G. Ikegami
Nagoya Math. J.
Vol. 90 (1983), 63–76

ON EXISTENCE OF TOLERANCE STABLE DIFFEOMORPHISMS*

GIKŌ IKEGAMI

§ 1. Introduction

We consider a compact smooth manifold M. $\mathrm{Diff}^1(M)$ denotes the space of C^1-diffeomorphisms of M onto itself with the usual C^1-topology. In the research of the qualitative theory of dynamical systems there is a desire to find a concept of stability of geometric global structure of orbits such that this stable systems are dense in the space of dynamical systems on M. Structural stability does not satisfy the density condition in $\mathrm{Diff}^1(M)$. Tolerance stability (see Section 2 for definition) is a candidate for the density property [7, p. 294]. Concerning tolerance stability there are researches as [6], [7], [8], and [2].

If $f \in \mathrm{Diff}^1(M)$ is structurally stable in strong sense, f is topologically stable in $\mathrm{Diff}^1(M)$ (see Section 2 for definition). Moreover, topological stability implies tolerance stability [A. Morimoto, 2]. The proof of this property will be introduced in Section 2.

The main result of this paper is to show the existence of diffeomorphisms on any compact manifold M which are tolerance stable but not topologically stable in $\mathrm{Diff}^1(M)$, so that, not structurally stable in strong sense. This will be proved in Sections 3, 4 and 5.

§ 2. Definitions and statement of results

We denote by Homeo (M) the set of homeomorphisms of M onto itself; the topology on Homeo (M) is given by the neighborhood $N_\varepsilon(f)$ of $f \in$ Homeo (M)

$$N_\varepsilon(f) = \{g; d(f, g) < \varepsilon\}, \qquad \varepsilon > 0 .$$

Here, for a Riemannian metric d on M, $d(f, g) < \varepsilon$ means

Received October 12, 1981.
* The author is partly supported by Grant in Aid for Scientific Research Project No. 546004.

https://doi.org/10.1017/S0027763000020341 Published online by Cambridge University Press

$$d(f(x), g(x)) < \varepsilon \qquad \text{for } x \in M .$$

To state the definition of tolerance stability, we need the following definition:

DEFINITION (2.1). f, $g \in$ Homeo (M) are *orbit-ε-equivalent*, $\varepsilon > 0$, if

1. for every f-orbit O_f, there is a g-orbit O_g such that
   (a) $O_f \subset U_\varepsilon(O_g)$
   (b) $O_g \subset U_\varepsilon(O_f)$, and
2. for every g-orbit O_g, there is a f-orbit O_f such that
   (a) $O_g \subset U_\varepsilon(O_f)$
   (b) $O_f \subset U_\varepsilon(O_g)$.

Here, $U_\varepsilon(*)$ is the ε-neighborhood of $*$.

Suppose that a subset $\mathscr{D}$ of Homeo (M) is given a topology not coarser than that of Homeo (M).

DEFINITION (2.2). An element $f \in \mathscr{D}$ is *tolerance-stable* in $\mathscr{D}$ if for every $\varepsilon > 0$ there is a neighborhood N of f in $\mathscr{D}$ (with respect to the given topology on $\mathscr{D}$) such that, for every $g \in N$, f and g are orbit-ε-equivalent.

DEFINITION (2.3). An element $f \in \mathscr{D}$ is *topologically stable* in $\mathscr{D}$, if for any $\varepsilon > 0$ there is a neighborhood N of f in $\mathscr{D}$ such that for every $g \in N$ there is a continuous map $h: M \to M$ satisfying

(a) $d(h, i_M) < \varepsilon$, where i_M is the identity map of M,
(b) $hg = fh$.

The following property is mentioned and proved by A. Morimoto in [2]. We introduce this:

PROPOSITION. *If M is a compact topological manifold and $f \in$ Homeo (M) is topologically stable in $\mathscr{D}$ then f is tolerance stable in $\mathscr{D}$, for any subset $\mathscr{D} \subset$ Homeo (M).*

Proof. For closed non-empty subsets A and B of M, let

$$\bar{d}(A, B) = \max \left\{ \max_{a \in A} d(a, B), \max_{b \in B} d(b, A) \right\},$$

where $d(a, B) = \min_{b \in B} d(a, b)$. $O_f(x)$ denotes the f-orbit of x; $O_f(x) = \{f^i(x);\ i \in Z\}$. Put $\bar{O}_f(x) = \mathrm{Cl}(O_f(x))$. By the assumption, for every $\varepsilon > 0$,

https://doi.org/10.1017/S0027763000020341 Published online by Cambridge University Press

there is a neighborhood N of f in $\mathscr{D}$ such that for every $g \in N$ there is $h: M \to M$ satisfying (a) and (b) in Definition (2.3). By (b), $h(O_g(x)) = O_f(h(x))$ for every $x \in M$. Hence,

$$\bar{d}(\overline{O}_g(x), \overline{O}_f(h(x))) = \bar{d}(\overline{O}_g(x), h(\overline{O}_g(x))) < \varepsilon .$$

Therefore, for any g-orbit O_g there is f-orbit O_f such that $O_g \subset U_\varepsilon(O_f)$. and $O_f \subset U_\varepsilon(O_g)$. Since M is a compact manifold, we can prove that $d(h, i_M) < \varepsilon$ implies that $h: M \to M$ is a surjection if $\varepsilon > 0$ is sufficiently small. We may assume that ε is taken so small that this property is satisfied. Hence for every $x \in M$ there is $y \in M$ such that $h(y) = x$. Then

$$\begin{aligned}\bar{d}(\overline{O}_f(x), \overline{O}_g(y)) &= \bar{d}(\overline{O}_f(h(y)), \overline{O}_g(y)) \\ &= \bar{d}(h(\overline{O}_g(y)), \overline{O}_g(y)) < \varepsilon .\end{aligned}$$

Hence, for any f-orbit O_f there is g-orbit O_g such that $O_f \subset U_\varepsilon(O_g)$ and $O_g \subset U_\varepsilon(O_f)$. Therefore, f is tolerance stable in $\mathscr{D}$.

DEFINITION (2.4). Two elements $f, g \in \mathrm{Diff}^1(M)$ are *topologically ε-conjugate* if there is a homeomorphism $h: M \to M$ such that $hg = fh$ and $d(h(x), x) < \varepsilon$ for every $x \in M$. f, g are *topologically conjugate* if there is a homeomorphism h such that $hg = fh$.

DEFINITION (2.5). An element $f \in \mathrm{Diff}^1(M)$ is *structurally stable in strong sense* if for every $\varepsilon > 0$ there is a neighborhood N of f in $\mathrm{Diff}^1(M)$ such that every $g \in N$ are topologically ε-conjugate to f. f is *structurally stable*, if there is N such that, for every $g \in N$, f and g are topologically conjugate.

Structural stability in strong sense implies structural stability and topological stability in $\mathrm{Diff}^1(M)$. If $f \in \mathrm{Diff}^1(M)$ satisfies Axiom A and strong transversality condition then f is structurally stable in strong sense [4].

THEOREM. *Let M be a compact differentiable manifold. There is a diffeomorphism f, in the boundary $\partial\Sigma$ of the set Σ of all structurally stable elements in* $\mathrm{Diff}^1(M)$, *such that*

(a) *f is tolerance-stable in* $\mathrm{Diff}^1(M)$, *and*

(b) *f is not topologically stable in* $\mathrm{Diff}^1(M)$, *so that, f is not structurally stable in strong sense.*

§ 3. Construction of f

Theorem is proved in Sections 3, 4 and 5. In these sections M is assumed to have $\dim M \geqq 2$. But to the readers of these sections the proof of Theorem in the case $\dim M = 1$ will be obvious.

f will be constructed as follows. If f_0 is a diffeomorphism which is structurally stable in strong sense and has a periodic point p that is a sink or source, then f will be obtained from f_0 by isotopically replacing f_0 on a small neighborhood of p.

Let f_0 be a time-one map of the flow of the vector field Y obtained by Theorem 2.1 of [5]. Then f_0 is a Morse-Smale diffeomorphism having a fixed point p which is a sink. By [3], f_0 is structurally stable in strong sense.

By replacing f_0 by an isotopy on a small neighborhood U of p we obtain f_1 such that

(i) every point in a small closed ball neighborhood B in U, with center p, is a fixed point of f_1, and

(ii) for every x in $U - B$, $\lim_{k\to\infty} f_1^k(x)$ exists in ∂B.

Let B_r be a closed ball in the euclidean space $\boldsymbol{R}^m$ of the same dimension as M, centered on the origin with radius r. Let $S_r = \partial B_r$, a $(m-1)$-sphere. After this, we regard B as a closed ball B_{r_0} in $\boldsymbol{R}^m$, and p as the origin of $\boldsymbol{R}^m$.

To construct f we will define a vector field V on B. On a neighborhood of p, f will be the time-one map of the flow of V.

(1) Construction of V.

For this purpose we at first define a vector field X. Let

$$\varphi_0(r) = e^{-1/r^2} \sin \frac{1}{r}, \qquad r > 0 .$$

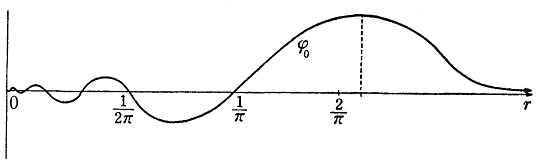

Fig. 1.

Take $r_1 \in \boldsymbol{R}_+$ such that $r_1 < r_0$, $\varphi_0'(r_1) > 0$, and

$$\frac{1}{2n\pi} < r_1 < \frac{1}{(2n-1)\pi} \quad \text{for a fixed } n \in \boldsymbol{Z}_+ . \tag{3.1}$$

Let $\alpha\colon [r_1, \infty) \to \boldsymbol{R}$ be a C^1-function such that $\alpha(r) < 0$ and $\alpha'(r) > 0$ for every $r \in [r_1, \infty)$, and that the function defined by

$$\varphi(r) = \begin{cases} 0 & \text{if } r = 0 \\ \varphi_0(r) & \text{if } 0 < r < r_1 \\ \alpha(r) & \text{if } r_1 \leqq r \end{cases}$$

is C^1.

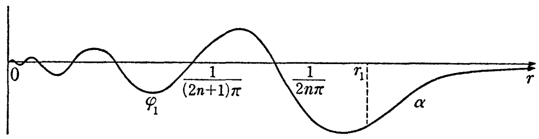

Fig. 2.

Define a vector field X on B by

$$X_x = \begin{cases} \varphi(\|x\|)\dfrac{x}{\|x\|} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Here, $\|\cdot\|$ is the euclidean norm on $\boldsymbol{R}^m$.

We show that X is C^1. Let $x = {}^t(x_1, \cdots, x_m) \in \boldsymbol{R}^m$ be a row vector, i.e. the transposition of $(x_1, \cdots, x_m)$. If $x \neq 0$

$$\begin{aligned} \frac{\partial}{\partial x_i} X_x &= \frac{\partial}{\partial x_i}\left(\frac{\varphi(\|x\|)}{\|x\|}\right)x + \frac{\varphi(\|x\|)}{\|x\|}\frac{\partial}{\partial x_i}x \\ &= \frac{\partial}{\partial x_i}\|x\|\frac{\varphi'(\|x\|)\|x\| - \varphi(\|x\|)}{\|x\|^2}x + \frac{\varphi(\|x\|)}{\|x\|}\frac{\partial}{\partial x_i}x \\ &= \frac{x_i}{\|x\|}\frac{\varphi'(\|x\|)\|x\| - \varphi(\|x\|)}{\|x\|^2}x + \frac{\varphi(\|x\|)}{\|x\|}\frac{\partial}{\partial x_i}x \\ &= x_i\left(\frac{\varphi'(\|x\|)}{\|x\|^2} - \frac{\varphi(\|x\|)}{\|x\|^3}\right)x + \frac{\varphi(\|x\|)}{\|x\|}\frac{\partial}{\partial x_i}x\,. \end{aligned}$$

Hence, for $x \neq 0$

$$DX_x = \left(\frac{\varphi'(\|x\|)}{\|x\|^2} - \frac{\varphi(\|x\|)}{\|x\|^3}\right)x\cdot{}^t x + \frac{\varphi(\|x\|)}{\|x\|}I\,,$$

where DX_x is the Jacobian matrix, and I is the unit matrix. For a matrix $A = (a_1, \cdots, a_m)$ with row vectors $a_1, \cdots, a_m$, we define the norm of A by

$$\|A\| = \max_j \|a_j\| .$$

Then,

$$\|DX_x\| \leqq \left| \frac{\varphi'(\|x\|)}{\|x\|^2} - \frac{\varphi(\|x\|)}{\|x\|^3} \right| \cdot \|x\|^2 + \left| \frac{\varphi(\|x\|)}{\|x\|} \right| .$$

$DX_0 = 0$ since $\varphi'(0) = 0$. Therefore, since φ is C^1, X is a C^1-vector field.

Next we define a vector field Y on B. Let $\mu: [0, \infty) \to [0, \infty)$ be a C^1-function such that

$$\begin{cases} \mu \geqq 0, & \text{and} \\ \mu(r) = 0 \quad \text{and} \quad \mu'(r) = 0 & \text{if } r = 0 \quad \text{or} \quad r \geqq r_1 . \end{cases}$$

Let C be a C^1-vector field, on the unit sphere S^{m-1}, such that C has two singular points p_+ and p_-, where p_+ is a source at the north pole and p_- is a sink at the south pole, and such that every other trajectory of C goes out of p_+ and into p_-. Then Y is defined by

$$Y_x = \begin{cases} \mu(\|x\|) C_{x/\|x\|} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 . \end{cases}$$

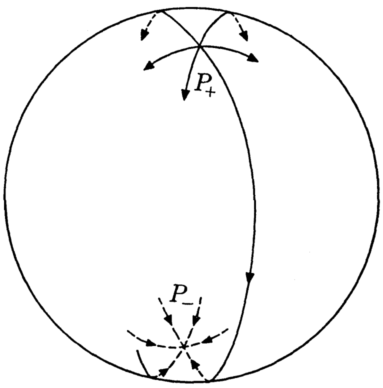

Fig. 3.

For the calculation of the derivative of Y_x, we take a C^1-extension $\tilde{C}: U(S^{m-1}) \to \boldsymbol{R}^m$ of $C: S^{m-1} \to \boldsymbol{R}^m$, where $U(S^{m-1})$ is a neighborhood of S^{m-1} in $\boldsymbol{R}^m$. Then, for $x \neq 0$, we have

$$\mu(\|x\|) C_{x/\|x\|} = \mu(\|x\|) \tilde{C}_{x/\|x\|} .$$

Let e_i be the i-th row vector of the unit matrix I. Let $y = x/\|x\|$, and let D be the notation of the derivative of variable x. Since

$$\frac{\partial}{\partial x_i}\frac{x}{\|x\|} = -\frac{x_i}{\|x\|^3}x + \frac{1}{\|x\|}e_i, \qquad \text{and}$$

$$DY_x = D\mu(\|x\|)\cdot \tilde{C}_y + \mu(\|x\|)\cdot D\tilde{C}_y \cdot D\Big(\frac{x}{\|x\|}\Big),$$

we have

$$\begin{aligned}\frac{\partial}{\partial x_i}Y_x &= \frac{\partial}{\partial x_i}(\mu(\|x\|))\tilde{C}_y + \mu(\|x\|)\cdot D\tilde{C}_y\cdot\frac{\partial}{\partial x_i}\frac{x}{\|x\|}\\ &= \frac{x_i}{\|x\|}\mu'(\|x\|)\tilde{C}_y + \mu(\|x\|)\cdot D\tilde{C}_y\cdot\Big(-\frac{x_i}{\|x\|^3}x + \frac{1}{\|x\|}e_i\Big).\end{aligned}$$

Consequently, if $x \neq 0$ then

$$DY_x = \mu'(\|x\|)\tilde{C}_y\cdot t_y + \mu(\|x\|)\cdot D\tilde{C}_y\cdot\Big(-\frac{1}{\|x\|^3}x\cdot {}^t x + \frac{1}{\|x\|}I\Big).$$

Since $\mu(0) = \mu'(0) = 0$ we have $DY_0 = 0$. Therefore Y is a C^1-vector field.

The C^1-vector field V on B is defined by

$$V_x = X_x + Y_x .$$

Fig. 4 shows the orbit structure of V. Here, we denote $B(k) = B_{1/k\pi}$ and $S(k) = \partial B(k)$. Every singular point of V is hyperbolic except p.

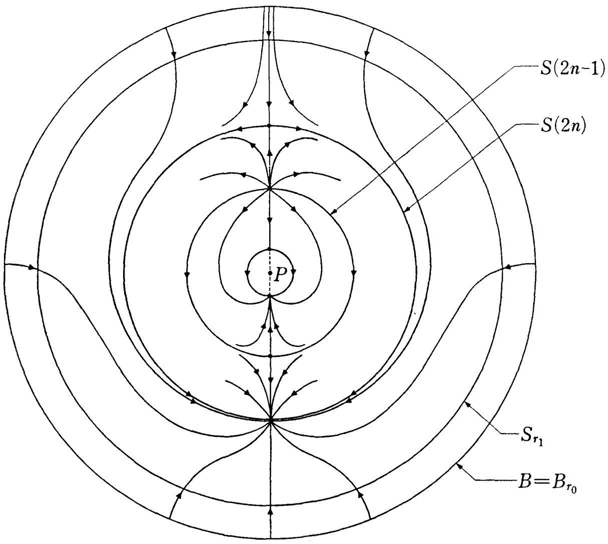

Fig. 4.

(2) Construction of f.

Let $\Psi_1: B \to B$ be the time one map of the flow Ψ of V. Ψ_1 is a C^1-diffeomorphism such that $B - \Psi_1(B)$ is an annulus which is diffeomorphic to $\partial B \times [0, 1)$. Every fixed point of Ψ_1 is hyperbolic except p. The property (ii) of f_1 and the orbit structure of V enable us to obtain a diffeomorphism $f: M \to M$ satisfying the following properties;

(i) $f|B = \Psi_1$,

(ii) $f|(M - U) = f_1(M - U)$,

(iii) if $x \in U - B$ then $\lim_{k\to\infty} f^k(x)$ is the north pole or the south pole of $S(2n)$.

Moreover, $f|(M - B)$ is obtained from f_1 by an isotopy supported by U. Since Ψ_1 is isotopic to $i_B = f_1|B$ by the isotopy Ψ_t, $t \in [0, 1]$, f is isotopic to f_1 by an isotopy supported by U.

In Sections 4 and 5 it is proved that f possesses the desired properties (a), (b) of Theorem.

§ 4. Proof of tolerance-stability of f in $\mathrm{Diff}^1(M)$

Let sufficiently small $\varepsilon > 0$ be given.

LEMMA. *There is a diffeomorphism $h: M \to M$ such that* (i) $h =$ *identity on* $M - B_{\varepsilon/4}$, *and* (ii) $f_\varepsilon = hf$ *is structurally stable in strong sense.*

Proof. We may assume

$$\frac{\varepsilon}{3} < r_1 . \tag{4.1}$$

Let ℓ be a sufficiently large integer satisfying the following inequalities.

$$\frac{1}{2\ell\pi} + e^{-(\ell\pi)^2} < \frac{1}{(2\ell - 1)\pi} < \frac{\varepsilon}{4} . \tag{4.2}$$

Put $1/2\ell\pi + e^{-(\ell\pi)^2} = r_2$. Define a discontinuous function $\eta_0: (0, r_2) \to \boldsymbol{R}_+$ by

$$\eta_0(r) = \begin{cases} r - e^{-(k\pi)^2} & \text{if } \dfrac{1}{(k+1)\pi} < r \leqq \dfrac{1}{k\pi}, \\ r - e^{-4(2\ell\pi)^2} & \text{if } \dfrac{1}{2\ell\pi} < r \leqq r_2, \end{cases}$$

where $k = 2\ell,\ 2\ell + 1,\ 2\ell + 3, \cdots$. Let $\eta: \boldsymbol{R}_+ \to \boldsymbol{R}_+$ be a C^1-function satisfying

$$(4.3)\quad \begin{cases} 0 \leqq \eta(r) \leqq r\,, & \\ \eta(r) = r & \text{if } r > \dfrac{1}{(2\ell-1)\pi}\,, \\ \eta(r) < \eta_0(r) & \text{if } 0 < r \leqq r_2\,, \\ \eta(0) = 0\,, & \\ \eta'(r) > 0 & \text{for every } r \geqq 0\,, \\ \eta'(0) < 1\,. & \end{cases}$$

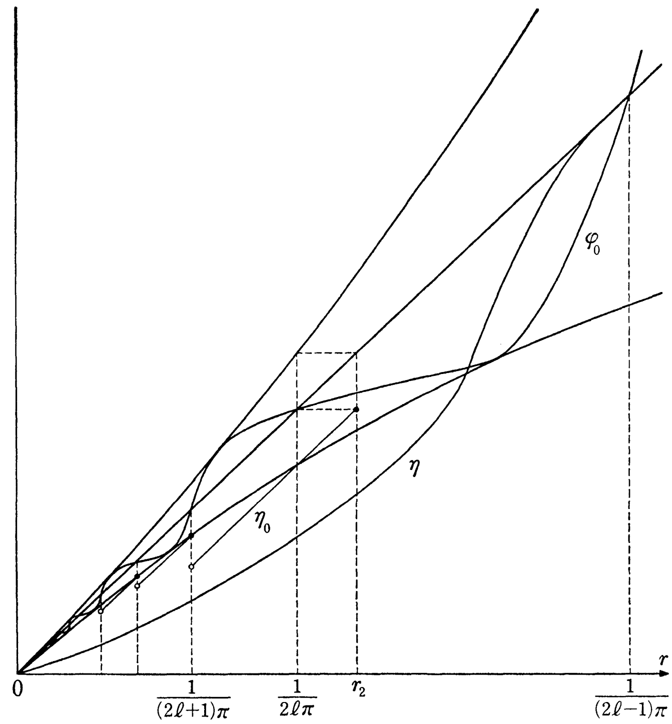

Fig. 5.

In fact, η exists. Especially we can find η such that $0 < \eta'(0) < 1$, since in a neighborhood of 0 the following properties hold.

$$(4.4)\quad \eta_0(r) > r - e^{-(1/r-\pi)^2}\,,$$

$$(4.5)\quad \lim_{r\to 0} \frac{1}{r}\,(r - e^{-(1/r-\pi)^2}) = 1\,.$$

Define $h: M \to M$ by

https://doi.org/10.1017/S0027763000020341 Published online by Cambridge University Press

(4.6)
$$h(x) = \begin{cases} \eta(\|x\|)\dfrac{x}{\|x\|} & \text{if } x \in B \\ x & \text{if } x \notin B\,. \end{cases}$$

Since $B = B_{r_0}$ and $r_1 < r_0$, the map h is well defined by (4.1), (4.2) and (4.3). h is a diffeomorphism. Define f_ε by

$$f_\varepsilon(x) = hf(x)\,.$$

By (4.3) $f_\varepsilon(x) = f(x)$ if $\|x\| \geqq 1/(2\ell - 1)\pi$.

Next, we show

(4.7)
$$\|f_\varepsilon(x)\| < \|x\| \qquad \text{if } \|x\| < \frac{1}{(2\ell - 1)\pi}\,.$$

Remember the definition of the vector field X, then we observe that $\|f(x)\| \leqq \|x\|$ when $1/2k\pi < \|x\| < 1/(2\ell - 1)\pi$. Since $\eta(\|x\|) \leqq \|x\|$, it follows that

(4.8)
$$\|f_\varepsilon(x)\| < \|x\| \qquad \text{if } \frac{1}{2k\pi} < \|x\| < \frac{1}{(2k-1)\pi}\,, \quad k \geqq \ell\,.$$

Next, let $1/(2k+1)\pi < \|x\| < 1/2k\pi$. Let $\overline{\Psi}_t(x)$ be the flow of X, so that $\overline{\Psi}_0(x) = x$. Since $V_x = X_x + Y_x$ and $\|f(x)\| = \|\Psi_1(x)\| = \|\overline{\Psi}_1(x)\|$, we have

(4.9)
$$\|f(x)\| = \|x\| + \int_0^1 \varphi(\|\overline{\Psi}_t(x)\|)dt\,,$$

where $\varphi(r) = \mathrm{e}^{-1/r^2} \sin 1/r$ as before. $1/(2k+1)\pi \leqq \|x\| \leqq 1/2k\pi$ implies $0 \leqq \sin(1/\|x\|) \leqq 1$.

Hence,

$$\varphi(\|x\|) \leqq \mathrm{e}^{-1/\|x\|^2} \leqq \mathrm{e}^{-(2k\pi)^2}\,.$$

Therefore, by (4.9),

$$\|f(x)\| \leqq \|x\| + \mathrm{e}^{-(2k\pi)^2}\,.$$

Using this and the definition of η_0 we have

$$\begin{aligned} \|f_\varepsilon(x)\| &= \|hf(x)\| = \eta(\|f(x)\|) \\ &\leqq \eta(\|x\| + \mathrm{e}^{-(2k\pi)^2}) < \eta_0(\|x\| + \mathrm{e}^{-(2k\pi)^2}) \\ &< (\|x\| + \mathrm{e}^{-(2k\pi)^2}) - \mathrm{e}^{-(2k\pi)^2} = \|x\|\,. \end{aligned}$$

Hence,

https://doi.org/10.1017/S0027763000020341 Published online by Cambridge University Press

$$(4.10) \qquad \|f_\varepsilon(x)\| < \|x\| \qquad \text{if } \frac{1}{(2k+1)\pi} \leqq \|x\| \leqq \frac{1}{2k\pi} .$$

By (4.8) and (4.10) we have (4.7).

Hence f_ε contracts to p in Int $B(2\ell - 1)$. We have $f_\varepsilon = f$ in $M - B_{1/(2\ell-1)\pi}$ by (4.3). By the definition of f, $f|(M - B_{1/(2\ell-1)\pi})$ is Morse-Smale and $\partial B_{1/(2\ell-1)\pi}$ is f-invariant. Therefore f_ε is Morse-Smale. Since a Morse-Smale diffeomorphism is structurally stable in strong sense by [3] this completes the proof of Lemma.

Since f_ε is structurally stable in strong sense there is a neighborhood N_0 of f_ε in $\mathrm{Diff}^1(M)$ such that every element in N_0 is topologically $\varepsilon/24$-conjugate to f_ε.

Since h is a C^1-diffeomorphism the map $h_*: \mathrm{Diff}^1(M) \to \mathrm{Diff}^1(M)$ defined by $h_*(g) = hg$ is continuous [1, p. 229, (B.8)]. Hence, for the neighborhood N_0 of $hf = f_\varepsilon$, there is a neighborhood N of f in $\mathrm{Diff}^1(M)$ such that

$$g \in N \Rightarrow hg = g_\varepsilon \in N_0 .$$

Hereafter, let g be included in N. Since $h =$ identity on $M - B_{\varepsilon/4}$ by (4.2), (4.3) and (4.6), we have

$$(4.11) \qquad \begin{array}{l} f_\varepsilon = f \ \text{ and } \ g_\varepsilon = g \ \text{ in } \ M - B_{\varepsilon/4} \\ f_\varepsilon \text{ and } g_\varepsilon \text{ are topologically } \varepsilon/24\text{-conjugate:} \end{array}$$

there is a homeomorphism $h_g: M \to M$ such that

$$(4.12) \qquad h_g g_\varepsilon = f_\varepsilon h_g \quad \text{and} \quad d(h_g(x), x) < \varepsilon/24, \qquad \text{for all } x.$$

We may assume that ε is small, so that there is an integer k satisfying $3/\pi\varepsilon < k < 24/7\pi\varepsilon$. Then we have

$$(4.13) \qquad \frac{\varepsilon}{4} + \frac{\varepsilon}{24} < \frac{1}{k\pi} < \frac{\varepsilon}{3} .$$

(4.1), (4.13) and the definition of f imply that $S_{1/k\pi}$ is f-invariant. Denote $S_f = S_{1/k\pi}$. Since S_f is contained in the complement of $B_{\varepsilon/4}$, (4.11) implies that S_f is also f_ε-invariant. Since f_ε and g_ε are topologically $\varepsilon/24$-conjugate, (4.11) and (4.13) imply that $h_g(S_f)$ is contained in $M - B_{\varepsilon/4}$ and is both g and g_ε-invariant. Denote $h_g(S_f) = S_g$, $B_{1/k\pi} = B_f$ and $h_g(B_f) = B_g$. Since $\partial B_f = S_f$ and $\partial B_g = S_g$ we have

https://doi.org/10.1017/S0027763000020341 Published online by Cambridge University Press

$$(4.14)\quad \begin{cases} f_\varepsilon = f & \text{in } M - B_f\,, \\ g_\varepsilon = g & \text{in } M - B_g\,, \\ f|(M - B_f) \text{ and } g|(M - B_g) & \text{are topologically } \varepsilon/24\text{-conjugate.} \end{cases}$$

Precisely, the last part of (4.14) means that there is the commutative diagram

$$\begin{array}{ccc} (M - B_f) & \xrightarrow{f} & (M - B_f) \\ \downarrow h_g & & \downarrow h_g \\ (M - B_g) & \xrightarrow{g} & (M - B_g) \end{array}$$

and $d(h_g(x), x) < \varepsilon/24$ for every $x \in (M - B_f)$. (4.14) implies

$$(4.15)\quad \begin{cases} B_f & \text{is } f\text{-invariant}\,, \\ B_g & \text{is } g\text{-invariant.} \end{cases}$$

For every g in N, we must show that f and g are orbit-ε-equivalent. First, let $O_f \subset M - B_f$. Then, O_f is an f_ε-orbit O_{f_ε}. By (4.14), $h_g(O_f) = O_{g_\varepsilon}$ is contained in $M - B_g$ and O_{g_ε} is a g-orbit O_g. Since $d(h_g, i_M) < \varepsilon/24$ then the conditions (a) and (b) of 1 in Definition (2.1) are satisfied in this case.

Next, let $O_f \subset B_f$. Take any orbit O_g in B_g (by using (4.15)). Then (a) and (b) of 1 in Definition (2.1) are satisfied. In fact, for any $x \in B_f$ and $y \in B_g$, by (4.13) we have

$$\begin{aligned} \|x - y\| &\leqq \|x\| + \|y\| \\ &\leqq \frac{1}{k\pi} + \left(\frac{1}{k\pi} + \frac{\varepsilon}{24}\right) \\ &< \frac{\varepsilon}{3} + \left(\frac{\varepsilon}{3} + \frac{\varepsilon}{24}\right) < \varepsilon\,. \end{aligned}$$

Hence, the condition 1 in Definition (2.1) is satisfied. Similarly we can show the condition 2. Therefore f is tolerance-stable in $\mathrm{Diff}^1(M)$.

§ 5. Proof of topological unstability in $\mathrm{Diff}^1(M)$

Suppose that f is topologically stable in $\mathrm{Diff}^1(M)$. Then, for any $\varepsilon_1 > 0$ there is a neighborhood N of f in $\mathrm{Diff}^1(M)$ such that for every g in N there is a continuous map $\tau\colon M \to M$ satisfying

(a) $d(\tau, i_M) < \dfrac{\varepsilon_1}{2}$,

(b) $\tau g = f\tau$.

For the fixed integer n in (3.1), let

$$\varepsilon_1 = \frac{1}{2n\pi} .$$

To introduce a contradiction, we take following g;

$$g = hf ,$$

where h is a diffeomorphism defined by (4.6). But we must take g such that $g \in N$. By (4.4), (4.5) and the definition (4.3) of η we can choose η, by taking ℓ sufficiently large, such that $|\eta(r) - r|$ and $|\eta'(r) - 1|$ are arbitrarily small. Hence we may assume that $g \in N$ and $1/(2\ell - 1)\pi < \varepsilon_1$.

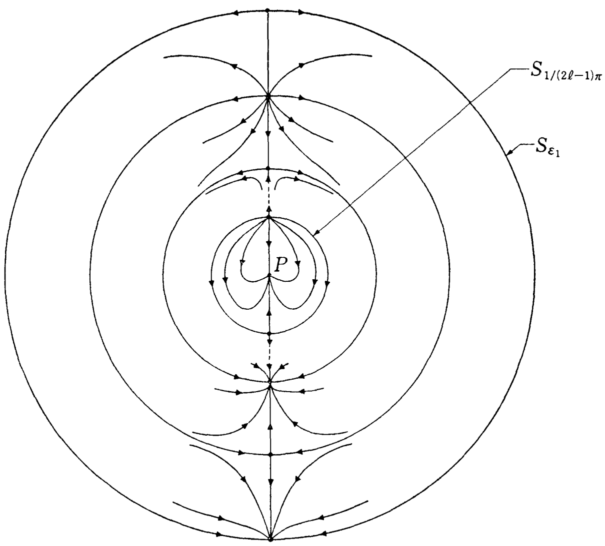

Fig. 6.

If y is a fixed point of f satisfying $\|y\| < \varepsilon_1/2$ then $\tau^{-1}(y)$ contains a fixed point of g. In fact, since $\tau g = f\tau$, $\tau^{-1}(y)$ is a g-invariant closed subset. By the condition (a) above, each x in $\tau^{-1}(y)$ satisfies

$$\begin{aligned} \|x\| &\leqq \|y\| + \|y - x\| \\ &= \|y\| + \|\tau x - x\| \\ &< \frac{\varepsilon_1}{2} + \frac{\varepsilon_1}{2} = \varepsilon_1 . \end{aligned}$$

Any invariant closed subset of g, included in B_{ε_1}, contains at last one fixed point (see Fig. 6). Hence, for each fixed point y of f in $B_{\varepsilon_1/2}$, there

https://doi.org/10.1017/S0027763000020341 Published online by Cambridge University Press

is a fixed point x of g such that $\tau(x) = y$ and $x \in B_{\varepsilon_1}$. There are infinitely many fixed points of f in $B_{\varepsilon_1/2}$, but there are at most finite fixed points of g in B_{ε_1}. This is a contradiction. Therefore f is topologically unstable in $\mathrm{Diff}^1(M)$.

References

[1] M. C. Irwin, Smooth dynamical systems, Academic press, 1980.
[2] A. Morimoto, The method of pseudo-orbit tracing and stability of dynamical systems, Seminar Note 39, Dept. of Math. Tokyo Univ., 1979, (in Japanese).
[3] J. Palis and S. Smale, Structural stability theorems, Proceedings of symposia in pure math. (Global Analysis), Amer. Math. Soc., **14** (1970), 223–231.
[4] C. Robinson, Structural stability of C^1 diffeomorphisms, J. Differential Equations, **22** (1976), 28–73.
[5] S. Smale, Morse inequalities for a dynamical system, Bull. Amer. Math. Soc., **66** (1960), 43–49.
[6] F. Takens, On Zeeman's tolerance stability conjecture, Manifolds-Amsterdam 1970, Lecture Notes in Math., Springer, **197** (1971), 209–219.
[7] ——, Tolerance stability, Dynamical Systems-Warwick 1974. Proceedings 1973/74 (Edited by A, Manning), Lecture Notes in Math., Springer, **468** (1975), 293–304.
[8] W. White, On the tolerance stability conjecture, Dynamical systems (Edited by M. Peixoto), Academic Press, 1973, 663–665.

Department of Mathematics
Faculty of General Education
Nagoya University
Chikusa-ku, Nagoya
464, Japan