NATURAL ACTION–ANGLE VARIABLES

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Since galaxies are collisionless relaxed systems, actions are an extremely useful tool for understanding their dynamics. There are many potential applications of actions: (1) When orbits in an N-body simulation are characterized by their actions, the six dimensional distribution function, $f(\vec{x},\vec{p})$, can be reduced to a more tractable three dimensional function, $f(J)$. (2) Actions are adiabatic invariants, and thus are useful for studying slowly evolving systems. Binney, May and Ostriker (1986) have applied this technique to study the response of the spheroid to the disc. (3) The spectral decomposition of an orbit can be used to help generate self–consistent galaxy models (Spergel 1987).

In an integrable potential, a star (or any collisionless particle) moves along a three dimensional torus imbedded in six dimensional phase space. The actions, J_k, are the projected areas of the torus, while the action angles, θ_k, are 2π periodic functions on the surface of the torus. The position and momentum of a particle on the torus can be expanded as a Fourier series:

$$q^j(\vec{\theta}) = \sum_{\vec{m}} C^j_{\vec{m}} \exp(i\vec{m}\cdot\vec{\theta}) = \sum_{\vec{m}} C^j_{\vec{m}} \exp[i(\vec{m}\cdot\vec{\omega})t] \tag{1a}$$

$$p^j(\vec{\theta}) = \sum_{\vec{m}} B^j_{\vec{m}} \exp(i\vec{m}\cdot\vec{\theta}) = \sum_{\vec{m}} B^j_{\vec{m}} \exp[i(\vec{m}\cdot\vec{\omega})t] \tag{1b}$$

where I have utilized the relation $\dot{\theta} = \partial H(J)/\partial J_k = \omega_k$. For further discussion, see Arnold (1977). The actions can now be expressed in terms of the Fourier coefficents of the orbit:

$$J_k = \int_{\gamma_k} \sum_j p_j dq_j = \sum_{\text{lines}} \sum_j m_k B^j_{\vec{m}} C^j_{\vec{m}'} \delta^{(3)}(\vec{m} - \vec{m}') \tag{2}$$

The coefficents $C_{\vec{k}}$ can be found by (1) numerically integrating an orbit, (2) representing its Fourier transform as a series of lines, and (3) then choosing the ω_k's (Binney and Spergel 1984).

While steps (1) and (2) are well defined, the problem remains of how to make the "right" choice of frequencies. Each set of frequency choices corresponds to a set of action–angle coordinates. If different sets of angle coordinates are used on

T. de Zeeuw (ed.), Structure and Dynamics of Elliptical Galaxies, 483–484.

https://doi.org/10.1017/S0074180900185857 Published online by Cambridge University Press

each torus, then action–angle coordinates are no longer useful global descriptions of orbits. We wish to choose angle coordinates so that all but one of the actions of the closed orbits that spawn the major orbital families are zero. This is equivalent to requiring that the frequency choices made in step (3) of the program go smoothly over to the frequencies associated with each of the closed orbits.

In confocal ellipsoidal coordinates, the global structure of the tori is apparent. Two of the actions are always zero for any of the closed orbits and the four classes of orbits can be neatly classified in action space (de Zeeuw 1984). I have explored the implications of de Zeeuw (1984)'s suggestion that the use of confocal coordinates would simplify the calculations of actions. The results have been most gratifying.

A computer program has been developed which calculates an orbit in *any* potential, and then switches from cartesian to confocal ellipsoidal coordinates, where the Fourier spectra are remarkably simple. Most of the power in each of the spectra $[\lambda(\omega), \mu(\omega)$, and $\nu(\omega)]$ lies in one line! (A different line in each coordinate's spectrum.) The frequency of these lines are the fundamental frequencies.

The simple spectra have many computational benefits. In practice, the presence of many lines with similar frequencies requires long integration times so that the two lines can be resolved. Since the choice of coordinates *nearly* decouples the three oscillations, there is not much power in lines that are not harmonics of a single frequency.

Since the momentum in confocal coordinates is often singular near the planes $x = 0$, $y = 0$, and $z = 0$, a coordinate transformation is necessary to remove these singularities. A different change of coordinates is required for each of the orbital families. However, since it is straightforward to classify an orbit, avoiding these numerical singularities is not a problem. The new conjugate momenta and positions can again be expanded in a Fourier series and the actions can be determined.

This technique has been applied successfully in three dimensions to orbits in a Schwarzschild potential. In two dimensions, a primitive form of this technique has been successfully applied to orbits computed in a Bahcall–Soneira galaxy model (Aguilar and Spergel 1987).

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