dots, y_N]$. By Theorem 3.1 (iii), (iv), since $l < N$,

$$\mathrm{Dlcop}(P_{i_1,\dots,i_l} x_{i_1} \cdots x_{i_l} \otimes 1) = 0.$$

Suppose now by induction that the lemma is true for $r - 1$. Then by Theorem 3.1 (i),

$$\begin{aligned} \mathrm{Dlcop}(x_{k_1} \cdots x_{k_q} \otimes x_{k_{q+1}} \cdots x_{k_{q+r}}) &= x_{k_{q+1}} \mathrm{Dlcop}(x_{k_1} \cdots x_{k_q} \otimes x_{k_{q+2}} \cdots x_{k_{q+r}}) \\ &\quad + \mathrm{Dlcop}(x_{k_1} \cdots x_{k_{q+1}} \otimes x_{k_{q+2}} \cdots x_{k_{q+r}}) \\ &= x_{k_{q+1}} \cup 0 + 0. \end{aligned}$$

■

Let $I = \{i_1, \dots, i_l\} \subset \{1, \dots, N\}$. In $\underline{\wedge}(x_1, \dots, x_N)$, denote the generator $x_{i_1} \cup x_{i_2} \cup \cdots \cup x_{i_l}$ by x_I. Since we consider the algebra over $\mathbb{F}_2$, the cup product is commutative, so we do not need to assume that $i_1 < i_2 < \cdots < i_l$.

Theorem 5.4 *Let I and J be two subsets of $\{1, \dots, N\}$. Then*

$$\mathrm{Dlcop}(x_I \otimes x_J) = \begin{cases} \mathrm{Dlcop}(x_1 \cdots x_N \otimes x_{I \cap J}) & \text{if } I \cup J = \{1, \dots, N\}, \\ 0 & \text{otherwise.} \end{cases}$$

In particular $\{x_I, x_J\} = \Delta(\mathrm{Dlcop}(x_I \otimes x_J)) = \Delta(\mathrm{Dlcop}(x_{I \cup J} \otimes x_{I \cap J})) = \{x_{I \cup J}, x_{I \cap J}\}$.

Proof Let $i_1, \dots, i_l$ denote the elements of the relative complement $I - J$, $j_1, \dots, j_m$ denote the elements of the relative complement $J - I$, and $k_1, \dots, k_u$ denote the elements of the intersection $I \cap J$.

By Lemma 5.3, $\mathrm{Dlcop}(x_{i_1} \dots x_{i_l} x_{k_1} \dots x_{k_u} \otimes x_{j_2} \dots x_{j_m} x_{k_1} \dots x_{k_u}) = 0$. So by Theorem 3.1 (i),

$$\begin{aligned} &\mathrm{Dlcop}(x_{i_1} \cdots x_{i_l} x_{k_1} \cdots x_{k_u} \otimes x_{j_1} \cdots x_{j_m} x_{k_1} \cdots x_{k_u}) \\ &\qquad = x_{j_1} \cup 0 + \mathrm{Dlcop}(x_{i_1} \cdots x_{i_l} x_{j_1} x_{k_1} \cdots x_{k_u} \otimes x_{j_2} \cdots x_{j_m} x_{k_1} \cdots x_{k_u}). \end{aligned}$$

By induction on m, this is equal to $\mathrm{Dlcop}(x_{i_1} \cdots x_{i_l} x_{j_1} \cdots x_{j_m} x_{k_1} \cdots x_{k_u} \otimes x_{k_1} \cdots x_{k_u})$. So we have proved that $\mathrm{Dlcop}(x_I \otimes x_J) = \mathrm{Dlcop}(x_{I \cup J} \otimes x_{I \cap J})$. By Lemma 5.3, if $I \cup J \neq \{1, \dots, N\}$, then $\mathrm{Dlcop}(x_I \otimes x_J) = 0$. ■

Theorem 5.5 *Let X be a simply-connected space such that $H^*(X; \mathbb{F}_2)$ is the polynomial algebra $\mathbb{F}_2[y_1, \dots, y_N]$. The dual of the loop coproduct admits*

$$\mathrm{Dlcop}(x_1 \cdots x_N \otimes x_1 \cdots x_N) \in H^d(LX; \mathbb{F}_2)$$

as a unit.

Lemma 5.6 *Let $a \in H^*(LX; \mathbb{F}_2)$.*

(i) *For $1 \leq i \leq N$, $x_i \cup \mathrm{Dlcop}(a \otimes a) = 0$.*
(ii) *For any $b \in H^*(LX; \mathbb{F}_2)$,*

$$\mathrm{Dlcop}(\mathrm{Dlcop}(a \otimes a) \otimes b) = b \cup \mathrm{Dlcop}(\mathrm{Dlcop}(a \otimes a) \otimes 1).$$

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Proof (i) By Theorem 3.1 (i),

$$\operatorname{Dlcop}(a\otimes ax_i)=x_i\operatorname{Dlcop}(a\otimes a)+\operatorname{Dlcop}(ax_i\otimes a).$$

Since Dlcop is graded commutative [6], $\operatorname{Dlcop}(a\otimes ax_i)=\operatorname{Dlcop}(ax_i\otimes a)$. So $x_i\operatorname{Dlcop}(a\otimes a)=0$.

(ii) By (i) and Theorem 3.1 (i),

$$\operatorname{Dlcop}(\operatorname{Dlcop}(a\otimes a)\otimes bx_i)=x_i\operatorname{Dlcop}(\operatorname{Dlcop}(a\otimes a)\otimes b)+0.$$

Therefore by induction,

$$\operatorname{Dlcop}(\operatorname{Dlcop}(a\otimes a)\otimes x_{i_1}\cdots x_{i_l})=x_{i_1}\cdots x_{i_l}\operatorname{Dlcop}(\operatorname{Dlcop}(a\otimes a)\otimes 1).$$

Using Theorem 3.1 (iv), we obtain (ii). ■

Proof of Theorem 5.5 Since Dlcop is graded associative [6] and using Theorem 3.1 (ii) twice,

$$\begin{aligned}\operatorname{Dlcop}(\operatorname{Dlcop}(x_1\dots x_N\otimes x_1\dots x_N)\otimes 1)&=\operatorname{Dlcop}(x_1\dots x_N\otimes\operatorname{Dlcop}(x_1\dots x_N\otimes 1))\\&=\operatorname{Dlcop}(x_1\dots x_N\otimes 1)=1.\end{aligned}$$

Therefore using Lemma 5.6 (ii),

$$\begin{aligned}&\operatorname{Dlcop}(\operatorname{Dlcop}(x_1\cdots x_N\otimes x_1\cdots x_N)\otimes b)\\&\qquad=b\cup\operatorname{Dlcop}(\operatorname{Dlcop}(x_1\cdots x_N\otimes x_1\cdots x_N)\otimes 1)\\&\qquad=b\cup 1=b.\end{aligned}$$ ■

The simplest connected Lie group with non-trivial Steenrod operation Sq_1 in the cohomology of its classifying space is $SO(3)$.

Theorem 5.7 *The cup product and the dual of the loop coproduct on the mod 2 free loop cohomology of the classifying space of* $SO(3)$ *are given by*

$$\begin{aligned}H^*(LBSO(3);\mathbb{F}_2)&\cong\triangle(x_1,x_2)\otimes\mathbb{F}_2[y_2,y_3]\\&\cong\mathbb{F}_2[x_1,x_2]\otimes\mathbb{F}_2[y_2,y_3]/\left(\begin{matrix}x_1^2+x_2\\x_2^2+x_2y_2+y_3x_1\end{matrix}\right)\end{aligned}$$

as algebras over $H^*(BSO(3);\mathbb{F}_2)\cong\mathbb{F}_2[y_2,y_3]$, *where* $\deg x_i=i$ *and* $\deg y_j=j$.

$$\begin{aligned}&\operatorname{Dlcop}(x_1x_2\otimes 1)=\operatorname{Dlcop}(x_1\otimes x_2)=1,\\&\operatorname{Dlcop}(x_1x_2\otimes x_1)=x_1,\\&\operatorname{Dlcop}(x_1x_2\otimes x_2)=x_2+y_2,\\&\operatorname{Dlcop}(x_1x_2\otimes x_1x_2)=x_1x_2+x_1y_2+y_3.\end{aligned}$$

Proof The cohomology $H^*(BSO(3);\mathbb{F}_2)$ is the polynomial algebra on the Stiefel–Whitney classes y_2 and y_3 of the tautological bundle γ^3 [37, Theorem 7.1], [38, III Corollary 5.10]. By Wu's formula [38, III.Theorem 5.12(1)], $\mathrm{Sq}^1 y_2=y_3$ and $\mathrm{Sq}^2 y_3=y_2y_3$. Now the computation of the cup product and of the dual of the loop coproduct follows from Theorem 3.1. ■

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In the following proof, we detail the computation of the cup product and the dual of the loop coproduct following Theorem 3.1 for a more complicated example of Lie group.

Proof of Theorem 5.1. Observe that $\mathrm{Sq}^2\, y_4 = y_6$, $\mathrm{Sq}^1\, y_6 = y_7$ [38, VII.Corollary 6.3] and hence $\mathrm{Sq}^3\, y_4 = \mathrm{Sq}^1\,\mathrm{Sq}^2\, y_4 = y_7$. From [28, Proof of Theorem 1.7], $\mathrm{Sq}^5\, y_6 = y_4 y_7$ and $\mathrm{Sq}^6\, y_7 = y_6 y_7$. Therefore the computation of the cup product on $H^*(\mathrm{LBG}_2;\mathbb{F}_2)$ follows from Theorem 3.1: $x_3^2 = x_6$, $x_5^2 = x_3 y_7 + y_4 x_6$, and $x_6^2 = x_5 y_7 + y_6 x_6$.

Lemma 5.3 implies that monomials with non-trivial loop coproduct are only the ones listed in the theorem.

By Theorem 3.1 (ii),

$$\mathrm{Dlcop}(x_3x_5x_6 \otimes 1) = \mathrm{Dlcop}(x_3x_5 \otimes x_6) = \mathrm{Dlcop}(x_3x_6 \otimes x_5) = \mathrm{Dlcop}(x_5x_6 \otimes x_3) = 1.$$

By Lemma 5.3, $\mathrm{Dlcop}(x_3x_5^2 \otimes 1) = 0$. By Theorem 3.1 (i),

$$\mathrm{Dlcop}(x_3x_5x_6 \otimes x_6) = x_6\,\mathrm{Dlcop}(x_3x_5x_6 \otimes 1) + \mathrm{Dlcop}(x_3x_5x_6^2 \otimes 1).$$

Since $x_3x_5x_6^2 = x_3x_5(x_5y_7 + y_6x_6)$, by Theorem 3.1 (iv),

$$\mathrm{Dlcop}(x_3x_5x_6^2 \otimes 1) = y_7\,\mathrm{Dlcop}(x_3x_5^2 \otimes 1) + y_6\,\mathrm{Dlcop}(x_3x_5x_6 \otimes 1) = y_7 \cup 0 + y_6 \cup 1$$

So finally $\mathrm{Dlcop}(x_3x_5x_6 \otimes x_6) = x_6 + y_6$.

By Theorem 5.4, $\mathrm{Dlcop}(x_3x_6 \otimes x_5x_6) = \mathrm{Dlcop}(x_3x_5x_6 \otimes x_6)$.

Since $x_3x_5^2x_6 = x_5y_7^2 + x_6y_6y_7 + x_3x_5y_7y_4 + x_3x_6y_6y_4$, by Theorem 3.1 (i) and Lemma 5.3,

$$\begin{aligned}\mathrm{Dlcop}(x_3x_5x_6 \otimes x_5x_6) &= x_5\,\mathrm{Dlcop}(x_3x_5x_6 \otimes x_6) + \mathrm{Dlcop}(x_3x_5^2x_6 \otimes x_6)\\ &= x_5(x_6 + y_6) + y_7^2 \cup 0 + y_6y_7 \cup 0 + y_7y_4 \cup 1 + y_6y_4 \cup 0.\end{aligned}$$

The other computations are left to the reader. ∎

We would like to emphasize that at the same time Theorem 5.1 gives the cup product and the dual of the loop coproduct on $H^*(\mathrm{LBG}_2;\mathbb{F}_2)$. As mentioned in the introduction, if we forget the cup product, then the following theorem shows that the dual of the loop coproduct is really simple.

Theorem 5.8 *Let X be a simply-connected space such that $H^*(X;\mathbb{F}_2)$ is the polynomial algebra $\mathbb{F}_2[V]$. Then with respect to the dual of the loop coproduct, there is an isomorphism of graded algebras between $H^{*+d}(LX;\mathbb{F}_2)$ and the tensor product of algebras $H^*(X;\mathbb{F}_2) \otimes H_{-*}(\Omega X;\mathbb{F}_2) \cong \mathbb{F}_2[V] \otimes \Lambda(sV)^\vee$.*

Lemma 5.9 *Let X be a simply-connected space such that $H^*(X;\mathbb{F}_2) = \mathbb{F}_2[V]$. Let $x_1, \dots, x_N$ be a basis of sV.*

(i) *Suppose that $\{i_1, \dots, i_l\} \cup \{j_1, \dots, j_m\} = \{1, \dots, N\}$. Let*

$$\{k_1, \dots, k_u\} := \{i_1, \dots, i_l\} \cap \{j_1, \dots, j_m\}.$$

Then $H^*(\iota) \circ \mathrm{Dlcop}(x_{i_1} \cdots x_{i_l} \otimes x_{j_1} \cdots x_{j_m}) = x_{k_1} \cdots x_{k_u}$.

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(ii) *Let* $\Theta\colon H_{-*}(\Omega X) = \wedge(sV)^\vee \xrightarrow{\cong} H^{*+d}(\Omega X) = \underline{\wedge}(sV)$ *be the linear isomorphism defined by* $\Theta(x_{j_1}^\vee \wedge \cdots \wedge x_{j_p}^\vee) = x_1 \cup \cdots \cup \widehat{x_{j_1}} \cup \cdots \cup \widehat{x_{j_p}} \cup \cdots \cup x_N$. *Here* $^\vee$ *denotes the dual and* $\widehat{\ }$ *denotes omission. Then the composite*

$$\Theta^{-1} \circ H^*(\iota)\colon H^{*+d}(LX) \longrightarrow H^{*+d}(\Omega X) \xrightarrow{\cong} H_{-*}(\Omega X)$$

is a morphism of graded algebras preserving the unit.

Proof of Lemma 5.9 (i) Suppose that $|x_{k_1}| \geq \cdots \geq |x_{k_u}|$. There exist polynomials $P_1, \dots, P_N \in \mathbb{F}_2[y_1, \dots, y_N]$, possibly null, such that

$$x_{k_1}^2 = \Delta \circ H^*(\mathrm{ev}) \circ \mathrm{Sq}^{|y_{k_1}|-1}(y_{k_1}) = \sum_{i=1}^N x_i P_i.$$

If P_i is of degree 0, since $|x_i| > |x_{k_1}|$, x_i is not one of the elements $x_{k_1}, \dots, x_{k_u}$ and so by Lemma 5.3, $\mathrm{Dlcop}(x_{i_1} \cdots \widehat{x_{k_1}} \cdots x_{i_l} x_i \otimes x_{j_1} \cdots \widehat{x_{k_1}} \cdots x_{j_m}) = 0$.

If P_i is of degree ≥ 1, by Theorem 3.1 (iv),

$$H^*(\iota) \circ \mathrm{Dlcop}(P_i x_{i_1} \cdots \widehat{x_{k_1}} \cdots x_{i_l} x_i \otimes x_{j_1} \cdots \widehat{x_{k_1}} \cdots x_{j_m}) = 0.$$

Therefore $H^*(\iota) \circ \mathrm{Dlcop}(x_{i_1} \cdots \widehat{x_{k_1}} \cdots x_{i_l} x_{k_1}^2 \otimes x_{j_1} \cdots \widehat{x_{k_1}} \cdots x_{j_m}) = 0$. Now the same proof as the proof of Theorem 4.1 shows (i).

(ii) Since $H^*(\Omega X; \mathbb{F}_2)$ is generated by the $x_i := \sigma(y_i)$, $1 \leq i \leq N$, which are primitives, $H_*(\Omega X; \mathbb{F}_2)$ is commutative and by [36, Proposition 4.20], all squares vanish in $H_*(\Omega X; \mathbb{F}_2)$. Therefore $H_*(\Omega X; \mathbb{F}_2)$ is the exterior algebra $\Lambda\sigma(y_i)^\vee$.

Let $I = \{i_1, \dots, i_l\} \subset \{1, \dots, N\}$. Recall from Theorem 5.4 that in $\underline{\wedge}(x_1, \dots, x_N)$, x_I denotes the generator $x_{i_1} \cup x_{i_2} \cup \cdots \cup x_{i_l}$. Denote also in the exterior algebra $\Lambda(x_1^\vee, \dots, x_N^\vee)$ by $x_I^\vee$ the element $x_{i_1}^\vee \wedge x_{i_2}^\vee \wedge \cdots \wedge x_{i_l}^\vee$. Then with this notation, $\Theta(x_I^\vee) = x_{I^c}$, where I^c is the complement of I in $\{1, \dots, N\}$. Let

$$\mathrm{Comp}^!\colon H^{*+d}(\Omega X) \otimes H^{*+d}(\Omega X) \longrightarrow H^{*+d}(\Omega X)$$

be the multiplication defined by $\mathrm{Comp}^!(x_I \otimes x_J) = x_{I \cap J}$ if $I \cup J = \{1, \dots, N\}$ and 0 otherwise. By (i) and Lemma 5.3, $H^*(\iota)\colon H^{*+d}(LX) \to H^{*+d}(\Omega X)$ commutes with the products Dlcop and $\mathrm{Comp}^!$. Since $x_{(I \cup J)^c} = x_{I^c \cap J^c}$, $\Theta\colon H_{-*}(\Omega X) \to H^{*+d}(\Omega X)$ commutes with the exterior product and $\mathrm{Comp}^!$.

By Theorem 5.5, $\mathrm{Dlcop}(x_1 \dots x_N \otimes x_1 \dots x_N)$ is the unit of Dlcop. By (i),

$$\Theta^{-1} \circ H^*(\iota) \circ \mathrm{Dlcop}(x_1 \dots x_N \otimes x_1 \cdots x_N) = \Theta^{-1}(x_1 \cdots x_N) = 1.$$

Therefore $\Theta^{-1} \circ H^*(\iota)$ also preserves the unit. ■

Proof of Theorem 5.8 Denote the unit of $H^{*+d}(LX; \mathbb{F}_2)$ (Theorem 5.5) by

$$\mathbb{I} := \mathrm{Dlcop}(x_1 \dots x_N \otimes x_1 \dots x_N).$$

By Theorem 2.2 (vii), the map $s^!\colon H^*(X) \to H^{*+d}(LX)$, $a \mapsto H^*(\mathrm{ev})(a)\mathbb{I}$, is a morphism of unital commutative graded algebras.

By Lemma 5.3, we have $\mathrm{Dlcop}(x_1 \dots \widehat{x_i} \dots x_N \otimes x_1 \dots \widehat{x_i} \dots x_N) = 0$. So let

$$\sigma\colon H^{*+d}(\Omega X) \longrightarrow H^{*+d}(LX)$$

be the unique linear map such that for $1 \leq i \leq N$, $\sigma(x_1 \dots \widehat{x_i} \cdots x_N) = x_1 \cdots \widehat{x_i} \cdots x_N$ and such that $\sigma \circ \Theta\colon H_{-*}(\Omega X) = \Lambda(sV)^\vee \to H^{*+d}(LX)$ is a morphism of unital

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commutative graded algebras. For $1 \le i \le N$, $\Theta^{-1} \circ H^*(\iota) \circ \sigma \circ \Theta(x_i^\vee) = x_i^\vee$. By Lemma 5.9, the composite $\Theta^{-1} \circ H^*(\iota)\colon H^{*+d}(LX) \to H^{*+d}(\Omega X) \overset{\cong}{\to} H_{-*}(\Omega X)$ is a morphism of graded algebras. So the composite $\Theta^{-1} \circ H^*(\iota) \circ \sigma \circ \Theta$ is the identity map and σ is a section of $H^*(\iota)$. So by the Leray–Hirsch theorem, the linear morphism of $H^*(X)$-modules $H^*(X) \otimes H^*(\Omega X) \to H^*(LX)$, $a \otimes g \mapsto H^*(\mathrm{ev})(a)\sigma(g)$, is an isomorphism.

The composite

$$\varphi\colon H^*(X) \otimes H_{-*}(\Omega X) \xrightarrow{s^! \otimes \sigma \circ \Theta} H^{*+d}(LX) \otimes H^{*+d}(LX) \xrightarrow{\mathrm{Dlcop}} H^{*+d}(LX)$$

is a morphism of commutative graded algebras with respect to the dual of the loop coproduct. By Theorem 3.1 (iv) and since $\mathbb{I}$ is the unit for Dlcop,

$$\varphi(a \otimes g) = \mathrm{Dlcop}(H^*(\mathrm{ev})(a)\mathbb{I} \otimes \sigma \circ \Theta(g)) = H^*(\mathrm{ev})(a)\sigma \circ \Theta(g).$$

Therefore, φ is an isomorphism. ∎

Example 5.10 With respect to the dual of the loop coproduct, there is an isomorphism of algebras between $H^{*+3}(\mathrm{LBSO}(3); \mathbb{F}_2)$ and

$$H_{-*}(\mathrm{SO}(3); \mathbb{F}_2) \otimes H^*(BSO(3); \mathbb{F}_2) \cong \wedge(u_{-1}, u_{-2}) \otimes \mathbb{F}_2[v_2, v_3].$$

Proof By Theorem 5.5, $\mathrm{Dlcop}(x_1x_2 \otimes x_1x_2) = x_1x_2 + x_1y_2 + y_3$ is the unit for the dual of the loop coproduct on $H^{*+3}(\mathrm{LBSO}(3); \mathbb{F}_2)$. By Lemma 5.3,

$$\mathrm{Dlcop}(x_1 \otimes x_1) = \mathrm{Dlcop}(x_2 \otimes x_2) = 0.$$

So let $\varphi\colon \wedge(u_{-1}, u_{-2}) \otimes \mathbb{F}_2[v_2, v_3] \to H^{*+3}(\mathrm{LBSO}(3); \mathbb{F}_2)$ be the unique morphism of algebras such that $\varphi(u_{-2}) = x_1$, $\varphi(u_{-1}) = x_2$, $\varphi(v_2) = y_2(x_1x_2 + x_1y_2 + y_3)$, and $\varphi(v_3) = y_3(x_1x_2 + x_1y_2 + y_3)$.

For all i, $j \ge 0$, we see that $\varphi(v_2^i v_3^j) = y_2^i y_3^j(x_1x_2 + x_1y_2 + y_3)$, $\varphi(u_{-1}u_{-2}v_2^i v_3^j) = y_2^i y_3^j$, $\varphi(u_{-1}v_2^i v_3^j) = x_2 y_2^i y_3^j$, and $\varphi(u_{-2}v_2^i v_3^j) = x_1 y_2^i y_3^j$. Therefore φ sends a linear basis of $\wedge(u_{-1}, u_{-2}) \otimes \mathbb{F}_2[v_2, v_3]$ to a linear basis $H^{*+3}(\mathrm{LBSO}(3); \mathbb{F}_2)$. So φ is an isomorphism. ∎

Example 5.11 With respect to the dual of the loop coproduct, there is an isomorphism of algebras between $H^{*+14}(\mathrm{LBG}_2; \mathbb{F}_2)$ and

$$H_{-*}(G_2; \mathbb{F}_2) \otimes H^*(BG_2; \mathbb{F}_2) \cong \wedge(u_{-3}, u_{-5}, u_{-6}) \otimes \mathbb{F}_2[v_4, v_6, v_7].$$

Proof By Theorem 5.5, $\mathrm{Dlcop}(x_3x_5x_6 \otimes x_3x_5x_6) = x_3x_5x_6 + x_3x_5y_6 + x_3y_4y_7 + y_7^2$ is the unit for the dual of the loop coproduct on $H^{*+14}(\mathrm{LBG}_2; \mathbb{F}_2)$. By Lemma 5.3,

$$\mathrm{Dlcop}(x_5x_6 \otimes x_5x_6) = \mathrm{Dlcop}(x_3x_6 \otimes x_3x_6) = \mathrm{Dlcop}(x_3x_5 \otimes x_3x_5) = 0.$$

So let $\varphi\colon \wedge(u_{-3}, u_{-5}, u_{-6}) \otimes \mathbb{F}_2[v_4, v_6, v_7] \to H^{*+14}(\mathrm{LBG}_2; \mathbb{F}_2)$ be the unique morphism of algebras such that $\varphi(u_{-3}) = x_5x_6$, $\varphi(u_{-5}) = x_3x_6$, $\varphi(u_{-6}) = x_3x_5$, $\varphi(v_4) = y_4(x_3x_5x_6 + x_3x_5y_6 + x_3y_4y_7 + y_7^2)$, $\varphi(v_6) = y_6(x_3x_5x_6 + x_3x_5y_6 + x_3y_4y_7 + y_7^2)$, and $\varphi(v_7) = y_7(x_3x_5x_6 + x_3x_5y_6 + x_3y_4y_7 + y_7^2)$.

For all i, j, and $k \ge 0$, we see that

$$\varphi(v_4^i v_6^j v_7^k) = y_4^i y_6^j y_7^k (x_3x_5x_6 + x_3x_5y_6 + x_3y_4y_7 + y_7^2),$$
$$\varphi(u_{-3}u_{-5}u_{-6}v_4^i v_6^j v_7^k) = y_4^i y_6^j y_7^k,$$
$$\varphi(u_{-3}u_{-5}v_4^i v_6^j v_7^k) = (x_6 + y_6) y_4^i y_6^j y_7^k,$$
$$\varphi(u_{-3}u_{-6}v_4^i v_6^j v_7^k) = x_5 y_4^i y_6^j y_7^k,$$
$$\varphi(u_{-5}u_{-6}v_4^i v_6^j v_7^k) = x_3 y_4^i y_6^j y_7^k,$$
$$\varphi(u_{-3}v_4^i v_6^j v_7^k) = x_5x_6 y_4^i y_6^j y_7^k,$$
$$\varphi(u_{-5}v_4^i v_6^j v_7^k) = x_3x_6 y_4^i y_6^j y_7^k$$
$$\varphi(u_{-6}v_4^i v_6^j v_7^k) = x_3x_5 y_4^i y_6^j y_7^k.$$

Therefore φ sends a linear basis of $\wedge(u_{-3}, u_{-5}, u_{-6}) \otimes \mathbb{F}_2[v_4, v_6, v_7]$ to a linear basis $H^{*+14}(\mathrm{LBG}_2; \mathbb{F}_2)$. So φ is an isomorphism. ∎

Lemma 5.12 *Let $(A, \odot)$ be a commutative unital associative graded algebra. Let $x \in A$ such that $x \odot x = 1$. Let $\psi\colon A \to A$ be the linear morphism mapping a to $x \odot a$. Then ψ is an involutive isomorphism such that for any a, b in A, $\psi(a) \odot \psi(b) = a \odot b$.*

Proof $\psi(a) \odot \psi(b) = (x \odot a) \odot (x \odot b) = (x \odot x) \odot (a \odot b) = 1 \odot (a \odot b) = a \odot b.$ ∎

Second proof of Theorem 5.8 This proof gives another (better?) algebra isomorphism. By commutativity and associativity of Dlcop and Theorem 5.5, applying Lemma 5.12, $\psi\colon H^*(X) \otimes H^{*+d}(\Omega X) \to H^{*+d}(LX)$ defined by

$$\psi(a \otimes x_{k_1} \cdots x_{k_u}) = \mathrm{Dlcop}(x_1 \cdots x_N \otimes a x_{k_1} \cdots x_{k_u})$$

is an involutive isomorphism such that

$$\mathrm{Dlcop}(\psi(a \otimes x_I) \otimes \psi(b \otimes x_J)) = \mathrm{Dlcop}(a x_I \otimes b x_J)$$

for any subsets I and J of $\{1, \dots, N\}$.

Case $I \cup J = \{1, \dots, N\}$. By Theorem 5.4,

$$\mathrm{Dlcop}(a x_I \otimes b x_J) = \mathrm{Dlcop}(x_1 \dots x_N \otimes ab x_{I \cap J}) = \psi(ab \otimes x_{I \cap J})$$
$$= \psi(ab \otimes \mathrm{Comp}^!(x_I \otimes x_J)).$$

Case $I \cup J \neq \{1, \dots, N\}$. By Theorem 5.4,

$$\mathrm{Dlcop}(a x_I \otimes b x_J) = 0 \quad \text{and} \quad \mathrm{Comp}^!(x_I \otimes x_J) = 0.$$

Therefore ψ is a morphism of graded algebras. One can show that

$$\{\psi(1 \otimes \Theta(x_i^\vee)), \psi(1 \otimes \Theta(x_j^\vee))\} = 0.$$

That is why this isomorphism might be better. ∎

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Theorem 5.13 *As a BV-algebra,*

$$H^{*+3}(\mathrm{LBSO}(3);\mathbb{F}_2) \cong \wedge(u_{-1},u_{-2})\otimes\mathbb{F}_2[v_2,v_3]$$

where for all $i,j\geq 0$,

$$\begin{aligned}
\Delta(v_2^iv_3^j) &= 0,\\
\Delta(u_{-1}u_{-2}v_2^iv_3^j) &= iu_{-2}v_2^{i-1}v_3^j + ju_{-1}v_2^iv_3^{j-1},\\
\Delta(u_{-2}v_2^iv_3^j) &= iu_{-1}v_2^{i-1}v_3^j + jv_2^iv_3^{j-1} + ju_{-2}v_2^{i+1}v_3^{j-1} + ju_{-1}u_{-2}v_2^iv_3^j,\\
\Delta(u_{-1}v_2^iv_3^j) &= iv_2^{i-1}v_3^j + (i+j)u_{-2}v_2^iv_3^j + iu_{-1}u_{-2}v_2^{i-1}v_3^{j+1} + ju_{-1}v_2^{i+1}v_3^{j-1}.
\end{aligned}$$

In particular, $1\notin \mathrm{Im}\,\Delta$.

Proof Theorem 5.7 gives the BV-algebra $H^{*+3}(\mathrm{LBSO}(3);\mathbb{F}_2)$, since Δ is a derivation with respect to the cup product. In the proof of Example 5.10, the isomorphism of algebras $\varphi\colon \wedge(u_{-1},u_{-2})\otimes\mathbb{F}_2[v_2,v_3]\to H^{*+3}(\mathrm{LBSO}(3);\mathbb{F}_2)$ of Theorem 5.8 is made explicit on generators. We now transport the operator Δ using φ.

In degree 1, the Δ operator is given by $\Delta(u_{-1}u_{-2}v_2^2)=0$ and

$$\Delta(u_{-2}v_3) = \Delta(u_{-1}v_2) = 1 + u_{-2}v_2 + u_{-1}u_{-2}v_3. \qquad \blacksquare$$

Theorem 5.14 *As a BV-algebra,*

$$H^{*+14}(\mathrm{LBG}_2;\mathbb{F}_2) \cong \wedge(u_{-3},u_{-5},u_{-6})\otimes\mathbb{F}_2[v_4,v_6,v_7]$$

where for all $i,j,k\geq 0$, $\Delta(v_4^iv_6^jv_7^k)=0$,

$$\begin{aligned}
\Delta(u_{-3}u_{-5}u_{-6}v_4^iv_6^jv_7^k) &= iu_{-5}u_{-6}v_4^{i-1}v_6^jv_7^k + ju_{-3}u_{-6}v_4^iv_6^{j-1}v_7^k\\
&\quad + ku_{-3}u_{-5}v_4^iv_6^jv_7^{k-1} + ku_{-3}u_{-5}u_{-6}v_4^iv_6^{j+1}v_7^{k-1},\\
\Delta(u_{-5}u_{-6}v_4^iv_6^jv_7^k) &= iu_{-3}u_{-5}v_4^{i-1}v_6^jv_7^k + iu_{-3}u_{-5}u_{-6}v_4^{i-1}v_6^{j+1}v_7^k\\
&\quad + ju_{-6}v_4^iv_6^{j-1}v_7^k + ku_{-5}v_4^iv_6^jv_7^{k-1},\\
\Delta(u_{-3}u_{-6}v_4^iv_6^jv_7^k) &= iu_{-6}v_4^{i-1}v_6^jv_7^k + ju_{-5}u_{-6}v_4^iv_6^{j-1}v_7^{k+1} + ju_{-3}u_{-5}v_4^{i+1}v_6^{j-1}v_7^k\\
&\quad + ju_{-3}u_{-5}u_{-6}v_4^{i+1}v_6^jv_7^k + ku_{-3}v_4^iv_6^jv_7^{k-1},\\
\Delta(u_{-3}u_{-5}v_4^iv_6^jv_7^k) &= iu_{-5}v_4^{i-1}v_6^jv_7^k + iu_{-5}u_{-6}v_4^{i-1}v_6^{j+1}v_7^k + ju_{-3}v_4^iv_6^{j-1}v_7^k\\
&\quad + (j+1+k)u_{-3}u_{-6}v_4^iv_6^jv_7^k\\
\Delta(u_{-6}v_4^iv_6^jv_7^k) &= iu_{-3}v_4^{i-1}v_6^jv_7^k + ju_{-5}v_4^{i+1}v_6^{j-1}v_7^k + ju_{-3}u_{-5}v_4^iv_6^{j-1}v_7^{k+1}\\
&\quad + (j+k)u_{-3}u_{-5}u_{-6}v_4^iv_6^jv_7^{k+1} + kv_4^iv_6^jv_7^{k-1}\\
&\quad + ku_{-6}v_4^iv_6^{j+1}v_7^{k-1} + ku_{-5}u_{-6}v_4^{i+1}v_6^jv_7^k,\\
\Delta(u_{-3}v_4^iv_6^jv_7^k) &= iv_4^{i-1}v_6^jv_7^k + iu_{-6}v_4^{i-1}v_6^{j+1}v_7^k + (i+k)u_{-5}u_{-6}v_4^iv_6^jv_7^{k+1}\\
&\quad + iu_{-3}u_{-5}u_{-6}v_4^{i-1}v_6^jv_7^{k+2} + ju_{-5}v_4^iv_6^{j-1}v_7^{k+1}\\
&\quad + ju_{-3}u_{-6}v_4^{i+1}v_6^{j-1}v_7^{k+1} + (j+k)u_{-3}u_{-5}v_4^{i+1}v_6^jv_7^k\\
&\quad + (j+k)u_{-3}u_{-5}u_{-6}v_4^{i+1}v_6^{j+1}v_7^k + ku_{-3}v_4^iv_6^{j+1}v_7^{k-1},
\end{aligned}$$

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$$\Delta(u_{-5}v_4^i v_6^j v_7^k) = iu_{-3}u_{-5}v_4^{i-1}v_6^{j+1}v_7^k + iu_{-3}u_{-5}u_{-6}v_4^{i-1}v_6^{j+2}v_7^k + jv_4^i v_6^{j-1}v_7^k$$
$$+ (j+k)u_{-6}v_4^i v_6^j v_7^k + ju_{-5}u_{-6}v_4^{i+1}v_6^{j-1}v_7^{k+1}$$
$$+ ju_{-3}u_{-5}u_{-6}v_4^i v_6^{j-1}v_7^{k+2} + ku_{-5}v_4^i v_6^{j+1}v_7^{k-1}.$$

In particular, $1 \notin \operatorname{Im}\Delta$.

Proof Theorem 5.1 gives the BV-algebra $H^{*+14}(\mathrm{LBG}_2;\mathbb{F}_2)$, since Δ is a derivation with respect to the cup product. In the proof of Example 5.11, the isomorphism of algebras $\varphi\colon \wedge(u_{-3},u_{-5},u_{-6})\otimes\mathbb{F}_2[v_4,v_6,v_7] \to H^{*+14}(LG_2;\mathbb{F}_2)$ of Theorem 5.8 is made explicit on generators. We now transport the operator Δ using φ.

In degree 1, the Δ operator is given by $\Delta(u_{-5}u_{-6}v_6^2) = 0$,

$$\Delta(u_{-3}u_{-5}u_{-6}v_4^2v_7) = \Delta(u_{-5}u_{-6}v_4^3) = u_{-3}u_{-5}v_4^2 + u_{-3}u_{-5}u_{-6}v_4^2v_6,$$
$$\Delta(u_{-3}u_{-6}v_4v_6) = u_{-6}v_6 + u_{-5}u_{-6}v_4v_7 + u_{-3}u_{-5}v_4^2 + u_{-3}u_{-5}u_{-6}v_4^2v_6,$$
$$\Delta(u_{-6}v_7) = \Delta(u_{-5}v_6) = \Delta(u_{-3}v_4) = 1 + u_{-6}v_6 + u_{-5}u_{-6}v_4v_7$$
$$+ u_{-3}u_{-5}u_{-6}v_7^2.$$
∎

Note that $\varphi^{-1}\circ\Delta\circ\varphi(y_i\otimes x_i^\vee) = \varphi^{-1}(x_1\cdots x_N)$ is independent of i.

6 Relation to Hochschild Cohomology

Let $\mathbb{K}$ be any field. Let G be a connected compact Lie group of dimension d.

Conjecture 6.1 ([6, Conjecture 68]) *There is an isomorphism of Gerstenhaber algebras* $H^{*+d}(\mathrm{LBG}) \xrightarrow{\cong} HH^*(S_*(G),S_*(G))$.

Suppose that $H^*(BG;\mathbb{K})$ is a polynomial algebra $\mathbb{K}[V] = K[y_1,\dots,y_N]$. It follows from [40, Theorem 9, p. 572], [31, Proposition 8.21] that BG is $\mathbb{K}$-formal. Then BG is $\mathbb{K}$-coformal and $H_*(G;\mathbb{K})$ is the exterior algebra $\wedge(sV)^\vee$. Indeed, since BG is $\mathbb{K}$-formal, the Cobar construction $\Omega H_*(\mathrm{BG})$ is weakly equivalent as algebras to $S_*(G)$. Let A_i denote the exterior algebra $\Lambda s^{-1}(y_i^\vee)$. Then EZ, the Eilenberg–Zilber map, and ε, the counit of the adjunction between the Bar and the Cobar construction, give the quasi-isomorphims of algebras

$$\Omega H_*(BG) = \Omega\Big(\bigotimes_{i=1}^N BA_i\Big) \underset{\simeq}{\xleftarrow{EZ}} \bigotimes_{i=1}^N \Omega BA_i \underset{\simeq}{\xrightarrow{\otimes_{i=1}^N\varepsilon_i}} \Big]\bigotimes_{i=1}^N A_i = \Lambda s^{-1}V^\vee.$$

Alternatively, since BG is $\mathbb{K}$-formal, we can use the implication $(2)\Rightarrow(1)$ in [2, Theorem 2.14]. Therefore, we have the isomorphism of Gerstenhaber algebras

$$HH^*(S_*(G),S_*(G)) \cong HH^*(H_*(G;\mathbb{K}),H_*(G;\mathbb{K})) \cong HH^*(\wedge(sV)^\vee,\wedge(sV)^\vee).$$

By Theorem F.3 (i) and (ii) as graded algebras,

$$HH^*(\wedge(sV)^\vee,\wedge(sV)^\vee) \cong \wedge(sV)^\vee\otimes\mathbb{K}[V] \cong H_{-*}(G;\mathbb{K})\otimes H^*(\mathrm{BG};\mathbb{K}).$$

So in Theorem 5.8, we have checked only Conjecture 6.1 for the algebra structure when $\mathbb{K}=\mathbb{F}_2$. When $\mathbb{K}=\mathbb{F}_2$, we would like also to check Conjecture 6.1 also for the Gerstenhaber algebra structure.

The following theorem shows that the conjecture is true for the Gerstenhaber algebra structure when $\mathbb{K}$ is a field of characteristic different from 2.

Theorem 6.2 *Under hypothesis* (H)*, the free loop space cohomology of the classifying space of* G*,* $H^{*+\dim G}(\mathrm{LBG};\mathbb{K})$ *is isomorphic as BV-algebra to the Hochschild cohomology of* $H_*(G;\mathbb{K})$*,* $HH^*(H_*(G;\mathbb{K});H_*(G;\mathbb{K}))$*. In particular, the underlying Gerstenhaber algebras are isomorphic.*

Proof By hypothesis, $H^*(\mathrm{BG}) \cong \mathbb{K}[V] = \mathbb{K}[y_i]$ as algebras. Then

$$H_*(G) \cong \Lambda(sV)^\vee = \Lambda x_j^\vee$$

as algebras.

Let $\Psi\colon sV \to (sV)^{\vee\vee}$ be the canonical isomorphism of the graded vector space sV into its bidual. By definition, $\Psi(sv)(\varphi) = (-1)^{|\varphi||sv|}\varphi(sv)$ for any linear form φ on sV.

By Theorem F.3 (iii), we have the BV-algebra isomorphism

$$HH^*(H_*(G);H_*(G)) \cong \Lambda(sV)^\vee \otimes \mathbb{K}[s^{-1}(sV)^{\vee\vee}],$$

where for any $v \in V$ and $\varphi \in (sV)^\vee$,

$$\Delta((1\otimes s^{-1}\Psi(sv))(\varphi\otimes 1)) = (-1)^{|v|}\{s^{-1}\Psi(sv),\varphi\} = -\Psi(sv)(\varphi) = -(-1)^{|\varphi||sv|}\varphi(sv)$$

and where Δ is trivial on $\Lambda(sV)^\vee$ and on $\mathbb{K}[s^{-1}(sV)^{\vee\vee}]$.

The isomorphism of algebras

$$\mathrm{Id}\otimes\mathbb{K}[s^{-1}\Psi]\colon \Lambda(sV)^\vee \otimes \mathbb{K}[V] \longrightarrow \Lambda(sV)^\vee \otimes \mathbb{K}[s^{-1}(sV)^{\vee\vee}]$$

is an isomorphism of BV-algebras if for any $v \in V$ and $\varphi \in (sV)^\vee$, $\Delta((1\otimes v)(\varphi\otimes 1)) = -(-1)^{|\varphi||sv|}\varphi(sv)$ and if Δ is trivial on $\Lambda(sV)^\vee$ and on $\mathbb{K}[V]$.

Taking $v = y_i$ and $\varphi = \sigma(y_j)^\vee = x_j^\vee$, we obtained that $\Delta(y_i\otimes x_j^\vee) = 1$ if $i = j$ and 0 otherwise, as in Theorem 4.3. ∎

Theorem 6.3 *For* $G = \mathrm{SO}(3)$ *or* $G = G_2$*, the free loop space modulo* 2 *cohomology of the classifying space of* G*,* $H^{*+\dim G}(\mathrm{LBG};\mathbb{F}_2)$ *is not isomorphic as a BV-algebra to the Hochschild cohomology of* $H_*(G;\mathbb{F}_2)$*,* $HH^*(H_*(G;\mathbb{F}_2);H_*(G;\mathbb{F}_2))$*, although when* $G = \mathrm{SO}(3)$*, the underlying Gerstenhaber algebras are isomorphic.*

The main result of [34] is that the same phenomenon appears for Chas–Sullivan string topology even in the simple case of the two-dimensional sphere S^2.

Definition 6.4 Let A be an augmented graded algebra. Let $F^0(A) := A$ and $F^n(A) := \overline{A}\cdot\overline{A}\cdots\overline{A}$ for $n \geq 1$ be the *augmentation filtration* [36, 7.1]. We say that A is *Hausdorff* [31, Lemma 3.10] if $\bigcap_{n\in\mathbb{N}} F^n(A) = \{0\}$.

Lemma 6.5 *Let* A *and* B *be a morphism of graded algebras between two Hausdorff augmented graded algebras such that the only indecomposable elements of* A *and* B*,* $Q(A)$ *and* $Q(B)$*, are the zero vectors. Let* $f\colon A \to B$ *be a morphism of graded algebras. Then* f *preserves the augmentations. Let* $d \in \mathbb{N}$ *be a non-negative integer. Suppose* *moreover that $B = B_{\geq -d}$, i.e., B is concentrated in degrees greater than or equal to $-d$ and B is graded commutative. Then f is surjective if and only if $Q(f)$ is surjective.*

Proof When $d = 0$, $\overline{A}_0 = \{0\}$, and $\overline{B}_0 = \{0\}$, this lemma is Proposition 3.8 of [36], but the proof cannot be easily generalized. Therefore, we provide a proof.

Denote by $Q\colon \overline{A} \twoheadrightarrow Q(A) := \frac{\overline{A}}{\overline{A}\cdot\overline{A}}$ the quotient map. The sequence

$$\bigoplus_{i=1}^{n}\big(\overline{A}^{\otimes i-1}\otimes\overline{A}\cdot\overline{A}\otimes\overline{A}^{\otimes n-i}\big)\longrightarrow\overline{A}^{\otimes n}\xrightarrow{Q^{\otimes n}}\!\!\!\!\rightarrow Q(A)^{\otimes n}\longrightarrow 0$$

is exact. Alternatively, since over a field $\mathbb{K}$, $\overline{A} = \overline{A}\cdot\overline{A}\oplus Q(A)$,

$$0\longrightarrow +_{i=1}^{n}\big(\overline{A}^{\otimes i-1}\otimes\overline{A}\cdot\overline{A}\otimes\overline{A}^{\otimes n-i}\big)\hookrightarrow\overline{A}^{\otimes n}\xrightarrow{Q^{\otimes n}}\!\!\!\!\rightarrow Q(A)^{\otimes n}\longrightarrow 0$$

is a short exact sequence. Therefore, the iterated multiplication of $\overline{A}$ induces a natural map $Q(A)^{\otimes n} \twoheadrightarrow F^n(A)/F^{n+1}(A)$ that is obviously surjective.

Let $x \in \overline{A} = F^1(A)$ with $x \neq 0$. Since $\bigcap_{n\in\mathbb{N}} F^n(A) = \{0\}$, there exists $r \geq 1$ such that $x \in F^r(A)$ and $x \notin F^{r+1}(A)$. Therefore x is the product of r elements of $\overline{A}$, $x_1\cdots x_r$ such that $Q(x_1)\otimes\cdots\otimes Q(x_r) \neq 0$. By hypothesis, $Q(A)_0 = \{0\}$. So x_i and $f(x_i)$ are of degrees different from 0. So $f(x_i) \in \overline{B}$. And $f(x) = \Pi_i f(x_i) \in \overline{B}$: we have proved that f preserves the augmentations.

Let $y \in F^n(B)$ with $y \neq 0$. Similarly, y is the product of $r \geq n$ elements of $\overline{B}$, $y_1\cdots y_r$ such that all the $Q(y_i)$ are non-zero. Since $Q(B)_0 = \{0\}$, the y_i are all of degrees different from 0. Since B is graded commutative, $B_{<-d} = \{0\}$ and $y \neq 0$, there are at most d elements y_i of negative degree in the product $y_1\cdots y_r$. So there is at least $r - d$ elements y_i of positive degree. Therefore, the degree of y is at least $d\times(-1) + (r-d)\times 1$; we have proved that the non-zero elements of $F^n(B)$ are all of degree greater than or equal to $n - 2d$.

Assume that $Q(f)$ is surjective. Then $Q(f)^{\otimes n}\colon Q(A)^{\otimes n} \twoheadrightarrow Q(B)^{\otimes n}$ is also surjective. Since the following square is commutative by naturality,

$$\begin{array}{ccc} Q(A)^{\otimes n} & \longrightarrow & F^n(A)/F^{n+1}(A) \\ {\scriptstyle Q(f)^{\otimes n}}\downarrow & & \downarrow{\scriptstyle Gr_n f} \\ Q(B)^{\otimes n} & \longrightarrow & F^n(B)/F^{n+1}(B), \end{array}$$

the map induced by f, $Gr_n f$, is also surjective. In a fixed degree, consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^{n+1}(A) & \longrightarrow & F^n(A) & \longrightarrow & F^n(A)/F^{n+1}(A) & \longrightarrow & 0 \\ & & {\scriptstyle f|F^{n+1}(A)}\downarrow & & \downarrow{\scriptstyle f|F^n(A)} & & \downarrow{\scriptstyle Gr_n f} & & \\ 0 & \longrightarrow & F^{n+1}(B) & \longrightarrow & F^n(B) & \longrightarrow & F^n(B)/F^{n+1}(B) & \longrightarrow & 0 \end{array}$$

with exact rows. Suppose by induction that the restriction of f to $F^{n+1}(A)$, $f|F^{n+1}(A)$, is surjective. Then by the five Lemma, $f|F^n(A)$, is also surjective. Since $F^n(B)$ is concentrated in degrees greater than or equal to $n - 2d$, in a fixed degree, for large n,

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$F^n(B)$ is trivial and we can start the induction. Therefore $f = f|F^0(A)$ is surjective. ∎

Proof of Theorem 6.3 Since $H_*(G)$ is an exterior algebra, by Example F.2 (ii), $1 \in \operatorname{Im}\Delta$ in the BV-algebra $HH^*(H_*(G); H_*(G))$. On the contrary, by Theorems 5.13 and 5.14, the unit 1 does not belong to the image of Δ in the BV-algebra

$$H^{*+\dim G}(\mathrm{LBG}; \mathbb{F}_2).$$

So the BV-algebras $HH^*(H_*(G); H_*(G))$ and $H^{*+\dim G}(\mathrm{LBG}; \mathbb{F}_2)$ are not isomorphic.

The BV-algebra $HH^*(H_*(\mathrm{SO}(3)), H_*(\mathrm{SO}(3)))$ was explicitly computed in the proof of Theorem 6.2 and is isomorphic to the tensor product of algebras $\Lambda(x_{-2}, x_{-1}) \otimes \mathbb{F}_2[y_2, y_3]$ with $\Delta(x_{-2}y_3) = 1$, $\Delta(x_{-2}y_2) = 0$, $\Delta(x_{-1}y_2) = 1$, $\Delta(x_{-1}y_3) = 0$, and Δ is trivial on $\Lambda(x_{-2}, x_{-1}) \otimes 1$ and on $1 \otimes \mathbb{F}_2[y_2, y_3]$. The BV-algebra $H^{*+3}(\mathrm{LBSO}(3); \mathbb{F}_2) \cong \Lambda(u_{-2}, u_{-1}) \otimes \mathbb{F}_2[v_2, v_3]$ is given explicitly by Theorem 5.13.

Let $\varphi\colon \Lambda(x_{-2}, x_{-1}) \otimes \mathbb{F}_2[y_2, y_3] \to \Lambda(u_{-2}, u_{-1}) \otimes \mathbb{F}_2[v_2, v_3]$ be any morphism of graded algebras. Since $\Lambda(x_{-2}, x_{-1}) \otimes \mathbb{F}_2[y_2, y_3]$ and $\Lambda(u_{-2}, u_{-1}) \otimes \mathbb{F}_2[v_2, v_3]$ are of the same finite dimension in each degree, φ is an isomorphism if and only if φ is surjective. By Lemma 6.5, φ is surjective if and only if $Q(\varphi)$ is surjective. Therefore, φ is an isomorphism of algebras if and only if

$$\begin{aligned}
&\varphi(x_{-2}) = u_{-2}, \quad \varphi(x_{-1}) = u_{-1} + \varepsilon u_{-1}u_{-2}v_2,\\
&\varphi(y_2) = v_2 + a u_{-2}v_2^2 + b u_{-1}u_{-2}v_2v_3 + c u_{-1}v_3,\\
&\varphi(y_3) = v_3 + \alpha u_{-2}v_2v_3 + \beta u_{-1}u_{-2}v_3^2 + \gamma u_{-1}u_{-2}v_2^3 + \delta u_{-1}v_2^2,
\end{aligned}$$

where $\varepsilon, a, b, c, \alpha, \beta, \gamma, \delta$ are eight elements of $\mathbb{F}_2$. Since

$$(u_{-2})^2 = 0 \quad \text{and} \quad (u_{-1} + \varepsilon u_{-1}u_{-2}v_2)^2 = 0,$$

the above four formulas always define a morphism φ of algebras.

By the Poisson rule, a morphism of algebras between Gerstenhaber algebras is a morphism of Gerstenhaber algebras if and only if the brackets are compatible on the generators.

Note that, modulo 2, in a BV-algebra, for any elements z and t, $\{z+t, z+t\} = \{z,z\} + \{t,t\}$ and $\{z,z\} = \Delta(z^2)$. Therefore it is easy to check that

$$\begin{aligned}
&\varphi(\{x_{-2}, x_{-2}\}) = 0 = \{\varphi(x_{-2}), \varphi(x_{-2})\}, \qquad \varphi(\{x_{-1}, x_{-1}\}) = 0 = \{\varphi(x_{-1}), \varphi(x_{-1})\},\\
&\varphi(\{y_2, y_2\}) = 0 = \{\varphi(y_2), \varphi(y_2)\}, \qquad \varphi(\{y_3, y_3\}) = 0 = \{\varphi(y_3), \varphi(y_3)\}.
\end{aligned}$$

Note that $\Delta\varphi(x_{-1}) = \varepsilon u_{-2}$, $\Delta\varphi(x_{-2}) = 0$, $\Delta\varphi(y_2) = (b+c)(u_{-2}v_3 + u_{-1}v_2)$, and $\Delta\varphi(y_3) = \alpha u_{-1}v_3 + \alpha v_2 + (\alpha+\gamma)u_{-2}v_2^2 + \alpha u_{-1}u_{-2}v_2v_3$.

Therefore

$$
\begin{aligned}
\varphi(\{x_{-2},y_2\}) &= 0,\\
\{\varphi(x_{-2}),\varphi(y_2)\} &= (1+c)u_{-1} + (b+c)u_{-1}u_{-2}v_2,\\
\varphi(\{x_{-1},y_2\}) &= 1,\\
\{\varphi(x_{-1}),\varphi(y_2)\} &= 1 + (1+\varepsilon)u_{-2}v_2 + (\varepsilon c + 1 + b + c)u_{-1}u_{-2}v_3,\\
\varphi(\{x_{-2},x_{-1}\}) &= 0 = \{\varphi(x_{-2}),\varphi(x_{-1})\},\\
\varphi(\{x_{-2},y_3\}) &= 1,\\
\{\varphi(x_{-2}),\varphi(y_3)\} &= 1 + (1+\alpha)u_{-2}v_2 + (1+\alpha)u_{-1}u_{-2}v_3,\\
\varphi(\{x_{-1},y_3\}) &= 0,\\
\{\varphi(x_{-1}),\varphi(y_3)\} &= (1+\alpha+\varepsilon+\alpha)u_{-1}v_2 + (\varepsilon+1+\alpha+\varepsilon)u_{-2}v_3\\
&\quad + (\varepsilon\delta+\alpha+\gamma+\varepsilon\alpha)u_{-1}u_{-2}v_2^2,\\
\varphi(\{y_2,y_3\}) &= 0,\\
\{\varphi(y_2),\varphi(y_3)\} &= \Delta\varphi(y_2)\varphi(y_3) + \Delta(\varphi(y_2)\varphi(y_3)) + \varphi(y_2)\Delta\varphi(y_3)\\
&= (b+c)(u_{-2}v_3^2 + u_{-1}v_2v_3 + (\alpha+\delta)u_{-1}u_{-2}v_2^2v_3)\\
&\quad + \Delta\big((a+\alpha)u_{-2}v_2^2v_3 + (b+c\alpha+\beta)u_{-1}u_{-2}v_2v_3^2 + \delta u_{-1}v_2^3\big)\\
&\quad + \varphi(y_2)\Delta\varphi(y_3)\\
&= (a+\alpha+\delta+\alpha)v_2^2 + (a+\alpha+\delta+\alpha+\gamma+a\alpha)u_{-2}v_2^3\\
&\quad + \big((b+c)(\alpha+\delta) + a+\alpha+\delta+\alpha+a\alpha+b\alpha+c\alpha+c\gamma\big)\\
&\qquad \times u_{-1}u_{-2}v_2^2v_3\\
&\quad + (b+c+\alpha+c\alpha)u_{-1}v_2v_3 + (b+c+b+c\alpha+\beta)u_{-2}v_3^2.
\end{aligned}
$$

Therefore, by symmetry of the Lie brackets, φ is a morphism of Gerstenhaber algebras if and only if $\varepsilon = b = c = \alpha = 1$, $\beta = 0$ and $a = \gamma = \delta$. Conclusion: we have found only two isomorphisms of Gerstenhaber algebras between $H^{*+3}(\mathrm{LBSO}(3);\mathbb{F}_2)$ and $HH^*(H_*(\mathrm{SO}(3)),H_*(\mathrm{SO}(3)))$. ■

7 Triviality of the Loop Product When $H^*(BG)$ Is Polynomial

This section is independent of the rest of the paper. Recall that the dual of the loop coproduct introduced by Sullivan for manifolds $H^*(LM)\otimes H^*(LM)\to H^{*+d}(LM)$ is (almost) trivial [44]. In this section, we prove that the loop product for classifying spaces of Lie groups $H_*(\mathrm{LBG})\otimes H_*(\mathrm{LBG})\to H_{*+d}(\mathrm{LBG})$ is trivial if the inclusion of the fibre in cohomology $H^*(j)\colon H^*(\mathrm{LBG};\mathbb{K})\twoheadrightarrow H^*(G;\mathbb{K})$ is surjective (Theorem 7.1). We also explain that the condition that $H^*(j)\colon H^*(\mathrm{LBG};\mathbb{K})\twoheadrightarrow H^*(G;\mathbb{K})$ is surjective is equivalent to our hypothesis $H^*(BG)$ polynomial (Theorem 7.3).

Theorem 7.1 *Let* BG *be the classifying space of a connected Lie group G. Suppose that the map induced in cohomology $H^*(\mathrm{LBG};\mathbb{K})\twoheadrightarrow H^*(G;\mathbb{K})$ is surjective. Then the loop product on $H_*(\mathrm{LBG};\mathbb{K})$ is trivial, while the loop coproduct is injective.*

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This result is a generalization of [12, Theorem D] in which it is assumed that the underlying field is of characteristic zero. If Char $\mathbb{K} \neq 2$, the triviality of the loop product was first proved by Tamanoi [43, Theorem 4.7 (2)]. David Chataur and the second author conjectured that the loop coproduct on $H_*(\mathrm{LBG})$ always has a counit. Assuming that the loop coproduct on $H_*(\mathrm{LBG})$ has a counit, obviously the loop coproduct is injective and it follows from [43, Theorem 4.5 (i)] that the loop product on $H_*(\mathrm{LBG})$ is trivial.

The injectivity described in Theorem 7.1 follows from a consideration of the Eilenberg–Moore spectral sequences associated with appropriate pullback diagrams. In fact, the induced maps $\mathrm{Comp}^!$ and $H(q)$ in the cohomology are epimorphisms; see Proposition 7.2.

Let $\Omega X \overset{\iota}{\hookrightarrow} LX \twoheadrightarrow X$ be the free loop fibration. The following proposition is key to proving Theorem 7.1.

Proposition 7.2 *Let X be a simply-connected space. Suppose that*

$$H^*(\iota)\colon H^*(LX) \longrightarrow H^*(\Omega X)$$

induced by the inclusion is surjective. Then one has the following.

(i) *The map $H^*(q)$ induced by the inclusion $q\colon LX \times_X LX \to LX \times LX$ is an epimorphism.*
(ii) *Suppose moreover that X is the classifying space of a connected Lie group G. Then for the map $\mathrm{Comp}\colon \mathrm{LBG} \times_{BG} \mathrm{LBG} \to \mathrm{LBG}$, $\mathrm{Comp}^!$ is an epimorphism.*

Proof of Theorem 7.1. By Proposition 7.2 (i) and (ii), we see that the dual to the loop coproduct $\mathrm{Dlcop} := \mathrm{Comp}^! \circ H^*(q)$ on $H^*(\mathrm{LBG})$ is surjective. Since $q^!$ is

$$H^*(\mathrm{LBG} \times \mathrm{LBG})\text{-linear}$$

and decreases the degrees, $q^! \circ H^*(q) = 0$. By Proposition 7.2 (i), $H^*(q)$ is an epimorphism. Therefore $q^!$ is trivial and the dual of the loop product $\mathrm{Dlp} := q^! \circ H^*(\mathrm{Comp})$ on $H^*(\mathrm{LBG})$ is also trivial. ∎

Proof of Proposition 7.2. Consider the two Eilenberg–Moore spectral sequences associated with the free loop fibration mentioned above and with the pull-back diagram

$$\begin{array}{ccc} LX \times_X LX & \xrightarrow{q} & LX \times LX \\ {\scriptstyle \mathrm{ev}}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}\times\mathrm{ev}} \\ X & \xrightarrow{\delta} & X \times X \end{array}$$

Since $H^*(LX)$ is a free $H^*(X)$-module by the Leray–Hirsch theorem, these two Eilenberg–Moore spectral sequences are concentrated on the 0-th column. So the two morphisms of graded algebras

$$H^*(\iota) \underset{H^*(X)}{\otimes} \eta : H^*(LX) \underset{H^*(X)}{\otimes} \mathbb{K} \xrightarrow{\cong} H^*(\Omega X),$$

$$H^*(q) \underset{H^*(X)^{\otimes 2}}{\otimes} H^*(\mathrm{ev}) : (H^*(LX) \otimes H^*(LX)) \underset{H^*(X)^{\otimes 2}}{\otimes} H^*(X) \xrightarrow{\cong} H^*(LX \times_X LX)$$

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are isomorphisms. In particular, $H^*(q)$ is an epimorphism and we have an isomorphism of graded vector spaces between $H^*(LX \times_X LX)$ and $H^*(LX) \otimes H^*(\Omega X)$.

Consider the Leray–Serre spectral sequence $\{\widehat{E}_r^{*,*}, \widehat{d}_r\}$ of the homotopy fibration

$$\Omega X \to LX \times_X LX \xrightarrow{\text{Comp}} LX.$$

Since $H^*(LX \times_X LX)$ is isomorphic to $H^*(LX) \otimes H^*(\Omega X)$, by [38, III.Lemma 4.5 (2)], $\{\widehat{E}_r^{*,*}, \widehat{d}_r\}$ collapses at the E_2-term. Then for $X = \mathrm{BG}$, the integration along the fibre $\mathrm{Comp}^!: H^*(\mathrm{LBG} \times_{\mathrm{BG}} \mathrm{LBG}) \to H^{*-\dim G}(\mathrm{LBG})$ is surjective. ■

Let G be a connected Lie group and $\mathbb{K}$ a field of arbitrary characteristic. Let $\mathcal{F}: G \xrightarrow{j} \mathrm{LBG} \to BG$ be the free loop fibration.

Theorem 7.3 *The induced map $H^*(j): H^*(\mathrm{LBG}; \mathbb{K}) \to H^*(G; \mathbb{K})$ is surjective if and only if $H^*(\mathrm{BG}; \mathbb{K})$ is a polynomial algebra.*

Proof The "if" part follows from the usual Eilenberg–Moore spectral sequence argument. In fact, suppose that $H^*(BG; \mathbb{K}) \cong \mathbb{K}[V]$. Then the Eilenberg–Moore spectral sequence for the universal bundle $\mathcal{F}': G \to EG \to \mathrm{BG}$ allows one to deduce that $H^*(G; \mathbb{K}) \cong \underline{\wedge}(sV)$. By using the Eilenberg–Moore spectral sequence for the fibre square ([26, Proof of Theorem 1.2] or [28, Proof of Theorem 1.6])

$$\begin{array}{ccc} \mathrm{LBG} & \longrightarrow & \mathrm{BG}^I \\ \downarrow & & \downarrow \\ \mathrm{BG} & \xrightarrow[\delta]{} & \mathrm{BG} \times \mathrm{BG}, \end{array}$$

we see that $H^*(\mathrm{LBG}; \mathbb{K}) \cong H^*(BG; \mathbb{K}) \otimes \underline{\wedge}(sV)$ as an $H^*(\mathrm{BG}) = \mathbb{K}[V]$-algebra. This implies that the Leray–Serre spectral sequence (LSSS) for $\mathcal{F}$ collapses at the E_2-term and hence $H^*(j)$ is surjective. See the beginning of Section 3 for an alternative proof that uses module derivations.

Suppose that $H^*(j)$ is surjective. We further assume that $\mathrm{Char}\,\mathbb{K} = 2$. By the argument in [28, Remark 1.4] or [21, Proof of Theorem 2.2], we see that the Hopf algebra $A = H^*(G; \mathbb{K})$ is cocommutative and so primitively generated, *i.e.*, the natural map $P(A) \to Q(A)$ is surjective. By [28, Lemma 4.3], this yields that $H^*(G; \mathbb{K}) \cong \underline{\wedge}(x_1, \dots, x_N)$, where x_i is primitive for any $1 \le i \le N$. The same argument as in the proof of [38, Chapter 7, Theorem 2.26(2)] allows us to deduce that each x_i is transgressive in the LSSS $\{E_r, d_r\}$ for $\mathcal{F}'$. To see this more precisely, we recall that the action of G on EG gives rise to a morphism of spectral sequence

$$\{\mu_r^*\}: \{E_r, d_r\} \longrightarrow \{E_r \otimes H^*(G; \mathbb{K}), d_r \otimes 1\}$$

for which

$$\mu_2^* = 1 \otimes \mu^*: H^*(BG; \mathbb{K}) \otimes H^*(G; \mathbb{K}) \longrightarrow H^*(BG; \mathbb{K}) \otimes H^*(G; \mathbb{K}) \otimes H^*(G; \mathbb{K}),$$

where $\mu: G \times G \to G$ denotes the multiplication on G [38, Chapter 7, §2].

Suppose that there exists an integer i such that x_j is transgressive for $j < i$, but not x_i. Then we see that for some $r < \deg x_i + 1$, $d_r(x_i) \neq 0$ and $d_p(x_i) = 0$ if $p < r$. We

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write $d_r(x_i) = \sum_l b_l \otimes x_{l_1} \cdots x_{l_{s_l}}$, where each b_l is a non-zero element of $H^*(BG;\mathbb{K})$ and $1 \le l_u \le N$ for any l and u. The equality $\mu_r^* d_r(x_i) = (d_r \otimes 1)\mu_r^*(x_i)$ implies that

$$\sum_l b_l \otimes x_{l_1} \cdots x_{l_{s_l-1}} \otimes x_{l_{s_l}} = d_r \otimes 1(1 \otimes x_i \otimes 1 + 1 \otimes 1 \otimes x_i)$$
$$= \sum_l b_l \otimes x_{l_1} \cdots x_{l_{s_l}} \otimes 1,$$

which is a contradiction. Observe that x_i and x_{l_u} are primitive. Thus it follows that x_i is transgressive for any $1 \le i \le N$.

In the case where Char $\mathbb{K} = p \ne 2$, since $H^*(j)$ is surjective by assumption, it follows from the argument in [28, Remark 1.4] that $H^*(G;\mathbb{Z})$ has no p-torsion. Observe that to obtain the result, the connectedness of the loop space is assumed. By virtue of [38, Chapter 7, Theorem 2.12], we see that $H^*(BG;\mathbb{K})$ is a polynomial algebra. This completes the proof. ∎

Theorem 7.4 gives another characterisation of our hypothesis that $H^*(\mathrm{BG})$ is polynomial.

Theorem 7.4 *Let G be a connected Lie group. Then the following three conditions are equivalent.*

(i) $H^*(\mathrm{BG};\mathbb{K})$ *is a polynomial algebra on even degree generators.*
(ii) BG *is* $\mathbb{K}$*-formal and* $H^*(\mathrm{BG};\mathbb{K})$ *is strictly commutative.*
(iii) *The singular cochain algebra* $S^*(\mathrm{BG};\mathbb{K})$ *is weakly equivalent, as algebra, to a strictly commutative differential graded algebra A.*

Strictly commutative means that $a^2 = 0$ if $a \in A^{\mathrm{odd}}$ ($\mathbb{K}$ can be a field of characteristic two). We conjecture that over a field of characteristic two, this theorem remains valid if we omit “on even degree generators” in (i), “and $H^*(\mathrm{BG};\mathbb{K})$ is strictly commutative” in (ii) and “strictly” in (iii).

Proof (i) ⇒ (ii). Suppose that $H^*(\mathrm{BG};\mathbb{K})$ is a polynomial algebra. Then by the beginning of Section 6, BG is $\mathbb{K}$-formal.

(ii) ⇒ (iii). Formality means that we can take $A = (H^*(\mathrm{BG};\mathbb{K}), 0)$ in (iii).

(iii) ⇒ (i). Let Y be a simply connected space such that $S^*(Y;\mathbb{K})$ is weakly equivalent as algebras to a strictly commutative differential graded algebra A. Let $(\Lambda V, d)$ be a minimal Sullivan model of A. Consider the semifree-$(\Lambda V, d)$ resolution of $(\mathbb{K}, 0)$, $(\Lambda V \otimes \Gamma sV, D)$ given in [16, Proposition 2.4] or [33, Lemma 7.2]. Then the tensor product of commutative differential graded algebras

$$(\mathbb{K}, 0) \underset{(\Lambda V, d)}{\otimes} (\Lambda V \otimes \Gamma sV, D) \cong (\Gamma sV, \overline{D})$$

has a trivial differential $\overline{D} = 0$ [16, Corollary 2.6]. Therefore we have the isomorphisms of graded vector spaces

$$H^*(\Omega Y) \cong \mathrm{Tor}^{S^*(Y;\mathbb{K})}(\mathbb{K}, \mathbb{K}) \cong \mathrm{Tor}^{(\Lambda V, d)}(\mathbb{K}, \mathbb{K}) \cong H_*(\Gamma sV, \overline{D}) \cong \Gamma sV.$$

If $H^*(\Omega Y)$ is of finite dimension, then the suspension of V, sV must be concentrated in odd degree and so V must be in even degree and $d = 0$; thus Y is $\mathbb{K}$-formal and $H^*(Y)$ is polynomial in even degree. ∎

A Review of [6] With Sign Corrections

In this appendix, we review the results of Chataur and the second author [6]. And we correct a sign mistake.

A.1 Integration Along the Fibre in Homology With Corrected Sign

Let $F \to E \xrightarrow{\text{proj}} B$ be an oriented fibration with B path-connected, *i.e.*, the homology $H_*(F;\mathbb{K})$ is concentrated in degree less than or equal to n, $\pi_1(B)$ acts on $H_n(F;\mathbb{K})$ trivially, and $H_n(F;\mathbb{K}) \cong \mathbb{K}$. In what follows, we write $H_*(X)$ for $H_*(X;\mathbb{K})$. We choose a generator ω of $H_n(F;\mathbb{K})$, which is called an orientation class. Then the integration along the fibre $\text{proj}_!^{\omega}\colon H_*(B) \to H_{*+n}(E)$ is defined by the composite

$$H_s(B) \xrightarrow{\eta} H_s(B) \otimes H_n(F) = E^2_{s,n} \twoheadrightarrow E^{\infty}_{s,n} = F^s/F^{s-1} = F^s \subset H_{s+n}(E),$$

where η sends the $x \in H_s(B)$ to the element $(-1)^{sn} x \otimes \omega \in H_s(B) \otimes H_n(F)$ and $\{F^l\}_{l\geq 0}$ denotes the filtration of the Leray–Serre spectral sequence $\{E^r_{*,*}, d^r\}$ of the fibration $F \to E \xrightarrow{\text{proj}} B$. This Koszul sign $(-1)^{sn}$ does not appear in the usual definition of integration along the fibre recalled in [6, 2.2.1].

A.2 Products

Let $F' \to E' \xrightarrow{\text{proj}'} B'$ be another oriented fibration with orientation class $\omega' \in H_{n'}(F')$. We will choose $\omega \otimes \omega' \in H_{n+n'}(F \times F')$ as an orientation class of the fibration

$$F \times F' \longrightarrow E \times E' \xrightarrow{\text{proj} \times \text{proj}'} B \times B'.$$

By [39, Theorem 3, p. 493], the cross product $\times$ induces a morphism of spectral sequences between the tensor product of the Serre spectral sequences associated with proj and proj′ and the Serre spectral sequence associated with proj × proj′. Therefore the interchange on $H_*(B) \otimes H_n(F) \otimes H_*(B') \otimes H_{n'}(F')$ between the orientation class $\omega \in H_n(F)$ and elements in $H_*(B')$ yields the formula given (without proof) in [6, §2.3]

$$(\text{proj} \times \text{proj}')_!^{\omega \times \omega'}(a \times b) = (-1)^{|\omega'||a|}\, \text{proj}_!^{\omega}(a) \times \text{proj}_!'^{\omega'}(b).$$

Note that with the usual definition of integration along the fibre recalled from [6, 2.2.1], the Koszul sign $(-1)^{|\omega'||a|}$ must be replaced by the awkward sign $(-1)^{|\omega||b|}$. Therefore there is a sign mistake in [6, §2.3].

A.3 Integration Along the Fibre in Cohomology With Corrected Sign

Let $F \overset{\text{incl}}{\hookrightarrow} E \overset{\text{proj}}{\twoheadrightarrow} B$ be an oriented fibration with orientation $\tau\colon H^n(F) \to \mathbb{K}$. By definition, $\text{proj}^!_{\tau}\colon H^{s+n}(E) \to H^s(B)$ is the composite

$$H^{s+n}(E) \twoheadrightarrow E^{s,n}_{\infty} \subset E^{s,n}_2 = H^s(B) \otimes H^n(F) \xrightarrow{\text{id} \otimes \tau} H^s(B),$$

where $(\text{id} \otimes \tau)(b \otimes f) = (-1)^{n|b|} b\tau(f)$. This Koszul sign $(-1)^{n|b|}$ does not appear in the usual definition of integration along the fibre recalled from [3, p. 268].

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By [3, IV.14.1], $\mathrm{proj}^!_\tau(H^*(\mathrm{proj})(\beta)\cup\alpha) = (-1)^{|\beta|n}\beta\cup\mathrm{proj}^!_\tau(\alpha)$ for $\alpha\in H^*(E)$ and $\beta\in H^*(B)$. This means that the degree $-n$ linear map $\mathrm{proj}^!_\tau\colon H^*(E)\to H^{*-n}(B)$ is a morphism of left $H^*(B)$-modules in the sense that $f(xm) = (-1)^{|f||x|}xf(m)$ as stated in [9, p. 44].

A.4 Example: Trivial Fibrations

Let $\omega\in H_n(F;\mathbb{K})$ be a generator. Define the orientation $\tau\colon H^n(F)\to\mathbb{K}$ as the image of ω by the natural isomorphism of the homology into its double dual, $\psi\colon H_n(F;\mathbb{K})\to \mathrm{Hom}(H^n(F;\mathbb{K}),\mathbb{K})$. Explicitly, $\tau(f) = (-1)^{n|f|}\langle f,\omega\rangle$, where $\langle\cdot,\cdot\rangle$ is the Kronecker bracket.

Let $\mathrm{proj}_1\colon B\times F\twoheadrightarrow B$ be the projection on the first factor. Then for any $f\in H^*(F)$ and $b\in H^*(B)$, $\mathrm{proj}^!_{1\tau}(b\times f) = (-1)^{|f||b|}b\tau(f)$. Let $\mathrm{proj}_2\colon F\times B\twoheadrightarrow B$ be the projection on the second factor. Since proj_2 is the composite of proj_1 and the exchange homeomorphism, by naturality of integration along the fibre,

$$\mathrm{proj}^!_{2\tau}(f\times b) = \mathrm{proj}^!_{1\tau}((-1)^{|f||b|}b\times f) = b\tau(f) = (-1)^{n|f|}\langle f,\omega\rangle b.$$

A.5 Main Dual Theorem With Signs

The main theorem of [6] states that $H_*(LX)$ is a d-dimensional (non-unital non counital) homological conformal field theory, *i.e.*, $H_*(\mathcal{L}X)$ is an algebra over the tensor product of graded linear props

$$\bigoplus_{F_{p+q}} \det H_1(F,\partial_{\mathrm{in}};\mathbb{Z})^{\otimes d}\otimes_{\mathbb{Z}} H_*(\mathrm{Bdiff}^+(F,\partial);\mathbb{K}).$$

See [6, §3 and 11] for the definition of this prop: here F (respectively F_{p+q}) denotes a non-necessarily connected cobordism (with p incoming circles and q outcoming circles). The prop $\det H_1(F,\partial_{\mathrm{in}};\mathbb{Z})$ manages the degree shift and the sign of each operation. In [6], Chataur and the second author did not pay attention to this prop $\det H_1(F,\partial_{\mathrm{in}};\mathbb{Z})$ (and neither did [1, p. 120], it seems). Therefore, in order to get the signs correct, we need to review all the results of [6] by taking this prop into account. Explicitly, we have maps

$$\vartheta(F_{q+p})\colon \det H_1(F_{q+p},\partial_{\mathrm{in}};\mathbb{Z})^{\otimes d}\otimes_{\mathbb{Z}} H_*(\mathrm{Bdiff}^+(F_{q+p},\partial))\otimes H_*(LX)^{\otimes q} \longrightarrow H_*(LX)^{\otimes p}$$

that assign $\vartheta^{s\otimes a}(F_{q+p})(v)$ to $s\otimes a\otimes v$.

Therefore (*cf.* [6, §6.3]), its dual $H^*(LX)$ is an algebra over the opposite prop $\bigoplus_{F_{p+q}}\det H_1(F,\partial_{\mathrm{in}};\mathbb{Z})^{\mathrm{op}\,\otimes d}\otimes_{\mathbb{Z}} H_*(\mathrm{Bdiff}^+(F,\partial))^{\mathrm{op}}$, which is isomorphic to the prop $\bigoplus_{F_{p+q}}\det H_1(F,\partial_{\mathrm{out}};\mathbb{Z})^{\otimes d}\otimes_{\mathbb{Z}} H_*(\mathrm{Bdiff}^+(F,\partial))$, since

$$\det H_1(F_{p+q},\partial_{\mathrm{out}};\mathbb{Z}) = \det H_1(F_{q+p},\partial_{\mathrm{in}};\mathbb{Z})$$

and $\mathrm{diff}^+(F_{p+q},\partial) = \mathrm{diff}^+(F_{q+p},\partial)$. Explicitly, the degree 0 map since

$$\nu(F_{p+q})\colon \det H_1(F_{q+p},\partial_{\mathrm{in}};\mathbb{Z})^{\otimes d}\otimes_{\mathbb{Z}} H_*(\mathrm{Bdiff}^+(F_{q+p},\partial))\otimes H^*(LX)^{\otimes p} \longrightarrow H^*(LX)^{\otimes q}$$

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sends the element $s \otimes a \otimes \alpha$ to

$$\nu^{s\otimes a}(F_{p+q})(\alpha) :=^{t} (\vartheta^{s\otimes a}(F_{q+p}))(\alpha) = (-1)^{|\alpha|(|s|+|a|)}\alpha \circ \vartheta^{s\otimes a}(F_{q+p}).$$

Note that here we have defined the transposition of a map f as ${}^{t}f(\alpha) = (-1)^{|\alpha||f|}\alpha \circ f$.

This yields the following five propositions: A.1, A.3, A.4, A.5.

Proposition A.1 (*Cf.* [6, Proposition 24]) *Let F and F' be two cobordisms with the same incoming boundary and the same outgoing boundary. Let $\phi\colon F \to F'$ be an orientation preserving diffeomorphism, fixing the boundary,* i.e., *an equivalence between the two cobordisms F and F'. Let $c_\phi\colon \operatorname{diff}^+(F,\partial) \to \operatorname{diff}^+(F',\partial)$ be the isomorphism of groups, mapping f to $\phi \circ f \circ \phi^{-1}$. Then for*

$$s \otimes a \in \det H_1(F, \partial\text{out}; \mathbb{Z})^{\otimes d} \otimes_{\mathbb{Z}} H_*(\operatorname{Bdiff}^+(F,\partial)),$$

$$\nu^{s\otimes a}(F) = \nu^{\det H_1(\phi,\partial_{\text{out}};\mathbb{Z})^{\otimes d}(s)\otimes H_*(Bc_\phi)(a)}(F').$$

Remark A.2. In Proposition A.1, suppose that $F = F'$. By a variant of [6, Proposition 19], $H_1(\phi, \partial_{\text{out}}; \mathbb{Z})$ is of determinant +1. Since the natural surjection

$$\operatorname{diff}^+(F,\partial)) \xrightarrow{\simeq} \pi_0(\operatorname{diff}^+(F,\partial))$$

is a homotopy equivalence [7] and $\pi_0(c_\phi)$ is the conjugation by the isotopy class of ϕ, $H_*(Bc_\phi)$ is the identity. So the conclusion of Proposition A.1 is just $\nu^{s\otimes a}(F) = \nu^{s\otimes a}(F)$.

Using Proposition A.1, it is enough to define the operation $\nu(F)$ for a set of representatives F of oriented classes of cobordisms (therefore, the direct sum over a set $\oplus_F$ in the above definition of the prop has a meaning). Conversely, if $\nu(F)$ is defined for a cobordism F, then using Proposition A.1, we can define $\nu(F')$ for any equivalent cobordism F' using an equivalence of cobordism $\phi\colon F \to F'$. Two equivalences of cobordism $\phi, \phi'\colon F \to F'$ define the same operation $\nu(F')$, since

$$\det H_1(\phi, \partial\text{out}) \circ \det H_1(\phi', \partial_{\text{out}})^{-1} = \det H_1(\phi \circ \phi'^{-1}, \partial_{\text{out}}) = \text{Id}$$

and $H_*(Bc_\phi) \circ H_*(Bc_{\phi'})^{-1} = H_*(Bc_{\phi\circ\phi'^{-1}}) = \text{Id}$ by Remark A.2.

Proposition A.3 (*Cf.* [6, Proposition 30, Monoidal]) *Let F and F' be two cobordisms. For*

$$s \otimes a \in \det H_1(F, \partial_{\text{out}}; \mathbb{Z})^{\otimes d} \otimes_{\mathbb{Z}} H_*(\operatorname{Bdiff}^+(F,\partial)),$$

and

$$t \otimes b \in \det H_1(F', \partial_{\text{out}}; \mathbb{Z})^{\otimes d} \otimes_{\mathbb{Z}} H_*(\operatorname{Bdiff}^+(F',\partial)),$$

we have $\nu^{(s\otimes t)\otimes(a\otimes b)}(F \coprod F') = (-1)^{|t||a|}\nu^{s\otimes a}(F) \otimes \nu^{t\otimes b}(F')$.

Proposition A.4 (*Cf.* [6, Proposition 31, Gluing]) *Let F_{p+q} and F_{q+r} be two composable cobordisms. Denote by $F_{q+r} \circ F_{p+q}$ the cobordism obtained by gluing. For*

$$s_1 \otimes m_1 \in \det H_1(F_{p+q}, \partial_{\text{out}}; \mathbb{Z})^{\otimes d} \otimes_{\mathbb{Z}} H_*(\operatorname{Bdiff}^+(F_{p+q},\partial)),$$

and

$$s_2 \otimes m_2 \in \det H_1(F_{q+r}, \partial_{\text{out}}; \mathbb{Z})^{\otimes d} \otimes_{\mathbb{Z}} H_*(\operatorname{Bdiff}^+(F_{q+r},\partial)),$$

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we have $\nu^{s_2\otimes m_2}(F_{q+r})\circ\nu^{s_1\otimes m_1}(F_{p+q}) = (-1)^{|m_2||s_1|}\nu^{(s_2\circ s_1)\otimes(m_2\circ m_1)}(F_{q+r}\circ F_{p+q})$*. Here*

$$\circ\colon H_*(\mathrm{Bdiff}^+(F_{q+r},\partial))\otimes H_*(\mathrm{Bdiff}^+(F_{p+q},\partial))\longrightarrow H_*(\mathrm{Bdiff}^+(F_{q+r}\circ F_{p+q},\partial)),$$

and mapping $m_2\otimes m_1$ *to* $m_2\circ m_1$ *is induced by the gluing of* F_{p+q} *and* F_{q+r}*.*

As noted in [20], with their notion of h-graph cobordism, Chatour and Menichi [6] never used the smooth structure of the cobordisms. So, in fact, our cobordisms are topological. Therefore the cobordism $F_{q+r}\circ F_{p+q}$ obtained by gluing is canonically defined [25, 1.3.2]. Note that by [7,17] the inclusion $\mathrm{diff}^+(F,\partial)\overset{\approx}{\hookrightarrow}\mathrm{Homeo}^+(F,\partial)$ is a homotopy equivalence since $\pi_0(\mathrm{diff}^+(F,\partial))\cong\pi_0(\mathrm{Homeo}^+(F,\partial))$ [8, p. 45].

Proposition A.5 (*Cf.* [6, Corollary 28 i), Identity]) *Let* $\mathrm{id}_1\in\det H_1(F_{0,1+1},\partial_{\mathrm{out}};\mathbb{Z})$ *and* $\mathrm{id}_1\in H_0(\mathrm{Bdiff}^+(F_{0,1+1},\partial))$ *be the identity morphisms of the object* 1 *in the two props. Then* $\nu^{\mathrm{id}_1^{\otimes d}\otimes\mathrm{id}_1}(F_{0,1+1}) = \mathrm{Id}_{H^*(LX)}$.

Proposition A.6 (*Cf.* [6, Corollary 28 ii), Symmetry]) *Let* C_ϕ *be the twist cobordism of* $S^1\coprod S^1$*. Let* $\tau\in\det H_1(C_\phi,\partial\mathrm{out};\mathbb{Z})$*,* $\tau\in H_0(\mathrm{Bdiff}^+(C_\phi,\partial))$*, and*

$$\tau\in\mathrm{End}(H^*(LX)^{\otimes 2})$$

be the exchange isomorphisms of the three props. Then $\nu^{\tau^{\otimes d}\otimes\tau}(C_\phi)=\tau$*.*

Let F be a cobordism. Let κ_F be the generator of $H_0(\mathrm{Bdiff}^+(F,\partial))$ represented by the connected component of $\mathrm{Bdiff}^+(F,\partial)$. We may write κ instead of κ_F for simplicity. If $\chi(F)=0$, then $H_1(F,\partial_{\mathrm{out}};\mathbb{Z})=\{0\}$ has a unique orientation class. This class corresponds to the generator $1\in\det H_1(F,\partial_{\mathrm{out}};\mathbb{Z})=\Lambda^{-\chi(F)}H_1(F,\partial_{\mathrm{out}};\mathbb{Z})=\mathbb{Z}$.

The identity morphim id_1 and the exchange isomorphism τ of the prop

$$\det H_1(F,\partial_{\mathrm{out}};\mathbb{Z})$$

correspond to these unique orientation classes of

$$H_1(F_{0,1+1},\partial_{\mathrm{out}};\mathbb{Z})\quad\text{and}\quad H_1(C_\phi,\partial_{\mathrm{out}};\mathbb{Z}).$$

The identity morphism id_1 and the exchange isomorphism τ of the prop

$$H_*(\mathrm{Bdiff}^+(F,\partial))$$

are just $\kappa_{F_{0,1+1}}$ and κ_{C_ϕ}.

B Commutativity and Associativity of the Dual to the Loop Coproduct

The connected cobordism of genus g with p incoming circles and q outgoing circles is denoted $F_{g,p+q}$. In particular, $F_{0,2+1}$ is the pair of pants.

Theorem B.1 *Let* $d\geq 0$*. Let* H^* *(upper graded) be an algebra over the (lower graded) prop* $\det H_1(F,\partial_{\mathrm{out}};\mathbb{Z})^{\otimes d}\otimes_{\mathbb{Z}}H_0(\mathrm{Bdiff}^+(F,\partial))$*. Let* $s\in\det H_1(F_{0,2+1},\partial_{\mathrm{out}};\mathbb{Z})^{\otimes d}$ *be a chosen orientation. Let* $\mathrm{Dlcop}:=\nu^{s\otimes\kappa}(F_{0,2+1})$*. Let* m *be the product defined by*

$$a\odot b=(-1)^{d(i-d)}\,\mathrm{Dlcop}(a\otimes b)$$

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for $a \otimes b \in H^i \otimes H^j$. Let $\mathbb{H}^* := H^{*+d}$. Then $(\mathbb{H}^*, \odot)$ is a graded associative and commutative algebra.

Proof Using Propositions A.3, A.4, and A.5,

$$\mathrm{Dlcop} \circ (\mathrm{Dlcop} \otimes 1) = \nu^{s\circ(s\otimes \mathrm{id}_1)\otimes\kappa\circ(\kappa\otimes\mathrm{id}_1)}(F_{0,2+1} \circ (F_{0,2+1} \coprod F_{0,1+1})),$$
$$\mathrm{Dlcop} \circ (1 \otimes \mathrm{Dlcop}) = \nu^{s\circ(\mathrm{id}_1\otimes s)\otimes\kappa\circ(\mathrm{id}_1\otimes\kappa)}(F_{0,2+1} \circ (F_{0,1+1} \coprod F_{0,2+1})).$$

The cobordisms $F_{0,2+1} \circ (F_{0,2+1} \coprod F_{0,1+1})$ and $F_{0,2+1} \circ (F_{0,1+1} \coprod F_{0,2+1})$ are equivalent. When we identify them, $\kappa \circ (\kappa \otimes \mathrm{id}_1) = \kappa \circ (\mathrm{id}_1 \otimes \kappa)$. Also $F_{0,2+1} \circ C_\phi = F_{0,2+1}$ and $\kappa \circ \tau = \kappa$.

Let $\beta \in \det H_1(F_{0,2+1}, \partial_{\mathrm{out}}; \mathbb{Z})$ the generator such that $\beta^{\otimes d} = s$. The compositions of the $\mathbb{Z}$-linear prop $\det H_1(F, \partial_{\mathrm{out}}; \mathbb{Z})$ are isomorphisms. Therefore, they send generators to generators. Moreover, $\det H_1(F, \partial_{\mathrm{out}}; \mathbb{Z}) := \Lambda^{-\chi(F)} H_1(F, \partial_{\mathrm{out}}; \mathbb{Z})$ is an abelian group on a single generator of lower degree $-\chi(F)$. So $\beta \circ (\beta \otimes \mathrm{id}_1) = \varepsilon_{\mathrm{ass}} \beta \circ (\mathrm{id}_1 \otimes \beta)$ and $\beta \circ \tau = \varepsilon_{\mathrm{com}} \beta$ for given signs $\varepsilon_{\mathrm{ass}}$ and $\varepsilon_{\mathrm{com}} \in \{-1, 1\}$. Therefore

$$s \circ (s \otimes \mathrm{id}_1) = \beta^{\otimes d} \circ (\beta \otimes \mathrm{id}_1)^{\otimes d} = (-1)^{\frac{d(d-1)}{2}|\beta|^2} (\beta \circ (\beta \otimes \mathrm{id}_1))^{\otimes d} = \varepsilon_{\mathrm{ass}}^d s \circ (\mathrm{id}_1 \otimes s),$$
$$s \circ \tau = \beta^{\otimes d} \circ \tau^{\otimes d} = (\beta \circ \tau)^{\otimes d} = (\varepsilon_{\mathrm{com}} \beta)^{\otimes d} = \varepsilon_{\mathrm{com}}^d \beta^{\otimes d} = \varepsilon_{\mathrm{com}}^d s.$$

Therefore, by Proposition A.1

$$\mathrm{Dlcop} \circ (\mathrm{Dlcop} \otimes 1) = \varepsilon_{\mathrm{ass}}^d \, \mathrm{Dlcop} \circ (1 \otimes \mathrm{Dlcop}),$$
$$\mathrm{Dlcop} \circ \tau = \varepsilon_{\mathrm{com}}^d \, \mathrm{Dlcop}.$$

This means that for $a, b, c \in H^*(LX)$,

$$(a \odot b) \odot c = \varepsilon_{\mathrm{ass}}^d (-1)^d a \odot (b \odot c),$$
$$b \odot a = \varepsilon_{\mathrm{com}}^d (-1)^{(|a|-d)(|b|-d)+d} a \odot b,$$

since

$$(a \odot b) \odot c = (-1)^{d|b|+d} \, \mathrm{Dlcop} \circ (\mathrm{Dlcop} \otimes 1)(a \otimes b \otimes c),$$
$$a \odot (b \odot c) = (-1)^{d(|a|+|b|)} \, \mathrm{Dlcop}(a \otimes \mathrm{Dlcop}(b \otimes c))$$
$$= (-1)^{d|b|} \, \mathrm{Dlcop} \circ (1 \otimes \mathrm{Dlcop})(a \otimes b \otimes c).$$

Godin [14, Proof of Proposition 21] showed geometrically that $\varepsilon_{\mathrm{ass}} = -1$ for the prop $\det H_1(F, \partial_{\mathrm{in}}; \mathbb{Z})$. To determine the signs $\varepsilon_{\mathrm{ass}}$ and $\varepsilon_{\mathrm{com}}$ for the prop $\det H_1(F, \partial_{\mathrm{out}}; \mathbb{Z})$, we prefer to use our computations of $\odot$.

Consider a particular connected compact Lie group G of a particular dimension d and a particular field $\mathbb{K}$ of characteristic different from 2 such that $H^*(\mathrm{BG}; \mathbb{K})$ is a polynomial, for example $G = (S^1)^d$ or $\mathbb{K} = \mathbb{Q}$. Then $H^*(\mathrm{LBG}; \mathbb{Q})$ is an algebra over our prop and we can apply Theorem 3.1 (ii) or Corollary 4.2. Taking $a = x_1 \cdots x_N$, $b = 1$, and $c = x_1 \cdots x_N$, we obtain $1 = \varepsilon_{\mathrm{ass}}^d (-1)^d$ and $1 = \varepsilon_{\mathrm{com}}^d (-1)^d$. So if we chose d odd, $\varepsilon_{\mathrm{ass}} = \varepsilon_{\mathrm{com}} = -1$ and $\odot$ is associative and graded commutative. ∎

Remark B.2. When d is even, the d-th power of the prop $\det H_1(F, \partial_{\mathrm{in}}; \mathbb{Z})$ is isomorphic to the d-th power of the trivial prop with a degree shift $-\chi(F)$.

More precisely, let $\mathcal{P}$ be the prop such that $\mathcal{P}(p,q) := \bigoplus_{F_{p+q}} s^{-\chi(F_{p+q})}\mathbb{Z}$,

$$s^{-\chi(F')}1 \circ s^{-\chi(F)}1 = s^{-\chi(F' \circ F)}1,$$

and $s^{-\chi(F)}1 \otimes s^{-\chi(F')}1 = s^{-\chi(F \amalg F')}1$. This prop $\mathcal{P}$ is the the trivial prop with a degree shift $-\chi(F)$.

For any cobordism F, let $\Theta_F\colon s^{-\chi(F)}\mathbb{Z} \to \det H_1(F, \partial_{\text{in}}; \mathbb{Z})$ be a chosen isomorphism. Then $\Theta_F^{\otimes d}\colon \mathcal{P}^{\otimes d} \to \det H_1(F, \partial_{\text{in}}; \mathbb{Z})^{\otimes d}$ is an isomorphim of props if d is even. This prop $\mathcal{P}^{\otimes d}$ is the d-th power of the trivial prop with a degree shift $-\chi(F)$ and is not isomorphic to the trivial prop with a degree shift $-d\chi(F)$.

Proof The following upper square always commutes, while the lower square commutes if d is even.

$$\begin{array}{ccc}
(s^{-\chi(F')}\mathbb{Z})^{\otimes d} \otimes (s^{-\chi(F)}\mathbb{Z})^{\otimes d} & \xrightarrow{\Theta_{F'}^{\otimes d} \otimes \Theta_F^{\otimes d}} & \det H_1(F', \partial_{\notin}; \mathbb{Z})^{\otimes d} \otimes \det H_1(F, \partial_{\text{in}}; \mathbb{Z})^{\otimes d} \\
\downarrow \tau & & \downarrow \tau \\
(s^{-\chi(F')}\mathbb{Z} \otimes s^{-\chi(F)}\mathbb{Z})^{\otimes d} & \xrightarrow[(\Theta_{F'} \otimes \Theta_F)^{\otimes d}]{} & (\det H_1(F', \partial_{\text{in}}; \mathbb{Z}) \otimes \det H_1(F, \partial_{\text{in}}; \mathbb{Z}))^{\otimes d} \\
\circ^{\otimes d} \downarrow & & \downarrow \circ^{\otimes d} \\
(s^{-\chi(F' \circ F)}\mathbb{Z})^{\otimes d} & \xrightarrow[(\Theta_{F' \circ F})^{\otimes d}]{} & \det H_1(F' \circ F, \partial_{\text{in}}; \mathbb{Z})^{\otimes d}
\end{array}$$

Replacing $\circ$ by the tensor product $\otimes$ of props, we have proved that $\Theta_F^{\otimes d}$ is an isomorphism of props if d is even. ∎

Observe that the dual of the loop coproduct Dlcop on $H^*(LX)$ satisfies the same commutative and associative formulae as those of the Chas–Sullivan loop product on the loop homology of M [42, Remark 3.6], [29, Proposition 2.7]. So we wonder if the prop $\det H_1(F, \partial_{\text{out}}; \mathbb{Z})$ is isomorphic to the prop $\det H_1(F, \partial_{\text{in}}; \mathbb{Z})$.

Corollary B.3 *Let X be a simply connected space such that $H_*(\Omega X; \mathbb{K})$ is finite-dimensional. The shifted cohomology $\mathbb{H}^*(LX) := H^{*+d}(LX)$ is a graded commutative, associative algebra endowed with the product $\odot$ defined by*

$$a \odot b = (-1)^{d(i-d)} \operatorname{Dlcop}(a \otimes b),$$

for $a \in H^i(LX)$ and $b \in H^j(LX)$.

C The Batalin–Vilkovisky Identity

For any simple closed curve γ in a cobordism F, let us denote by $\overline{\gamma}$ the image of the Dehn twist T_γ by the Hurewicz map Θ

$$\pi_0(\operatorname{diff}^+(F, \partial)) \underset{\cong}{\xrightarrow{\partial^{-1}}} \pi_1(\operatorname{Bdiff}^+(F, \partial)) \xrightarrow{\Theta} H_1(\operatorname{Bdiff}^+(F, \partial)).$$

In this appendix, we prove the following theorem.

Theorem C.1 *Let H^* be an algebra over the prop*

$$\det H_1(F, \partial_{\text{out}}; \mathbb{Z})^{\otimes d} \otimes_{\mathbb{Z}} H_*(\operatorname{Bdiff}^+(F, \partial)).$$

Consider the graded associative and commutative algebra $(\mathbb{H}^*, \odot)$ given by Theorem B.1. Let α be a closed curve in the cylinder $F_{0,1+1}$ parallel to one of the boundary components. Let $\Delta = \nu^{\mathrm{id}_1 \otimes \overline{\alpha}}(F_{0,1+1})$. Then $(\mathbb{H}^*, \odot, \Delta)$ is a BV-algebra.

When $d = 0$, Wahl [46, Remark 2.2.4] and Kupers [27, 4.1, p. 158] gave an incomplete proof that we complete. Moreover, we pay attention to signs.

We conjecture that Theorem C.1 is true if we replace the prop $\det H_1(F, \partial_{\mathrm{out}}; \mathbb{Z})$ by the (isomorphic?) prop $\det H_1(F, \partial_{\mathrm{in}}; \mathbb{Z})$. A d-homological conformal field theory should have a structure of a BV-algebra. The dual of a d-homological conformal field theory should be a d-homological conformal field theory. All this is well known if we do not take into accounts the signs hidden in the prop $\det H_1(F, \partial_{\mathrm{in}}; \mathbb{Z})$. But the problem is to do a correct proof with signs.

The shifted cohomology algebra $(\mathbb{H}^*, \odot)$ equipped with the operator Δ is a BV-algebra if and only if $\Delta \circ \Delta = 0$ and if the Batalin–Vilkovisky identity holds; that is, for any elements a, b, and c in $\mathbb{H}^*$,

$$\begin{aligned}\Delta(a \odot b \odot c) = \Delta(a \odot b) \odot c + (-1)^{\|a\|} a \odot \Delta(b \odot c) + (-1)^{\|b\|\|a\| + \|b\|} b \odot \Delta(a \odot c)\\ - \Delta(a) \odot b \odot c - (-1)^{\|a\|} a \odot \Delta(b) \odot c\\ - (-1)^{\|a\| + \|b\|} a \odot b \odot \Delta(c),\end{aligned}$$

where $\|\alpha\|$ stands for the degree of an element α in $\mathbb{H}^*$, namely $\|\alpha\| = |\alpha| - d$.

Since $\mathrm{Bdiff}^+(F_{0,1+1})$ is $B\mathbb{Z}$, $\overline{\alpha} \circ \overline{\alpha} \in H_2(\mathrm{Bdiff}^+(F_{0,1+1})) = \{0\}$. Therefore $\Delta \circ \Delta = \pm \nu^{\mathrm{id}_1 \otimes \overline{\alpha} \circ \overline{\alpha}}(F_{0,1+1}) = 0$

The Batalin–Vilkovisky identity will arise up to signs from the lantern relation [46, Remark 2.2.4], [27, 4.1, p. 158].

Proposition C.2 ([22], [8, §5.1]) *Let $a_1, \dots, a_4$ and x, y, z be the simple closed curves described in* [27, Figure 6.89]. *Then one has $T_{a_1} T_{a_2} T_{a_3} T_{a_4} = T_x T_y T_z$ in the mapping class group of the sphere with four holes, $F_{0,3+1}$, where T_γ denotes the Dehn twist around a simple closed curve γ in the surface.*

In order to prove Theorem C.3, we represent each term of the Batalin–Vilkovisky identity in terms of elements of the prop with a homological conformal field theoretical way. This means using the horizontal (coproduct) composition $\otimes$ and the vertical composition $\circ$ on the prop. We start with the most complicated element $b \odot \Delta(a \odot c)$.

By Propositions A.3, A.4, A.5, and A.6,

$$\begin{aligned}&\mathrm{Dlcop} \circ [\mathrm{Id} \otimes (\Delta \circ \mathrm{Dlcop})] \circ (\tau \otimes \mathrm{Id})\\ &\quad = \nu^{s \otimes \kappa}(F_{0,2+1}) \circ \left[\nu^{\mathrm{id}_1 \otimes \mathrm{id}_1}(F_{0,1+1}) \otimes (\nu^{\mathrm{id}_1 \otimes \overline{\alpha}}(F_{0,1+1}) \circ \nu^{s \otimes \kappa}(F_{0,2+1})) \right]\\ &\qquad \circ (\nu^{\tau \otimes \tau}(C_\phi) \otimes \nu^{\mathrm{id}_1 \otimes \mathrm{id}_1}(F_{0,1+1}))\\ &= \pm \nu^{s \circ [\mathrm{id}_1 \otimes s] \circ (\tau \otimes \mathrm{id}_1) \otimes \kappa \circ [\mathrm{id}_1 \otimes (\overline{\alpha} \circ \kappa)] \circ (\tau \otimes \mathrm{id}_1)}(F_{0,2+1} \circ (F_{0,1+1} \coprod F_{0,2+1}) \circ (C_\phi \coprod F_{0,1+1}))\end{aligned}$$

Here $\pm$ is the Koszul sign $(-1)^{|s||\overline{\alpha}|} = (-1)^d$, since only s and $\overline{\alpha}$ have positive degrees.

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We choose $s' = s \circ (s \otimes \mathrm{id}_1)$. In the proof of Theorem B.1, we have seen that $\varepsilon_{\mathrm{ass}} = \varepsilon_{\mathrm{com}} = -1$ and hence $s \circ (s \otimes \mathrm{id}_1) = (-1)^d s \circ (\mathrm{id}_1 \otimes s)$ and $s \circ \tau = (-1)^d s$. Therefore,

$$\begin{aligned} s \circ (\mathrm{id}_1 \otimes s) \circ (\tau \otimes \mathrm{id}_1) &= (-1)^d s \circ (s \otimes \mathrm{id}_1) \circ (\tau \otimes \mathrm{id}_1) \\ &= (-1)^d s \circ [(s \circ \tau) \otimes (\mathrm{id}_1 \circ \mathrm{id}_1)] = s'. \end{aligned}$$

Since $\kappa \circ [\mathrm{id}_1 \otimes (\overline{\alpha} \circ \kappa)] \circ (\tau \otimes id_1)$ coincides with $\overline{z}$ by Proposition D.1, we have proved that $\mathrm{Dlcop} \circ (\mathrm{Id} \otimes (\Delta \circ \mathrm{Dlcop})) \circ (\tau \otimes \mathrm{Id}) = (-1)^d \nu^{s' \otimes \overline{z}}(F_{0,3+1})$. Similar computations show that

$$\begin{aligned}
&\mathrm{Dlcop} \circ (\mathrm{Id} \otimes (\Delta \circ \mathrm{Dlcop})) = \\
&\quad \pm \nu^{s \circ [\mathrm{id}_1 \otimes s] \otimes \kappa \circ [\mathrm{id}_1 \otimes (\overline{\alpha} \circ \kappa)]}(F_{0,2+1} \circ (F_{0,1+1} tsl \coprod F_{0,2+1})) = \nu^{s' \otimes \overline{x}}(F_{0,3+1}), \\
&\mathrm{Dlcop} \circ ((\Delta \circ \mathrm{Dlcop}) \otimes \mathrm{Id}) = \\
&\quad \pm \nu^{s \circ [s \otimes \mathrm{id}_1] \otimes \kappa \circ [(\overline{\alpha} \circ \kappa) \otimes \mathrm{id}_1]}(F_{0,2+1} \circ (F_{0,2+1} \coprod F_{0,1+1})) = (-1)^d \nu^{s' \otimes \overline{y}}(F_{0,3+1}), \\
&\Delta \circ \mathrm{Dlcop} \circ (\mathrm{Dlcop} \circ \mathrm{Id}) = \\
&\quad \nu^{s \circ [s \otimes \mathrm{id}_1] \otimes \overline{\alpha} \circ \kappa \circ (\kappa \otimes \mathrm{id}_1)}(F_{0,2+1} \circ (F_{0,2+1} \coprod F_{0,1+1})) = \nu^{s' \otimes \overline{a_4}}(F_{0,3+1}), \\
&\mathrm{Dlcop} \circ (\Delta \otimes \mathrm{Dlcop}) = \\
&\quad \pm \nu^{s \circ [\mathrm{id}_1 \otimes s] \otimes \kappa \circ [\overline{\alpha} \otimes \kappa]}(F_{0,2+1} \circ (F_{0,1+1} \coprod F_{0,2+1})) = \nu^{s' \otimes \overline{a_1}}(F_{0,3+1}), \\
&\mathrm{Dlcop} \circ (\mathrm{Id} \otimes \mathrm{Dlcop}) \circ (\mathrm{Id} \otimes \Delta \otimes \mathrm{Id}) = \\
&\quad \nu^{s \circ [\mathrm{id}_1 \otimes s] \otimes \kappa \circ (\mathrm{id}_1 \otimes \kappa) \circ (\mathrm{id}_1 \otimes \overline{\alpha} \otimes \mathrm{id}_1)}(F_{0,2+1} \circ (F_{0,1+1} \coprod F_{0,2+1})) = (-1)^d \nu^{s' \otimes \overline{a_2}}(F_{0,3+1}) \\
&\mathrm{Dlcop} \circ (\mathrm{Dlcop} \otimes \Delta) = \\
&\quad \nu^{s \circ [s \otimes \mathrm{id}_1] \otimes \kappa \circ [\kappa \otimes \overline{\alpha}]}(F_{0,2+1} \circ (F_{0,1+1} \coprod F_{0,2+1})) = \nu^{s' \otimes \overline{a_3}}(F_{0,3+1}).
\end{aligned}$$

Therefore, using the definition of the product $\odot$, straightforward computations show that

$$\begin{aligned}
\Delta((a \odot b) \odot c) &= (-1)^{d|b|+d} \nu^{s' \otimes \overline{a_4}}(F_{0,3+1})(a \otimes b \otimes c), \\
\Delta(a) \odot b \odot c &= (-1)^{d|b|+d} \nu^{s' \otimes \overline{a_1}}(F_{0,3+1})(a \otimes b \otimes c), \\
(-1)^{\|a\|} a \odot \Delta(b) \odot c &= (-1)^{d|b|+d} \nu^{s' \otimes \overline{a_2}}(F_{0,3+1})(a \otimes b \otimes c), \\
(-1)^{\|a\|+\|b\|} a \odot b \odot \Delta(c) &= (-1)^{d|b|+d} \nu^{s' \otimes \overline{a_3}}(F_{0,3+1})(a \otimes b \otimes c), \\
\Delta(a \odot b) \odot c &= (-1)^{d|b|+d} \nu^{s' \otimes \overline{y}}(F_{0,3+1})(a \otimes b \otimes c), \\
(-1)^{\|a\|} a \odot \Delta(b \odot c) &= (-1)^{d|b|+d} \nu^{s' \otimes \overline{x}}(F_{0,3+1})(a \otimes b \otimes c), \\
(-1)^{\|b\|\|a\|+\|b\|} b \odot \Delta(a \odot c) &= (-1)^{d|b|+d} \nu^{s' \otimes \overline{z}}(F_{0,3+1})(a \otimes b \otimes c).
\end{aligned}$$

The lantern relation gives rise to the equality

$$\begin{aligned}
\nu^{s' \otimes \overline{a_4}}(F_{0,3+1}) + \nu^{s' \otimes \overline{a_1}}(F_{0,3+1}) + \nu^{s' \otimes \overline{a_2}}(F_{0,3+1}) + \nu^{s' \otimes \overline{a_3}}(F_{0,3+1}) \\
= \nu^{s' \otimes \overline{x}}(F_{0,3+1}) + \nu^{s' \otimes \overline{y}}(F_{0,3+1}) + \nu^{s' \otimes \overline{z}}(F_{0,3+1}),
\end{aligned}$$

since the Hurewicz map is a morphism of groups. Thus,

$$\Delta(a \odot b \odot c) + \Delta(a) \odot b \odot c + (-1)^{\|a\|} a \odot \Delta(b) \odot c + (-1)^{\|a\|+\|b\|} a \odot b \odot \Delta(c)$$

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$$= \Delta(a \odot b) \odot c + (-1)^{\|a\|} a \odot \Delta(b \odot c) + (-1)^{\|b\|\|a\|+\|b\|} b \odot \Delta(a \odot c).$$

Corollary C.3 *Let G be a connected compact Lie group of dimension d. Consider the graded associative and commutative algebra $(\mathbb{H}^*(\mathrm{LBG}), \odot)$ given by Corollary B.3. Let Δ be the operator induced by the action of the circle on* LBG *(see our definition in Appendix E). Then the shifted cohomology $\mathbb{H}^*(\mathrm{LBG})$ carries the structure of a BV-algebra.*

Proof By Proposition E.1 and by [6, Proposition 60]), $\Delta = \nu^{\mathrm{id}_1 \otimes \overline{\alpha}}(F_{0,1+1})$. ■

D Seven Prop Structure Equalities on the Homology of Mapping Class Groups Proving the Batalin–Vilkovisky Identity

Recall that for any simple closed curve γ in a cobordism F, we write $\overline{\gamma}$ for the image of the Dehn twist T_α by the Hurewicz map Θ

$$\pi_0(\mathrm{diff}^+(F,\partial)) \xrightarrow[\cong]{\partial^{-1}} \pi_1(\mathrm{Bdiff}^+(F,\partial)) \xrightarrow{\Theta} H_1(\mathrm{Bdiff}^+(F,\partial)).$$

Here ∂ is the connecting homomorphism associated wwith the universal principal fibration.

Let α be a closed curve in the cylinder $F_{0,1+1}$ parallel to one of the boundary components. Let $a_1, \dots, a_4$ and x, y, z be the simple closed curves in $F_{0,3+1}$ described in [27, Figure 6.89]. In what follows, we denote by $\circ$ the vertical product in the prop

$$\bigoplus_F H_*(\mathrm{Bdiff}^+(F,\partial); \mathbb{K}),$$

which acts (up to signs) on $H^{*+\dim G}(\mathrm{LBG}; \mathbb{K})$. The goal of this appendix is to show the following equalities needed in the proof of the BV-identity given in Appendix C.

Proposition D.1

$$\overline{z} = \kappa \circ [\mathrm{id}_1 \otimes (\overline{\alpha} \circ \kappa)] \circ [\tau \otimes \mathrm{id}_1], \quad \overline{x} = \kappa \circ [\mathrm{id}_1 \otimes (\overline{\alpha} \circ \kappa)], \quad \overline{y} = \kappa \circ [(\overline{\alpha} \circ \kappa) \otimes \mathrm{id}_1],$$

$$\overline{a_4} = \overline{\alpha} \circ \kappa \circ (\kappa \otimes \mathrm{id}_1), \qquad \overline{a_1} = \kappa \circ [\overline{\alpha} \otimes \kappa],$$

$$\overline{a_2} = \kappa \circ (\mathrm{id}_1 \otimes \kappa) \circ (\mathrm{id}_1 \otimes \overline{\alpha} \otimes \mathrm{id}_1), \qquad \overline{a_3} = \kappa \circ [\kappa \otimes \overline{\alpha}].$$

Let $\widetilde{F}$ denote the group $\mathrm{diff}^+(F,\partial)$ (or the mapping class group of a surface F with boundary ∂). Recall that κ_F or simply κ denotes the generator of $H_0(B\widetilde{F})$ that is represented by the connected component of $B\widetilde{F}$.

Proposition D.2 *Let F and F' be two cobordisms. In* (i) *and* (ii)*, suppose that F and F' are gluable. Let $\circ\colon \widetilde{F} \times \widetilde{F'} \to \widetilde{F \circ F'}$ be the map induced by gluing on diffeomorphisms. Let $\mathrm{id}_F \in \widetilde{F}$ be the identity map of F. For D in $\pi_0(\widetilde{F})$ and D' in $\pi_0(\widetilde{F'})$,*

(i) $\Theta\partial^{-1}(id_F \circ D') = \kappa_F \circ \Theta\partial^{-1}D'$,
(ii) $\Theta\partial^{-1}(D \circ \mathrm{id}_{F'}) = \Theta\partial^{-1}D \circ \kappa_{F'}$,
(iii) $\Theta\partial^{-1}(id_F \sqcup D') = \kappa_F \otimes \Theta\partial^{-1}D'$.

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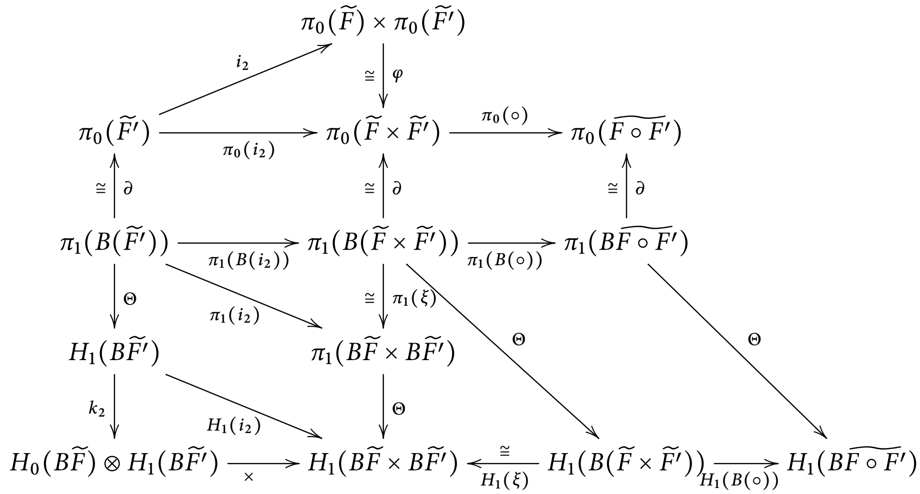
Proof We consider the diagram



Here φ is the natural isomorphism, $\times$ is the cross product,

$$\xi\colon B(\widetilde{F} \times \widetilde{F'}) \xrightarrow{\approx} B(\widetilde{F}) \times B(\widetilde{F'})$$

is the canonical homotopy equivalence, k_2 is the isomorphism defined by $k_2(x) = \kappa_F \otimes x$, and i_2 denotes various inclusions on the second factor. Note that by the definition of the prop structure, the bottom line coincides with

$$\circ\colon H_0(B\widetilde{F}) \otimes H_1(B\widetilde{F'}) \longrightarrow H_1(B\widetilde{F \circ F'}).$$

The commutativity of the diagram shows (i).

Replacing i_2 and k_2 by inclusions on the first factor, we obtain (ii). Replacing $\circ\colon \widetilde{F} \times \widetilde{F'} \to \widetilde{F \circ F'}$ by the map $\widetilde{F} \times \widetilde{F'} \to \widetilde{F \coprod F'}$, $(D, D') \mapsto D \sqcup D'$, we obtain (iii). ∎

Proof of Proposition D.1 Let $F = (F_{0,1+1} \coprod F_{0,2+1}) \circ (C_\phi \coprod F_{0,1+1})$. We can identify $F_{0,3+1}$ with $F_{0,2+1} \circ (F_{0,1+1} \coprod F_{0,1+1}) \circ F$. Let $\mathrm{emb}_2\colon F_{0,1+1} \hookrightarrow F_{0,3+1}$ be the second embedding due to this identification. The composite of the curve α and of emb_2, $S^1 \xrightarrow{\alpha} F_{0,1+1} \xhookrightarrow{\mathrm{emb}_2} F_{0,3+1}$, coincides with the curve z. Taking the same tubular neighborhood around α and z, the Dehn twists of α and z, T_α and T_z, coincide on this tubular neighborhood. Outside of this tubular neighborhood, T_α and T_z coincide with the identity maps of $F_{0,1+1}$ and of $F_{0,3+1}$, $\mathrm{id}_{F_{0,1+1}}$ and $\mathrm{id}_{F_{0,3+1}}$. Therefore

$$T_z = \mathrm{id}_{F_{0,2+1}} \circ (\mathrm{id}_{F_{0,1+1}} \sqcup T_\alpha) \circ \mathrm{id}_F\,.$$

By virtue of Proposition D.2 (i)–(iii),we have

$$\begin{aligned}
\overline{z} := \Theta\partial^{-1} T_z &= \Theta\partial^{-1}(\mathrm{id}_{F_{0,2+1}} \circ (\mathrm{id}_{F_{0,1+1}} \sqcup T_\alpha) \circ \mathrm{id}_F) \\
&= \kappa_{F_{0,2+1}} \circ \Theta\partial^{-1}((\mathrm{id}_{F_{0,1+1}} \sqcup T_\alpha) \circ \mathrm{id}_F) \\
&= \kappa_{F_{0,2+1}} \circ \Theta\partial^{-1}(\mathrm{id}_{F_{0,1+1}} \sqcup T_\alpha) \circ \kappa_F \\
&= \kappa_{F_{0,2+1}} \circ (\kappa_{F_{0,1+1}} \otimes \Theta\partial^{-1} T_\alpha) \circ \kappa_F \\
&= \kappa_{F_{0,2+1}} \circ [\mathrm{id}_1 \otimes \overline{\alpha}] \circ \kappa_F\,.
\end{aligned}$$

The prop structure on the 0-th homology gives $\kappa_F = [\mathrm{id}_1 \otimes \kappa_{F_{0,2+1}}] \circ [\tau \otimes \mathrm{id}_1]$. Finally, the prop structure on the homology of mapping class groups gives

$$\overline{z} = \kappa_{F_{0,2+1}} \circ [\mathrm{id}_1 \otimes \overline{\alpha}] \circ [\mathrm{id}_1 \otimes \kappa_{F_{0,2+1}}] \circ [\tau \otimes \mathrm{id}_1] = \kappa_{F_{0,2+1}} \circ [\mathrm{id}_1 \otimes (\overline{\alpha} \circ \kappa_{F_{0,2+1}})] \circ [\tau \otimes \mathrm{id}_1].$$

In a similar fashion, we have the other six equalities. ∎

E The Cohomological Batalin–Vilkovisky Operator Δ

The goal of this appendix is to give our definition of the Batalin–Vilkovisky operator Δ in cohomology and to compare it to others' definitions given in the literature.

Let $\Gamma\colon S^1 \times LX \to LX$ be the S^1-action map. Then in this paper the Batalin-Vilkovisky operator $\Delta\colon H^*(LX) \to H^{*-1}(LX)$ is defined [28, Proposition 3.3] by $\Delta := \int_{S^1} \circ \Gamma^*$, where $\int_{S^1}\colon H^*(S^1 \times LX) \to H^{*-1}(LX)$ denotes the integration along the fibre of the trivial fibration $S^1 \times LX \twoheadrightarrow LX$.

By our example in Appendix A (see also up to the sign [28, Proof of Proposition 3.3]), $\int_{S^1} f \times b = (-1)^{|f|}\langle f, [S^1]\rangle b$. Up to some signs, this is the slant with $[S^1]$ (*cf.* [24, Definition 1]).

Therefore for any $\beta \in H^*(LX)$, the image of β by Δ, $\Delta(\beta)$, is the unique element such that (see [42] up to the sign $-$)

$$\Gamma^*(\beta) = 1 \times \beta - \{S^1\} \times \Delta(\beta),$$

where $\{S^1\}$ is the fundamental class in cohomology defined by $\langle \{S^1\}, [S^1]\rangle = 1$.

So finally, we have proved that with our definition of integration along the fibre, since we define the BV-operator Δ using integration along the fibre as [28, Proposition 3.3], our Δ is exactly the opposite of the one defined by [42], [24, p. 648]. In particular, observe that Δ satisfies $\Delta^2 = 0$ and is a derivation on the cup product on $H^*(LX)$ [42, Proposition 4.1].

In Appendix C, we needed another characterisation of our Batalin–Vilkovisky operator Δ.

Proposition E.1 *The BV-operator $\Delta := \int_{S^1} \circ \Gamma^*$ is the dual (=transposition) of the composite*

$$H_*(LX) \xrightarrow{[S^1]\times -} H_{*+1}(S^1 \times LX) \xrightarrow{\Gamma_*} H_{*+1}(LX).$$

Proof For any space X, let $\mu_X\colon H^*(X;\mathbb{K}) \to H_*(X;\mathbb{K})^\vee$ be the map sending α to the form on $H_*(X;\mathbb{K})$, $\langle \alpha, \cdot\rangle$. Here $\langle \cdot, \cdot\rangle$ is the Kronecker bracket. By the universal coefficient theorem for cohomology, μ_X is an isomorphism. Consider the two squares

$$\begin{array}{ccccc} H^*(LX) & \xrightarrow{\Gamma^*} & H^*(S^1 \times LX) & \xrightarrow{\int_{S^1}} & H^{*-1}(LX) \\ \downarrow{\scriptstyle \mu_{LX}} & & \downarrow{\scriptstyle \mu_{S^1\times LX}} & & \downarrow{\scriptstyle \mu_{LX}} \\ H_*(LX)^\vee & \xrightarrow[(\Gamma_*)^\vee]{} & H_*(S^1 \times LX)^\vee & \xrightarrow[([S^1]\times -)^\vee]{} & H_{*-1}(LX)^\vee. \end{array}$$

The left square commutes by naturality of μ_X. For any $\alpha \in H^*(S^1)$, $\beta \in H^*(LX)$, and $y \in H_*(LX)$,

$$\Big(\mu_{LX} \circ \int_{S^1}\Big)(\alpha \times \beta)(y) = \mu_{LX}((-1)^{|\alpha||[S^1]|}\langle \alpha, [S^1]\rangle \beta)(y)$$
$$= (-1)^{|\alpha||[S^1]|}\langle \alpha, [S^1]\rangle \langle \beta, y\rangle$$

and

$$([S^1] \times -)^\vee(\mu_{S^1\times LX}(\alpha \times \beta))(y) = (-1)^{|\alpha\times\beta||[S^1]|}\mu_{S^1\times LX}(\alpha \times \beta) \circ ([S^1] \times -)(y)$$
$$= (-1)^{|\alpha||[S^1]|+|\beta||[S^1]|}\langle \alpha \times \beta, [S^1] \times y\rangle.$$

Since $\langle \alpha\times\beta, [S^1]\times y\rangle = (-1)^{|\beta||[S^1]|}\langle \alpha, [S^1]\rangle\langle \beta, y\rangle$, the right square commutes also. ■

F Hochschild Cohomology Computations

Proposition F.1 *Let A be a graded (or ungraded) algebra equipped with an isomorphism of A-bimodules $\Theta\colon A \xrightarrow{\cong} A^\vee$ between A and its dual of any degree $|\Theta|$. Denote by* $\mathrm{tr} := \Theta(1)$ *the induced graded trace on A. Let $a \in Z(A)$ be an element of the center of A. Let $d\colon A \to A$ be a derivation of A. Obviously $\overline{a} \in \mathcal{C}^0(A, A) = \mathrm{Hom}(\mathbb{K}, A)$ defined by $\overline{a}(1) = a$ and $d \circ s^{-1} \in \mathcal{C}^1(A, A) = \mathrm{Hom}(s\overline{A}, A)$ are two Hochschild cocycles. Then in the BV-algebra $HH^*(A, A) \cong HH^{*+|\Theta|}(A, A^\vee)$,*

(i) $\Delta([\overline{a}]) = 0$,

(ii) $\Delta([d \circ s^{-1}])$ *is equal to* $[\overline{a}]$, *the cohomology class of* $\overline{a}$, *if and only if for any* $a_0 \in A$, $(-1)^{1+|d|}\,\mathrm{tr} \circ d(a_0) = \mathrm{tr}(a a_0)$.

(iii) *In particular, the unit belongs to the image of Δ if and only if there exists a derivation $d\colon A \to A$ of degree 0 commuting with the trace:* $\mathrm{tr} \circ d(a_0) = \mathrm{tr}(a_0)$ *for any element a_0 in A.*

Proof By definition of Δ, the following diagram commutes up to the sign $(-1)^{|\Theta|}$ for any $p \ge 0$.

$$\begin{array}{ccccc}
\mathcal{C}^p(A, A) & \xrightarrow{\mathcal{C}^p(A,\Theta)} & \mathcal{C}^p(A, A^\vee) & \xrightarrow{Ad} & \mathcal{C}_p(A, A)^\vee \\
\downarrow{\scriptstyle\Delta} & & & & \downarrow{\scriptstyle B^\vee} \\
\mathcal{C}^{p-1}(A, A) & \xrightarrow[\mathcal{C}^{p-1}(A,\Theta)]{} & \mathcal{C}^{p-1}(A, A^\vee) & \xrightarrow[Ad]{} & \mathcal{C}_{p-1}(A, A)^\vee.
\end{array}$$

Taking $p = 0$, we obtain (i).

The image of the cocycle $d \circ s^{-1} \in \mathcal{C}^1(A; A)$ by $Ad \circ \mathcal{C}^*(A; \Theta)$ is the form $\widehat{\Theta}(d)$ on $\mathcal{C}_1(A; A) = A \otimes s\overline{A}$ defined by

$$\widehat{\Theta}(d)(a_0[sa_1]) = (-1)^{|sa_1||a_0|}(\Theta \circ d)(a_1)(a_0) = (-1)^{|sa_1||a_0|}tr(d(a_1)a_0),$$

(*cf.* [34, Proof of Proposition 20]). For any $a_0 \in A$,

$$(-1)^{|\Theta|+1+|d|}B^\vee(\widehat{\Theta}(d))(a_0) = (\widehat{\Theta}(d) \circ B)(a_0[\cdot]) = \widehat{\Theta}(d)(1[sa_0]) = tr \circ d(a_0).$$

The image of the cocycle $\overline{a} \in \mathcal{C}^0(A; A)$ by $Ad \circ \mathcal{C}^*(A; \Theta)$ is the form on A, mapping a_0 to $(\Theta \circ \overline{a})([\cdot])(a_0) = \Theta(a)(a_0) = \mathrm{tr}(a a_0)$. Therefore, $\Delta(d \circ s^{-1}) = a$ if and only if

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for any $a_0 \in A$, $(-1)^{|\Theta|+1+|d|}\,\mathrm{tr}\circ d(a_0) = (-1)^{|\Theta|} tr(aa_0)$. Since there is no coboundary in $\mathcal{C}^0(A, A)$, this proves (ii). ■

Example F.2 (a) Let $A = \Lambda x_{-d}$ be the exterior algebra on a generator of lower degree $-d \in \mathbb{Z}$. If $d \geq 0$, then $A = H^*(S^d; \mathbb{K})$. Denote by $1^\vee$ and $x^\vee$ the dual basis of $A^\vee$. The trace on A is $x^\vee$. Let $d\colon A \to A$ be the linear map such that $d(1) = 0$ and $d(x) = x$. Since $d(x \wedge x) = 0$ and $dx \wedge x + x \wedge dx = 2x \wedge x = 2 \times 0 = 0$, even over a field of characteristic different from 2, d is a derivation commuting with the trace. Therefore by Theorem F.1, $1 \in \mathrm{Im}\,\Delta$ in $HH^*(A; A)$. When $\mathbb{K} = \mathbb{F}_2$, compare with [34, Proposition 20].

(b) Let V be a graded vector space. Let $A := \Lambda(V)$ be the graded exterior algebra on V. If V is in non-positive degrees, then A is just the cohomology algebra of a product of spheres. Let $x_1, \dots, x_N$ be a basis of V. The trace of A is $(x_1 \cdots x_N)^\vee$. Let d_1 be the derivation on Λx_1 considered in the previous example. Then $d := d_1 \otimes \mathrm{id}$ is a derivation on $\Lambda x_1 \otimes \Lambda(x_2, \dots, x_N) \cong \Lambda V$. Obviously d commutes with the trace. So $1 \in \mathrm{Im}\,\Delta$.

(c) Let $A = \mathbb{K}[x]/x^{n+1}$, $n \geq 1$ be the truncated polynomial algebra on a generator x of even degree different from 0. If x is of upper degree 2, then $A = H^*(\mathbb{CP}^n; \mathbb{K})$. The trace of A is $(x^n)^\vee$. Let $d\colon A \to A$ be the unique derivation of A such that $d(x) = x$ (the case $n = 1$ was considered in Example F.2 (a)). Then $d(x^i) = ix^i$. For degree reason, d is a basis of the derivations of degree 0 of A. Then λd commutes with the trace if and only if $\lambda n = 1$ in $\mathbb{K}$. Therefore $1 \in \mathrm{Im}\,\Delta$ in $HH^*(A; A)$ if and only n is invertible in $\mathbb{K}$ (*cf.* [47] modulo 2 and with [48] otherwise).

Theorem F.3 *Let V be a graded vector space (non-negatively lower graded or concentrated in upper degree ≥ 1) such that in each degree, V is of finite dimension.*

(i) *Let $A = (\mathbf{S}(V), 0)$ be the free strictly commutative graded algebra on V,* i.e., *$A = \Lambda V^{\mathrm{odd}} \otimes \mathbb{K}[V^{\mathrm{even}}]$ is the graded tensor product on the exterior algebra on V^{odd} (the odd degree elements) and on V^{even} (the even degree elements)* [9, p. 46]. *Then the Hochschild cohomology of A, $HH^*(A, A)$, is isomorphic as Gerstenhaber algebras to $A \otimes \mathbf{S}(s^{-1}V^\vee)$. For φ, a linear form on V and $v \in V$, $\{1 \otimes s^{-1}\varphi, v \otimes 1\} = (-1)^{|\varphi|}\varphi(v)$. The Lie bracket is trivial on $(A \otimes 1) \otimes (A \otimes 1)$ and on $(1 \otimes \mathbf{S}(s^{-1}V^\vee)) \otimes (1 \otimes \mathbf{S}(s^{-1}V^\vee))$.*

(ii) *Suppose that $\mathbb{K}$ is a field of characteristic* 2. *Then we can extend* (i) *in the following way: let U and W be two graded vector spaces such that $U \oplus W = V$,* i.e., *we no longer assume that $U = V^{\mathrm{odd}}$ and $W = V^{\mathrm{even}}$. Let $A = \Lambda U \otimes \mathbb{K}[W]$. Then $HH^*(A, A)$ is isomorphic as Gerstenhaber algebra to $A \otimes \mathbb{K}[s^{-1}U^\vee] \otimes \Lambda(s^{-1}W^\vee)$, and the Lie bracket is the same as in* (i).

(iii) *Suppose that V is concentrated in odd degres or that $\mathbb{K}$ is a field of characteristic* 2. *Let $A = \Lambda V$ be the exterior algebra on V. Then the BV-algebra extending the Gerstenhaber algebra $HH^*(A, A) \cong A \otimes \mathbb{K}[s^{-1}V^\vee]$ has the trivial BV-operator Δ on A and on $\mathbb{K}[s^{-1}V^\vee]$.*

Proof (i) Recall that the Bar resolution $B(A, A, A) = A \otimes TsA \otimes A \overset{\simeq}{\twoheadrightarrow} A$ is a resolution of A as $A \otimes A^{\mathrm{op}}$-modules. When $A = (\mathbf{S}(V), 0)$, there is another smaller resolution $(A \otimes \Gamma(sV) \otimes A, D) \overset{\simeq}{\twoheadrightarrow} A$. Here $\Gamma(sV)$ is the free divided power graded algebra on

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sV and D is the unique derivation such that $D(\gamma^k(sv)) = v \otimes \gamma^{k-1}(sv) \otimes 1 - 1 \otimes \gamma^{k-1}(sv) \otimes v$ [32]. Since $\Gamma(sV)$ consists of the invariants of $T(sV)$ under the action of the permutation groups, there is a canonical inclusion of graded algebras [16, p. 278]

$$i\colon \Gamma(sV) \hookrightarrow T(sV) \hookrightarrow T(sA).$$

This map i maps $\gamma^k(sv)$ to $[sv|\cdots|sv]$. Since both $(A\otimes\Gamma(sV)\otimes A, D)$ and $B(A,A,A)$ are $A\otimes A$-free resolutions of A, the inclusion of differential graded algebras

$$A \otimes i \otimes A\colon (A \otimes \Gamma(sV) \otimes A, D) \overset{\simeq}{\hookrightarrow} B(A,A,A)$$

is a quasi-isomorphism. So by applying the functor $\mathrm{Hom}_{A\otimes A}(-,A)$,

$$\mathrm{Hom}(i,A)\colon \mathcal{C}^*(A,A) \overset{\simeq}{\twoheadrightarrow} (\mathrm{Hom}(\Gamma(sV),A),0)$$

is a quasi-isomorphism of complexes. The differential on

$$\mathrm{Hom}_{A\otimes A}((A\otimes\Gamma(sV)\otimes A, D),(A,0))$$

is zero since $f \circ D(\gamma^{k_1}(sv_1)\cdots\gamma^{k_r}(sv_r)) = 0$. The inclusion $i\colon \Gamma(sV) \hookrightarrow T(sA)$ is a morphism of graded coalgebras with respect to the diagonal [16, p. 279]

$$\Delta[sa_1|\cdots|sa_r] = \sum_{p=0}^{r}[sa_1|\cdots|sa_p]\otimes[sa_{p+1}|\cdots|sa_r].$$

Therefore the quasi-isomorphism of complexes

$$\mathrm{Hom}(i,A)\colon \mathcal{C}^*(A,A) \overset{\simeq}{\twoheadrightarrow} (\mathrm{Hom}(\Gamma(sV),A),0)$$

is also a morphism of graded algebras with respect to the cup product on the Hochschild cochain complex $\mathcal{C}^*(A,A)$ and the convolution product on $\mathrm{Hom}(\Gamma(sV),A)$.

The morphism of commutative graded algebras $j\colon A\otimes\Gamma(sV)^\vee \to \mathrm{Hom}(\Gamma(sV),A)$ mapping $a\otimes\phi$ to the linear map $j(a\otimes\phi)$ from $\Gamma(sV)$ to A defined by $j(a\otimes\phi)(\gamma) = \phi(\gamma)a$ is an isomorphim. By [16, (A.7)], the canonical map $(sV)^\vee \to \Gamma(sV)^\vee$ extends to an isomorphism of graded algebras $k\colon \mathbf{S}(sV)^\vee \overset{\cong}{\to} \Gamma(sV)^\vee$. The composite $\Theta\colon (sV)^\vee \overset{s^\vee}{\to} V^\vee \overset{s^{-1}}{\to} s^{-1}(V^\vee)$, mapping x to $\Theta(x) = (-1)^{|x|}s^{-1}(x\circ s)$, is a chosen isomorphism between $(sV)^\vee$ and $s^{-1}(V^\vee)$. Note that Θ^{-1} is the opposite of the composite $(s^{-1})^\vee \circ s$. Finally, the composite

$$A\otimes\mathbf{S}(s^{-1}(V^\vee)) \xrightarrow{A\otimes\mathbf{S}(\Theta)} A\otimes\mathbf{S}((sV)^\vee) \xrightarrow{A\otimes k} A\otimes(\Gamma(sV))^\vee \xrightarrow{j} \mathrm{Hom}(\Gamma(sV),A)$$

is an isomorphism of graded algebras. So we have obtained an explicit isomorphism of graded algebras $l\colon HH^*(A,A) \overset{\cong}{\to} A\otimes\mathbf{S}(s^{-1}(V^\vee))$. To compute the bracket, it is sufficient to compute it on the generators on $A\otimes\mathbf{S}(s^{-1}(V^\vee))$. For $m \in A$, let $\overline{m} \in \mathcal{C}^0(A,A) = \mathrm{Hom}((sA)^{\otimes 0},A)$ defined by $\overline{m}([\cdot]) = m$. Obviously, $l^{-1}(m\otimes 1)$ is the cohomology class of the cocycle $\overline{m}$. For any linear form φ on V, let $\overline{\varphi} \in \mathcal{C}^1(A,A) = \mathrm{Hom}(sA,A)$ be the map defined by

$$\overline{\varphi}([sv_1v_2\cdots v_n]) = \sum_{i=1}^{n}(-1)^{|\varphi||sv_1v_2\cdots v_{i-1}|}\varphi(v_i)v_1\cdots\widehat{v_i}\cdots v_n.$$

Since the composite $\overline{\varphi}\circ s$ is a derivation of A, $\overline{\varphi}$ is a cocycle. Since

$$\overline{\varphi}([sv_1]) = (-1)^{|\varphi|}\varphi(v_1)1,$$

the composite $\overline{\varphi} \circ i$ is the image of $1 \otimes s^{-1}\varphi$ by the composite

$$j \circ (A \otimes k) \otimes (A \otimes \mathbf{S}(\Theta)) \colon A \otimes \mathbf{S}(s^{-1}(V^{\vee})) \longrightarrow \operatorname{Hom}(\Gamma(sV), A).$$

Therefore $l^{-1}(1 \otimes s^{-1}\varphi)$ is the cohomology class of the cocycle $\overline{\varphi}$. By [10, p. 48–49], we have

(a) the Lie bracket is null on $\mathcal{C}^0(A, A) \otimes \mathcal{C}^0(A, A)$;
(b) the Lie bracket restricted to $\{\cdot, \cdot\} \colon \mathcal{C}^1(A, A) \otimes \mathcal{C}^0(A, A) \to \mathcal{C}^0(A, A)$ is given by $\{g, \overline{a}\} = \overline{g(sa)}$ for any $g \colon sA \to A$ and $a \in A$;
(c) the Lie bracket restricted to $\{\cdot, \cdot\} \colon \mathcal{C}^1(A, A) \otimes \mathcal{C}^1(A, A) \to \mathcal{C}^1(A, A)$ is given by

$$\{f, g, \}([sa]) = f \circ s \circ g \circ s(a) - (-1)^{(|f|+1)(|g|+1)} g \circ s \circ f \circ s(a).$$

By (a), the Lie bracket is trivial on $(A \otimes 1) \otimes (A \otimes 1)$. By (b), for $\varphi \in V^{\vee}$ and $v \in V$,

$$\{1 \otimes s^{-1}\varphi, v \otimes 1\} = (-1)^{|\varphi|}\varphi(v) 1 \otimes 1.$$

Let φ and φ' be two linear forms on V. Then

$$\overline{\varphi} \circ s \circ \overline{\varphi'} \circ s([v_1 \cdots v_n]) = \sum_{1 \le j < i \le n} \big((-1)^{|\varphi||\varphi'|} \varepsilon_{ij}(\varphi, \varphi') + \varepsilon_{ij}(\varphi', \varphi) \big) v_1 \cdots \widehat{v_j} \cdots \widehat{v_i} \cdots v_n,$$

where $\varepsilon_{ij}(\varphi, \varphi') = (-1)^{|\varphi||sv_1 \cdots v_{i-1}| + |\varphi'||sv_1 \cdots v_{j-1}|} \varphi(v_i)\varphi'(v_j)$. Therefore,

$$\overline{\varphi} \circ s \circ \overline{\varphi'} \circ s - (-1)^{|\varphi||\varphi'|} \overline{\varphi'} \circ s \circ \overline{\varphi} \circ s = 0.$$

So by (c), the Lie bracket $\{1 \otimes s^{-1}\varphi, 1 \otimes s^{-1}\varphi'\} = 0$.

(iii) By Proposition F.1 (i), $\Delta([\overline{m}]) = 0$ and so Δ is trivial on all $m \otimes 1 \in A \otimes 1$. Let $x_1, \dots, x_N$ be a basis of V. The trace of A is $(x_1 \cdots x_N)^{\vee}$. Therefore the trace vanishes on elements of wordlength strictly less than N. For any $\varphi \in V^{\vee}$, the derivation $\overline{\varphi} \circ s$ decreases wordlength by 1. So $\operatorname{tr} \circ \overline{\varphi} \circ s = 0$. By Proposition F.1 (ii), $\Delta(1 \otimes s^{-1}\varphi) = 0$. Since the Lie bracket is trivial on $(1 \otimes \mathbb{K}[s^{-1}V^{\vee}]) \otimes (1 \otimes \mathbb{K}[s^{-1}V^{\vee}])$, Δ is trivial on $1 \otimes \mathbb{K}[s^{-1}V^{\vee}]$.

(ii) The proof is the same as in (i). For example, $\Gamma(sV)$ is the graded tensor product of the free divided power algebra on sU and of the exterior algebra on sW. ■

Remark F.4. Suppose that V is concentrated in degree 0. We have obtained a quasi-isomorphism of differential graded algebras

$$\mathcal{C}^*(\mathbf{S}(V), \mathbf{S}(V)) \xrightarrow{\simeq} (\mathbf{S}(V) \otimes \Lambda(s^{-1}V^{\vee}), 0).$$

In particular, the differential graded algebra $\mathcal{C}^*(\mathbf{S}(V), \mathbf{S}(V))$ is formal.

It is easy to see that if V is of dimension 1, then the inclusion

$$(\mathbf{S}(V) \otimes \Lambda(s^{-1}V^{\vee}), 0) \hookrightarrow \mathcal{C}^*(\mathbf{S}(V), \mathbf{S}(V))$$

is a quasi-isomorphism of differential graded Lie algebras. In particular, the differential graded Lie algebra $\mathcal{C}^*(\mathbf{S}(V), \mathbf{S}(V))$ is formal. The Kontsevich formality theorem says that over a field $\mathbb{K}$ of characteristic zero, the differential graded Lie algebra $\mathcal{C}^*(\mathbf{S}(V), \mathbf{S}(V))$ is formal even if V is not of dimension 1 [23, Theorem 2.4.2].

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