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HITTING TIMES, OCCUPATION TIMES, TRIVARIATE LAWS AND THE FORWARD KOLMOGOROV EQUATION FOR A ONE-DIMENSIONAL DIFFUSION WITH MEMORY

MARTIN FORDE,* ** AND
ANDREY POGUDIN,* *** *King's College London*
HONGZHONG ZHANG,**** *Columbia University*

Abstract

We extend many of the classical results for standard one-dimensional diffusions to a diffusion process with memory of the form $\mathrm{d}X_t = \sigma(X_t, \underline{X}_t)\,\mathrm{d}W_t$, where $\underline{X}_t = m \wedge \inf_{0\le s\le t} X_s$. In particular, we compute the expected time for X to leave an interval, classify the boundary behavior at 0, and derive a new occupation time formula for X. We also show that $(X_t, \underline{X}_t)$ admits a joint density, which can be characterized in terms of two independent tied-down Brownian meanders (or, equivalently, two independent Bessel-3 bridges). Finally, we show that the joint density satisfies a generalized forward Kolmogorov equation in a weak sense, and we derive a new forward equation for down-and-out call options.

Keywords: One-dimensional diffusion; occupation time formula; stochastic functional differential equation; diffusion with memory

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1. Introduction

In [10] Forde constructed a weak solution to the stochastic functional differential equation $X_t = x + \int_0^t \sigma(X_s, M_s)\,\mathrm{d}W_s$, where $M_t = \sup_{0\le s\le t} X_s$. Using excursion theory, Forde then solved the following problem: for a natural class of joint density functions $\mu(y, b)$, specify $\sigma(\cdot,\cdot)$ so that X is a martingale, and the terminal level and supremum of X, when stopped at an independent exponential time ξ_λ, is distributed according to μ. The proof uses excursion theory for regular diffusions to compute an explicit expression for the Laplace transform of the joint density of the terminal level and the supremum of X at an independent exponential time, and the joint density satisfies a forward Kolmogorov equation. Integrating twice, Forde obtained a forward partial differential equation for the up-and-out put option payoff which was then used to back out σ from the pre-specified joint density. This was inspired by the earlier work of Carr [5] and Cox *et al.* [6], who showed how to construct a one-dimensional diffusion with a given marginal at an independent exponential time.

The main result, Theorem 3.6 of [4], shows that we can match the *joint* distribution at each fixed time of various functionals of an Itô process, including the maximum-to-date or the running average of one component of the Itô process. The mimicking process is also a weak solution to the stochastic functional differential equation (SFDE) and in the special case when

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* Postal address: Department of Mathematics, King's College London, Strand, London WC2R 2LS, UK.
** Email address: martin.forde@kcl.ac.uk
*** Email address: andrey.pogudin@kcl.ac.uk
**** Postal address: Department of Statistics, Columbia University, New York, NY 10027, USA.
Email address: hzhang@stat.columbia.edu

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we are mimicking the terminal level and the maximum, the mimicking process is of the form $X_t = x + \int_0^t \sigma(X_s, M_s, s)\,\mathrm{d}W_s$.

In this article we consider the case when the diffusion coefficient $\sigma(\cdot)$ depends only on X and its running minimum, and we assume that X is strictly positive, and $\sigma(x, m)$ is continuous with $0 < \sigma(x, m) < \infty$ for $x > 0$ and $m \geq 0$, $m \leq x$, and that $\sigma(0, 0) = 0$. The purpose of the article is to extend many of the standard well-known results for one-dimensional diffusions to the case when σ also depends on the running minimum (as opposed to solving one problem in particular), and we give financial motivation/applications where appropriate.

In Theorem 2.1 we prove weak existence and uniqueness in law for $\mathrm{d}X_t = \sigma(X_t, \underline{X}_t)\,\mathrm{d}W_t$ by extending the usual time-change argument for one-dimensional diffusions. In Proposition 3.1, we compute the expected length of time to hit either of two barriers for X, as a simple application of Itô's lemma and the optional sampling theorem. We then examine the nontrivial question of when the hitting time H_0 to 0 is finite or not (almost surely); specifically, in Theorem 4.1 we show that, for $\varepsilon \in (0, m)$,

$$\mathbb{P}(H_0 < \infty) = 0 \quad \text{if and only if} \quad \int_0^{\varepsilon} \int_0^{u} \tilde{m}(u, v)\,\mathrm{d}v\,\mathrm{d}u = \infty,$$

where $\tilde{m}(x, m) = 1/\sigma(x, m)^2$. For the case when $\tilde{m}$ is independent of m, this reduces to the well-known condition that $\mathbb{P}(H_0 < \infty) = 0$ if and only if $\int_0^{\varepsilon} v\tilde{m}(v)\,\mathrm{d}v = \infty$ (see, e.g. Theorem 51.2(i) of [21]). We then formulate an extension of the classical occupation time formula for the new X process (Theorem 5.1).

In Theorem 6.1, by adapting the argument in [18], using Girsanov's theorem, and conditioning on the terminal value *and* the minimum of X, we prove the existence of the joint density $p_t(x, m)$ for X and its minimum. We then further characterize this joint density in terms of two independent back-to-back Brownian meander bridges, which we can further represent in terms of two independent Bessel-3 bridges using standard results in, e.g. [1], [2], and [11]. Finally, in Section 8 we show that X is a weak solution to a forward Kolmogorov equation, and we also derive a new forward equation for down-and-out call options.

2. A one-dimensional diffusion with memory

In this section we construct a weak solution to the SFDE

$$X_t = x + \int_0^t \sigma(X_s, \underline{X}_s)\,\mathrm{d}W_s, \tag{2.1}$$

where $\underline{X}_t = m \wedge \inf_{0 \leq s \leq t} X_s$ and W is a standard Brownian motion, and we show that the solution X is unique in law. The m parameter allows us to include the possibility that X has accrued a previous historical minimum m which may be less than $X_0 = x$.

We make the following assumptions on σ throughout.

Assumption 2.1. (i) *σ is continuous, and strictly positive away from* $(0, 0)$.

(ii) $\sigma(0, 0) = 0$.

(iii) $\lim_{x \searrow 0} x/\sigma(x, x)^2 = 0$.

We let H_b denote the first hitting time to b, i.e.

$$H_b = \inf\{s \colon X_s = b\},$$

and define $\tilde{m}(u, v) = 1/\sigma(u, v)^2$.

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2.1. Weak existence and uniqueness in law

Theorem 2.1. *Equation (2.1) has a nonexploding weak solution for* $t < H_\delta$ *which is unique in law, where* $0 < \delta \leq m \leq x$.

Proof. Existence. Let (B_t, P_x) denote a standard Brownian motion defined on some $(\Omega, \mathcal{F}, (\mathcal{F}_t))$ with $B_0 = x > 0$ and $\underline{B}_t = \inf_{0 \leq s \leq t} B_s$, and assume that $\mathcal{F}_t$ satisfies the usual conditions (i.e. $\mathcal{F}_t$ is right continuous and $\mathcal{F}_0$ contains all $\mathcal{F}$ sets of measure 0). Let T_t denote the almost surely (a.s.), strictly increasing process

$$T_t = \int_0^t \tilde{m}(B_s, m \wedge \underline{B}_s)\, \mathrm{d}s \tag{2.2}$$

for $t < \tau_\delta$ and some $\delta > 0$, where

$$\tau_a = \inf\{s \colon B_s = a\}. \tag{2.3}$$

Let $A_t = \inf\{s \colon T_s = t\}$ denote the inverse of T_t, and set

$$X_t = B_{A_t}. \tag{2.4}$$

Then we have

$$\int_0^{A_t} \sigma^2(B_s, m \wedge \underline{B}_s)\, \mathrm{d}T_s = \int_0^{A_t} \mathrm{d}s = A_t.$$

If we make the change of variables $u = T_s$ so $\mathrm{d}u = \mathrm{d}T_s = \tilde{m}(B_s, m \wedge \underline{B}_s)\, \mathrm{d}s$ then we can rewrite the integral on the left-hand side as

$$A_t = \int_0^t \sigma^2(X_u, \underline{X}_u)\, \mathrm{d}u \quad \text{a.s.},$$

where we have used a pathwise application of the Lebesgue–Stieltjes change-of-variable formula. Thus, $\langle X \rangle_t = A_t$ a.s. Then, by Theorem 3.4.2 of [12], there exists a Brownian motion W on some extended probability space such that (2.1) is satisfied.

Uniqueness in law. We proceed along similar lines to Lemma V.28.7 of [21]. By Theorem IV.34.11 of [21], if X satisfies (2.1) then

$$B_t = X_{T_t}$$

is standard Brownian motion, where $T_t = \inf\{s \colon \langle X \rangle_s = t\}$, so

$$\int_0^{T_t} \sigma(X_s, \underline{X}_s)^2\, \mathrm{d}s = t.$$

Differentiating with respect to t we obtain

$$\sigma(X_{T_t}, \underline{X}_{T_t})^2 T_t' = 1 = \sigma(B_t, m \wedge \underline{B}_t)^2 T_t',$$

where $T_t' = \tilde{m}(B_t, m \wedge \underline{B}_t)$. Hence,

$$\langle X \rangle_t = \inf\biggl\{u \colon \int_0^u \tilde{m}(B_s, m \wedge \underline{B}_s)\, \mathrm{d}s = t\biggr\}.$$

Thus, X may be described explicitly in terms of the Brownian motion B, so the law of X is uniquely determined.

Finally, stopping X at H_δ means we are only running B until time τ_δ, and $\tau_\delta < \infty$ a.s., so $(X_{t \wedge H_\delta})$ cannot explode to ∞ a.s.

From here on we work on the canonical sample space $\Omega = C([0, \infty), \mathbb{R}^+)$ with the canonical process $X_t(\omega) = \omega(t)$ $(\omega \in \Omega, t \in [0, \infty))$ and its canonical filtration $\mathcal{F}_t = \sigma(X_s; s \leq t)$.

Let $\mathbb{P}_{x,m}$ denote the law on $(\Omega, \mathcal{B}(\Omega))$ induced by a weak solution to (2.1) (which is unique by Theorem 2.1).

Remark 2.1. If $\sigma \equiv \sigma(x, m, t)$ is time dependent, we can still obtain weak existence and uniqueness if the solution to the ordinary differential equation $\mathrm{d}T_t = \tilde{m}(B_t, m \wedge \underline{B}_t, T_t)\,\mathrm{d}t$ is uniquely determined a.s. This will be the case if $\tilde{m}$ is Lipschitz in the third argument (we thank Gerard Brunick for pointing this out).

We refer the reader to [14] and [15] for existence and uniqueness results for general SFDEs.

2.2. Application in financial modelling

We consider a time-homogeneous local volatility model with memory for a forward price process $(F_t)_{t\geq 0}$ which satisfies

$$\mathrm{d}F_t = F_t \mu\,\mathrm{d}t + F_t \sigma(F_t, \underline{F}_t)\,\mathrm{d}W_t$$

under the physical measure $\mathbb{P}$. This has the desirable feature of being a complete model, so, under the unique risk neutral measure $\mathbb{Q}$, F_t will satisfy $\mathrm{d}F_t = F_t\sigma(F_t, \underline{F}_t)\,\mathrm{d}W_t$, i.e. a diffusion-type process of the form in (2.1).

3. The expected time to leave an interval

The following proposition computes a closed-form expression for the expectation of the exit time from an interval, using Itô's lemma and a simple application of the optional sampling theorem. This proposition will be needed in the next section where we classify the boundary behaviour of X at 0. The proof is similar to that used for a regular diffusion in Section 5.5, part C of [12] and [13, p. 197].

Proposition 3.1. *The expected time for X to leave the interval (a, b) is given by*

$$\begin{aligned} h(x, m) &= \mathbb{E}_{x,m}(H_a \wedge H_b) \\ &= 2\int_m^x (u - x)\tilde{m}(u, m)\,\mathrm{d}u + \frac{2(x - m)}{b - m}\int_m^b (b - u)\tilde{m}(u, m)\,\mathrm{d}u + 2(b - x)C(m) \\ &< \infty \end{aligned} \tag{3.1}$$

for $0 < a \leq m \leq x \leq b < \infty$, where

$$C(m) = \int_a^b \int_a^{u\wedge m} \frac{b - u}{(b - v)^2}\tilde{m}(u, v)\,\mathrm{d}v\,\mathrm{d}u.$$

Proof. We can easily verify that $h(x, m)$ satisfies

$$\tilde{m}(x, m) = -\tfrac{1}{2}h_{xx}, \qquad h_m(m, m) = 0, \tag{3.2}$$

with endpoint condition $h(a, a) = h(b, m) = 0$ for all $a \leq m < b$.

Now let $\tau = H_a \wedge H_b$. Then by Itô's lemma we have

$$\begin{aligned} h(X_{t\wedge\tau}, \underline{X}_{t\wedge\tau}) - h(x, m) &= \int_0^{t\wedge\tau} h_x(X_s, \underline{X}_s)\,\mathrm{d}X_s + \frac{1}{2}\int_0^{t\wedge\tau} h_{xx}(X_s, \underline{X}_s)\sigma^2(X_s, \underline{X}_s)\,\mathrm{d}s \\ &\quad + \int_0^{t\wedge\tau} h_m(X_s, \underline{X}_s)\,\mathrm{d}\underline{X}_s \\ &= \int_0^{t\wedge\tau} h_x(X_s, \underline{X}_s)\,\mathrm{d}X_s + \frac{1}{2}\int_0^{t\wedge\tau} h_{xx}(X_s, \underline{X}_s)\sigma^2(X_s, \underline{X}_s)\,\mathrm{d}s \end{aligned}$$

using the second equation in (3.2) and the fact that $\mathrm{d}\underline{X}_t = 0$ if $X_t \neq \underline{X}_t$. The functions $h_x(u, v)$ and $\sigma(u, v)$ are bounded for $0 < a \leq v \leq u \leq b$, so taking expectations and applying the optional sampling theorem, and using the first equation in (3.2) yields

$$\mathbb{E}_{x,m}(h(X_{t\wedge\tau}, \underline{X}_{t\wedge\tau})) = h(x, m) - \mathbb{E}_{x,m}(t \wedge \tau). \tag{3.3}$$

Let $\tilde{m}(u, v) \leq K$ for $0 < a \leq v \leq u \leq b$ and some constant $K > 0$, so we have

$$\begin{aligned} h(x, m) &= \mathbb{E}_{x,m}(H_a \wedge H_b) \\ &= 2\int_m^x (u - x)\tilde{m}(u, m)\,\mathrm{d}u + \frac{2(x - m)}{b - m}\int_m^b (b - u)\tilde{m}(u, m)\,\mathrm{d}u + 2(b - x)C(m) \\ &\leq 2K\bigg[\int_m^x (x - u)\,\mathrm{d}u + \int_m^b (b - u)\,\mathrm{d}u + (b - x)\int_a^b \int_a^{u\wedge m} \frac{b - u}{(b - v)^2}\,\mathrm{d}v\,\mathrm{d}u\bigg] \\ &< \infty. \end{aligned}$$

Thus, $h(\cdot, \cdot)$ is continuous and bounded; so letting $t \to \infty$ in (3.3), applying the dominated convergence theorem to the left-hand side and the monotone convergence theorem to the right-hand side, and using the fact that $h(a, a) = h(b, m) = 0$, we obtain (3.1).

4. Absorption at zero

Theorem 4.1. *Let $\varepsilon \in (0, m)$. Then we have the following boundary behaviour for X:*

$$\mathbb{P}_{x,m}(H_0 < \infty) = 0 \quad \textit{if and only if} \quad \int_0^\varepsilon \int_0^u \tilde{m}(u, v)\,\mathrm{d}v\,\mathrm{d}u = \infty.$$

Remark 4.1. For the case when $\tilde{m}$ is independent of m, X is a regular one-dimensional diffusion, and Theorem 4.1 reduces to the well-known condition that

$$\mathbb{P}_x(H_0 < \infty) = 0 \quad \text{if and only if} \quad \int_{0+} v\tilde{m}(v)\,\mathrm{d}v = \infty$$

(see, e.g. Theorem 51.2(i) of [21]).

Proof of Theorem 4.1. Setting $a = 0$ in (3.1), we have

$$C(m) = \int_0^b \int_a^{u\wedge m} \frac{b - u}{(b - v)^2}\tilde{m}(u, v)\,\mathrm{d}v\,\mathrm{d}u$$

and $\mathbb{E}_{x,m}(H_0 \wedge H_b) < \infty$ if and only if $C(m) < \infty$, because $\tilde{m}(0, 0) = \infty$ and $\tilde{m} < \infty$ elsewhere, all the upper limits of integration are finite, and $1/(b - v)$ will not explode because the upper range of v is $m < b$. Noting that $(b - u)/(b - v)^2 \to 1$ as $u, v \searrow 0$ and replacing the upper limits of integration by $\varepsilon \in (0, m)$, we see that

$$\mathbb{E}_{x,m}(H_0 \wedge H_b) < \infty \quad \text{if and only if} \quad C^\varepsilon(m) = \int_0^\varepsilon \int_a^{u\wedge m} \tilde{m}(u, v)\,\mathrm{d}v\,\mathrm{d}u < \infty.$$

Thus, we have established that $\mathbb{E}_{x,m}(H_0 \wedge H_b) < \infty$ if and only if $\int_0^\varepsilon \int_0^u \tilde{m}(u, v)\,\mathrm{d}u\,\mathrm{d}v < \infty$. We now need to verify that $\mathbb{P}_{x,m}(H_0 < \infty) = 0$ if and only if $\int_0^\varepsilon \int_0^u \tilde{m}(u, v)\,\mathrm{d}v\,\mathrm{d}u = \infty$.

First assume that $\int_0^\varepsilon \int_0^u \tilde{m}(u, v)\,\mathrm{d}v\,\mathrm{d}u < \infty$. Then $\mathbb{E}_{x,m}(H_0 \wedge H_b) < \infty$, so $H_0 \wedge H_b < \infty$ a.s. and $\mathbb{P}_{x,m}(H_0 = H_b = \infty) = 0$. But, from the construction of X via the time-changed

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Brownian motion B in (2.4), we know that $\mathbb{P}_x(\tau_0 < \tau_b) > 0$, where τ_a is the first hitting time of B to a as defined in (2.3); hence, $\mathbb{P}_{x,m}(H_0 \le H_b) > 0$, $\mathbb{P}_{x,m}(H_0 < H_b) > 0$, and

$$\mathbb{P}_{x,m}(H_0 < \infty) \ge \mathbb{P}_{x,m}(H_0 < H_b \le \infty) > 0.$$

Conversely, assume that $\mathbb{P}_{x,m}(H_0 < \infty) > 0$. For this part, we proceed as in the proof of Lemma 6.2 of [13]. Then there exists a $t > 0$ for which

$$\mathbb{P}_{x,m}(H_0 < t) = \alpha > 0.$$

Every path starting at x and reaching 0 prior to time t visits every intervening state $\xi \in (0, x)$. Thus, we have

$$0 \le \alpha \le \mathbb{P}_{x,m}(H_0 - H_\xi < t) = \mathbb{P}_{\xi,\xi\wedge m}(H_0 < t) \le \mathbb{P}_{\xi,\xi\wedge m}(H_x \wedge H_0 < t)$$

for $0 < \xi \le x$. It follows that

$$\sup_{\xi\in(0,x]} \mathbb{P}_{\xi,,\xi\wedge m}(H_x \wedge H_0 \ge t) \le 1 - \alpha < 1,$$

and by induction we find that

$$\sup_{\xi\in(0,x)} \mathbb{P}_{\xi,,\xi\wedge m}(H_x \wedge H_0 \ge nt) \le (1 - \alpha)^n < 1.$$

We can rewrite this as

$$\mathbb{P}_{\xi,\xi\wedge m}(H_x \wedge H_0 \ge a) \le (1 - \alpha)^{[a/t]} \le (1 - \alpha)^{a/t-1}. \tag{4.1}$$

We now recall the general result in, e.g. [24, p. 79]. For any nonnegative random variable Y, we have

$$\mathbb{E}(Y) = \int_{[0,\infty)} \mathbb{P}(Y \ge y)\,\mathrm{d}y.$$

Thus, $\mathbb{E}(Y) < \infty$ if and only if $\int_{(R,\infty)} \mathbb{P}(Y \ge y)\,\mathrm{d}y < \infty$ for any $R > 0$. Thus, setting $Y = H_x \wedge H_0$ we have

$$\mathbb{E}_{\xi,\xi\wedge m}(H_x \wedge H_0) < \infty \quad \text{if and only if} \quad \int_{[R,\infty)} \mathbb{P}_{\xi,\xi\wedge m}(H_x \wedge H_0 \ge a)\,\mathrm{d}a < \infty.$$

But, from (4.1) we have

$$\int_{[R,\infty)} \mathbb{P}_{\xi,\xi\wedge m}(H_x \wedge H_0 \ge a)\,\mathrm{d}a \le \int_R^\infty (1 - \alpha)^{a/t-1}\,\mathrm{d}a = \frac{t(1 - \alpha)^{-1+R/t}}{\log(1 - \alpha)} < \infty.$$

Thus, $\mathbb{E}_{\xi,\xi\wedge m}(H_x \wedge H_0) < \infty$, and from the first part of the proof we know that $\mathbb{E}_{\xi,\xi\wedge m}(H_x \wedge H_0)$ is finite if and only if $\int_0^\varepsilon \int_0^u \tilde{m}(u, v)\,\mathrm{d}v\,\mathrm{d}u < \infty$ for all $\varepsilon \le m$.

Remark 4.2. For a stock price model of the form in (2.2), Theorem 4.1 allows us to compute whether or not the stock will default by hitting 0 or not in a finite time under the risk neutral measure $\mathbb{Q}$, which is relevant for the pricing of so-called *credit default swaps*, which pay 1 dollar at maturity T if the stock defaults before T.

5. The occupation time formula

From the continuity of σ, we see that, for any $R \in (1, \infty)$ and $0 < 1/R \le v \le u < R$, $\tilde{m}(u, v)$ is continuous in v, and, thus (by the Heine–Cantor theorem), is *uniformly* continuous in v on the compact set $0 < 1/R \le v \le u < R$ with fixed v. Using this property, we will construct an approximating sequence of processes (X^n) to the process X in (2.1) by 'freezing' the m-dependence on a small interval. We then derive a new occupation time formula for X by applying the standard occupation time formula for regular diffusions to the approximating process on each small interval, and then letting $n \to \infty$.

5.1. Almost-sure convergence for an approximating sequence of diffusion processes

Recall that $\tau_b = \inf\{s\colon B_s = b\}$. Set $0 < b \le m \le x$ and $\tilde{m}_n(u, v) = \tilde{m}(u, [vn]/n)$ for $n \ge 1$, so that $\tilde{m}_n(u, v)$ is piecewise constant in v, and define the process

$$X_t^n = B_{A_t^n}, \tag{5.1}$$

where A_t^n is the strictly increasing continuous inverse of

$$T_t^n = \int_0^{\tau_m \wedge t} \tilde{m}(B_s, m)\,\mathrm{d}s + \int_{\tau_m \wedge t}^t \tilde{m}_n(B_s, \underline{B}_s)\,\mathrm{d}s$$

for $0 \le t < \tau_0$. Note that $X_t = X_t^n$ for $0 \le t \le H_m$, because the m dependence in σ is 'frozen' until X sets a new minimum below m.

Proposition 5.1. *Let $H_b^n = \inf\{s\colon X_s^n = b\}$ and $H_b = \inf\{s\colon X_s = b\}$ as before for $b \in (0, m)$. Then $H_b^n \to H_b$ a.s. and $X_{t\wedge H_b} - X^n_{t\wedge H_b^n} \to 0$ a.s.*

Proof. Without loss of generality, we assume that $x = m$; otherwise, we just start from time H_m instead of time 0. From the time-change construction in the proof of Theorem 2.1, we know that $B_t = X_{T_t}$ and $B_{\tau_b} = X_{H_b}$, so we have

$$H_b = \int_0^{\tau_b} \tilde{m}(B_s, \underline{B}_s)\,\mathrm{d}s$$

and, similarly,

$$H_b^n = \int_0^{\tau_b} \tilde{m}_n(B_s, \underline{B}_s)\,\mathrm{d}s.$$

Using the uniform continuity of $\tilde{m}(u, v)$ on $\{(u, v)\colon 1/R \le v \le u \le R\}$ for any $R \in (1, \infty)$, and the fact that $\sup_{0\le s\le\tau_b} B_s(\omega) < \infty$ a.s., we know that, for any $\varepsilon > 0$, there exists a $N = N(\omega)$ such that, for all $n > N(\omega)$, we have

$$\begin{aligned} |H_b - H_b^n| &= \left| \int_0^{\tau_b} [\tilde{m}(B_s, \underline{B}_s) - \tilde{m}_n(B_s, \underline{B}_s)]\,\mathrm{d}s \right| \\ &= \left| \int_0^{\tau_b} \left[\tilde{m}(B_s, \underline{B}_s) - \tilde{m}\left(B_s, \frac{1}{n}[n\underline{B}_s] \right) \right] \mathrm{d}s \right| \\ &\le \varepsilon \tau_b \end{aligned}$$

and $\tau_b < \infty$, $\mathbb{P}_x$-a.s., so $H_b \to H_b^n$ a.s. Now, let

$$\tilde{m}_{\min}(\omega) = \inf_{0\le s\le\tau_b} \tilde{m}(B_s(\omega), \underline{B}_s(\omega)) < \infty \quad \text{a.s.}$$

By the definition of the inverse processes A_t and A_t^n, we have

$$t \wedge H_b = \int_0^{A_t \wedge \tau_b} \tilde{m}(B_s, \underline{B}_s)\, \mathrm{d}s \geq (A_t \wedge \tau_b)\tilde{m}_{\min}(\omega), \tag{5.2}$$

$$t \wedge H_b^n = \int_0^{A_t^n \wedge \tau_b} \tilde{m}_n(B_s, \underline{B}_s)\, \mathrm{d}s. \tag{5.3}$$

We first consider the case when $A_t \wedge \tau_b \leq A_t^n \wedge \tau_b$ (the other case is dealt with similarly). We know that $\sup_{0 \leq s \leq \tau_b \wedge A_t} B_s < \infty$ a.s. Subtracting (5.3) from (5.2), and again using the uniform continuity of $\tilde{m}$ in m, we see that

$$\begin{aligned} t \wedge H_b - t \wedge H_b^n &= \int_0^{A_t \wedge \tau_b} [\tilde{m}(B_s, \underline{B}_s) - \tilde{m}_n(B_s, \underline{B}_s)]\, \mathrm{d}s - \int_{A_t \wedge \tau_b}^{A_t^n \wedge \tau_b} \tilde{m}_n(B_s, \underline{B}_s)\, \mathrm{d}s \\ &\leq \varepsilon(A_t \wedge \tau_b) - \tilde{m}_{\min}(\omega)(A_t^n \wedge \tau_b - A_t \wedge \tau_b) \\ &\leq \frac{\varepsilon(t \wedge H_b^n)}{\tilde{m}_{\min}(\omega)} - \tilde{m}_{\min}(\omega)(A_t^n \wedge \tau_b - A_t \wedge \tau_b), \end{aligned}$$

where in the last line we have used the inequality in (5.2). Rearranging, we find that

$$0 \leq \tilde{m}_{\min}(A_t^n \wedge \tau_b - A_t \wedge \tau_b) \leq \frac{\varepsilon(t \wedge H_b^n)}{\tilde{m}_{\min}} - (t \wedge H_b - t \wedge H_b^n) \quad \text{a.s.}$$

But, we have already shown that $H_b^n \to H_b$ a.s, so the right-hand side can be made arbitrarily small, and, thus, $A_t^n \wedge \tau_b \to A_t \wedge \tau_b$ a.s. We proceed similarly for the case $A_t^n \wedge \tau_b \leq A_t \wedge \tau_b$. Then

$$X_{t \wedge H_b} - X^n_{t \wedge H_b^n} = B_{A_t \wedge \tau_b} - B_{A_t^n \wedge \tau_b}$$

and B is continuous, so

$$X_{t \wedge H_b} - X^n_{t \wedge H_b^n} \to 0 \quad \text{a.s.},$$

as required.

5.2. The occupation time formula

Let (l_t^x) denote the local time process for B in (2.4) at the level x.

Theorem 5.1. *Let $x = m$, $0 < \delta < x$, and $f \colon \mathbb{R}^2 \mapsto \mathbb{R}^+$ be a bounded, continuous function. Then we have the occupation time formula*

$$\int_0^{H_\delta \wedge t} f(X_s, \underline{X}_s)\, \mathrm{d}s = \sum_{\delta < m \leq x} \int_m^\infty f(x, m)\tilde{m}(x, m) l^{x,m}_{A_t \wedge \tau_\delta}\, \mathrm{d}x \quad \textit{a.s.}, \tag{5.4}$$

where $l_t^{x,m} = \int_0^t \mathbf{1}_{\{\underline{B}_s \in \{m\}\}}\, \mathrm{d}l_s^x = l^x_{\tau_{b-}} - l^x_{\tau_b} \geq 0$ is the local time that B spends at x when the minimum is exactly m, and the sum is taken over the (a.s. countable) m-values where B makes a nonzero upward excursion from a minimum at m (we know these m-values are a.s. countable from standard excursion theory for Brownian motion; see, e.g. [17, Chapter XII, Section 2]).

Proof. See Appendix A.

Remark 5.1. Theorem 5.1 is clearly more involved than the standard occupation time formula. However, it can be used to show that $\int_0^\epsilon \int_0^u \tilde{m}(u, v)\, \mathrm{d}v\, \mathrm{d}u < \infty$ implies that

$$\mathbb{P}_{x,m}(H_0 < \infty) = 1,$$

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which combined with Theorem 4.1 shows that $\mathbb{P}(H_0 < \infty)$ is either 1 or 0 depending on the finiteness of $\int_0^\epsilon \int_0^u \tilde{m}(u, v)\,\mathrm{d}v\,\mathrm{d}u$ (we defer the details to future work).

6. Transition densities

6.1. Existence of a joint transition density for $(X_t, \underline{X}_t)$

Theorem 6.1. *Define the function*

$$\tilde{\sigma}(y, \underline{y}) = \mathrm{e}^{-y}\sigma(\mathrm{e}^y, \mathrm{e}^{\underline{y}})$$

for all $y \geq \underline{y}$, and assume that

- *$\tilde{\sigma}(y, \underline{y})$ possesses bounded continuous partial derivatives of all orders up to and including 2;*
- *$\int_0^\varepsilon \int_0^u \tilde{m}(u, v)\,\mathrm{d}v\,\mathrm{d}u = \infty$ so $\mathbb{P}(H_0 < \infty) = 0$.*

Then, under $\mathbb{P}_{x,x}$, $(X_t, \underline{X}_t)$ defined in (2.1) admits a joint density $p_t(x', \underline{x}')$.

Remark 6.1. Note that, under $\mathbb{P}_{x,m}$ with $x > m$, there is a nonzero probability that $\underline{X}_t = m \wedge \inf_{0\leq s\leq t} X_s = m$, i.e. the law of $\underline{X}_t$ has an atom at m.

Proof of Theorem 6.1. Let $Y_t := \log X_t$ and $\underline{Y}_t := \log \underline{X}_t$, which are well defined because X cannot hit 0 in finite time a.s. We note that $Y_0 = \underline{Y}_0$. Using Itô's lemma, we have

$$\mathrm{d}Y_t = \tilde{\sigma}(Y_t, \underline{Y}_t)\,\mathrm{d}W_t - \tfrac{1}{2}\tilde{\sigma}^2(Y_t, \underline{Y}_t)\,\mathrm{d}t.$$

Let us define

$$\rho_t = \inf\{u \leq t : X_u = \underline{X}_t\}.$$

Because the log function is monotonically increasing, we have $\rho_t = \inf\{u \leq t : Y_u = \underline{Y}_t\}$. We now make a transformation of Y to a process with diffusion coefficient equal to 1. To this end, we first define

$$\eta(\underline{y}) = \int_{y_0}^{\underline{y}} \frac{\mathrm{d}u}{\tilde{\sigma}(u, u)}, \qquad \beta(y, \underline{y}) = \eta(\underline{y}) + \int_{\underline{y}}^{y} \frac{\mathrm{d}u}{\tilde{\sigma}(u, \underline{y})},$$

and consider the new processes $Z_t := \beta(Y_t, \underline{Y}_t)$ and $\underline{Z}_t := \inf_{s\leq t} Z_s$. Then $Z_0 = \beta(Y_0, \underline{Y}_0) = 0$. Note that, for all t,

$$Z_t = \beta(Y_t, \underline{Y}_t) = \eta(\underline{Y}_t) + \int_{\underline{Y}_t}^{Y_t} \frac{\mathrm{d}u}{\tilde{\sigma}(u, \underline{Y}_t)} \geq \eta(\underline{Y}_t),$$

and from this we see that

$$\underline{Z}_t = \inf_{s\leq t} Z_s \geq \eta(\underline{Y}_t). \tag{6.1}$$

It turns out that we have equality in (6.1), since at time $\rho_t \leq t$ we have $Y_{\rho_t} = \underline{Y}_t$. Using the monotonicity of $\eta(\cdot)$ and $\beta(\cdot, \underline{y})$, we have

$$\underline{Y}_t = \eta^{-1}(\underline{Z}_t), \tag{6.2}$$
$$Y_t = \beta^{-1}(Z_t, \eta^{-1}(\underline{Z}_t)), \tag{6.3}$$
$$\rho_t = \inf\{u \leq t : Z_u = \underline{Z}_t\},$$

where $\beta^{-1}(\cdot, \underline{y})$ is the inverse of function $\beta(\cdot, \underline{y})$.

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Since β is at least C^2, using Itô's lemma, we obtain

$$\mathrm{d}Z_t = \mathrm{d}W_t - \tfrac{1}{2}[\tilde{\sigma}(Y_t, \underline{Y}_t) + \tilde{\sigma}_y(Y_t, \underline{Y}_t)]\,\mathrm{d}t = \mathrm{d}W_t + b(Z_t, \underline{Z}_t)\,\mathrm{d}t,$$

where

$$b(z, \underline{z}) = -\tfrac{1}{2}[\tilde{\sigma}(\beta^{-1}(z, \eta^{-1}(\underline{z})), \eta^{-1}(\underline{z})) + \tilde{\sigma}_y(\beta^{-1}(z, \eta^{-1}(\underline{z})), \eta^{-1}(\underline{z}))].$$

In light of (6.2) and (6.3), it suffices to show that $(Z_t, \underline{Z}_t)$ has a density function.

We now mimic the proof of the main theorem in [18, p. 2], and consider a new measure $\tilde{\mathbb{P}}$ defined by

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\tilde{\mathbb{P}}}\bigg|_{\mathcal{F}_t} = \mathbb{E}p\bigg\{\int_0^t b(Z_s, \underline{Z}_s)\,\mathrm{d}Z_s - \frac{1}{2}\int_0^t b^2(Z_s, \underline{Z}_s)\,\mathrm{d}s\bigg\}.$$

By Girsanov's theorem, the process (Z_t) is a standard Brownian motion under the measure $\tilde{\mathbb{P}}$. Now define the C^2 function

$$h(z, \underline{z}) = \int_{\underline{z}}^{z} b(u, \underline{z})\,\mathrm{d}u + \int_0^{\underline{z}} b(u, u)\,\mathrm{d}u. \tag{6.4}$$

Using Itô's lemma, we have

$$\mathrm{d}h(Z_t, \underline{Z}_t) = b(Z_t, \underline{Z}_t)\,\mathrm{d}Z_t + \tfrac{1}{2}b_z(Z_t, \underline{Z}_t)\,\mathrm{d}t,$$

from which we obtain (note that $h(Z_0, \underline{Z}_0) = h(0, 0) = 0$)

$$h(Z_t, \underline{Z}_t) - \frac{1}{2}\int_0^t b_z(Z_s, \underline{Z}_s)\,\mathrm{d}s = \int_0^t b(Z_s, \underline{Z}_s)\,\mathrm{d}Z_s.$$

Now, for any bounded *bivariate* continuous function f, we have

$$\mathbb{E}(f(Z_t, \underline{Z}_t)) = \tilde{\mathbb{E}}\bigg(f(Z_t, \underline{Z}_t)\exp\bigg[h(Z_t, \underline{Z}_t) - \frac{1}{2}\int_0^t g(Z_s, \underline{Z}_s)\,\mathrm{d}s\bigg]\bigg),$$

where $g = b^2 + b_z$. Conditioning on $(Z_t, \underline{Z}_t) = (z, \underline{z})$ for $z > \underline{z}$, $\underline{z} < 0$, we obtain

$$\begin{aligned}\mathbb{E}(f(Z_t, \underline{Z}_t)) = \int_{-\infty}^0 \int_{\underline{z}}^{\infty} & f(z, \underline{z})\phi_t(z, \underline{z})\mathrm{e}^{h(z,\underline{z})} \\ & \times \tilde{\mathbb{E}}\bigg(\exp\bigg[-\frac{1}{2}\int_0^t g(Z_s, \underline{Z}_s)\,\mathrm{d}s\bigg]\,\bigg|\, Z_t = z,\ \underline{Z}_t = \underline{z}\bigg)\,\mathrm{d}z\,\mathrm{d}\underline{z},\end{aligned}$$

where $\phi_t(z, \underline{z})$ is the joint density of the standard Brownian motion (Z_t) and its minimum $\underline{Z}_t$. Thus, the pair $(Z_t, \underline{Z}_t)$ has joint density

$$p_t^{Z,\underline{Z}}(z, \underline{z}) = \phi_t(z, \underline{z})\mathrm{e}^{h(z,\underline{z})}\tilde{\mathbb{E}}\bigg(\exp\bigg[-\frac{1}{2}\int_0^t g(Z_s, \underline{Z}_s)\,\mathrm{d}s\bigg]\,\bigg|\, Z_t = z,\ \underline{Z}_t = \underline{z}\bigg). \tag{6.5}$$

It follows that the pair $(Y_t, \underline{Y}_t) = (\log X_t, \log \underline{X}_t)$ has joint density

$$p_t^{Y,\underline{Y}}(y, \underline{y}) = p_t^{Z,\underline{Z}}(\beta(y, \underline{y}), \eta(\underline{y}))\frac{\partial\beta}{\partial y}\frac{\partial\eta}{\partial\underline{y}} = \frac{p_t^{Z,\underline{Z}}(\beta(y, \underline{y}), \eta(\underline{y}))}{\tilde{\sigma}(y, \underline{y})\tilde{\sigma}(\underline{y}, \underline{y})}. \tag{6.6}$$

Remark 6.2. For a stock price model of the form in (2.2), the existence of a semiclosed-form density for $(X_t, \underline{X}_t)$ as proved above allows us to price general barrier option contracts with payoffs of the form $\varphi(X_t, \underline{X}_t)$ for a measurable function φ.

6.2. Characterizing the joint density in terms of Bessel-3 bridges

From (6.5) and (6.6), it is seen that the regularity of the joint density of $p_t^{Y,\underline{Y}}(y, \underline{y})$ depends on that of h in (6.4) and the following function ψ_t:

$$\psi_t(z, \underline{z}) = \tilde{\mathbb{E}}\biggl(\exp\biggl[-\frac{1}{2}\int_0^t g(Z_s, \underline{Z}_s)\,\mathrm{d}s\biggr] \biggm| Z_t = z,\ \underline{Z}_t = \underline{z}\biggr).$$

The function ψ_t depends on the law of a standard Brownian motion $(Z_s)_{0\le s\le t}$ given Z_t, and $\underline{Z}_t$. To this end, let us condition on $(Z_t, \underline{Z}_t, \rho_t) = (z, \underline{z}, u)$; $(Z_t, \underline{Z}_t, \rho_t)$ has a smooth density given by

$$\chi_t(z, \underline{z}, u) = 2f(\underline{z}, u)f(\underline{z} - z, t - u) = \frac{-\underline{z}(z - \underline{z})}{\pi u^{3/2}(t - u)^{3/2}} \exp\biggl[-\frac{\underline{z}^2}{2u} - \frac{(z - \underline{z})^2}{2(t - u)}\biggr],$$

where $f(y, t) = |y|\mathrm{e}^{-y^2/2t}/\sqrt{2\pi t^3}$ is the hitting time density from 0 to y for a standard Brownian motion (see, e.g. [11]). Moreover, given $(Z_t, \underline{Z}_t, \rho_t) = (z, \underline{z}, u)$, the path fragments

$$(Z_{u-s} - \underline{z})_{0\le s\le u} \quad \text{and} \quad (Z_{u+s} - \underline{z})_{0\le s\le t-u}$$

are two independent Brownian meanders of lengths u and $t - u$, starting at 0 and conditioned to end at $-\underline{z} > 0$ and $z - \underline{z} > 0$, respectively (see, e.g. [2]). A Brownian meander of length s is defined as the rescaled portion of a Brownian path following the last passage time at 0, $G_1 = \sup\{s \le 1\colon B_s = 0\}$:

$$B_u^{\mathrm{me}} = \frac{\sqrt{s}}{\sqrt{1 - G_1}} |B_{G_1 + u(1-G_1)/s}| \qquad 0 \le u \le s$$

(see [3, p. 63]). It is known that the law of a Brownian meander of length s is identical to that of a standard Brownian motion starting at 0 and conditioned to be positive for $t \in [0, s]$ (see, e.g. [8]). Moreover, the tied-down Brownian meander, i.e. the Brownian meander conditioned so that $B_1^{\mathrm{me}} = x > 0$, has the same law as a three-dimensional Bessel bridge R^{br} with $R_0^{\mathrm{br}} = 0$ and $R_1^{\mathrm{br}} = x$ (see, e.g. [1] and [11]). Hence, the path fragments $(Z_{u-s} - \underline{z})_{0\le s\le u}$ and $(Z_{u+s} - \underline{z})_{0\le s\le t-u}$ can be identified with two independent Bessel-3 bridges, starting at 0, and ending at $-\underline{z} > 0$ and $z - \underline{z} > 0$, respectively (see [2] and [23]). Thus, as in [16], we have

$$\begin{aligned}
\kappa_t(z, \underline{z}, u) &= \tilde{\mathbb{E}}\biggl(\exp\biggl[-\frac{1}{2}\int_0^u g(Z_s, \underline{Z}_s)\,\mathrm{d}s\biggr] \biggm| Z_t = z,\ \underline{Z}_t = \underline{z},\ \rho_t = u\biggr)\\
&\quad \times \tilde{\mathbb{E}}\biggl(\exp\biggl[-\frac{1}{2}\int_u^t g(Z_s, \underline{z})\,\mathrm{d}s\biggr] \biggm| Z_t = z,\ \underline{Z}_t = \underline{z},\ \rho_t = u\biggr)\\
&= \tilde{\mathbb{E}}\biggl(\exp\biggl[-\frac{1}{2}\int_0^u g(Z_s, \underline{Z}_s)\,\mathrm{d}s\biggr] \biggm| Z_t = z,\ \underline{Z}_t = \underline{z},\ \rho_t = u\biggr)\\
&\quad \times \tilde{\mathbb{E}}\biggl(\exp\biggl[-\frac{1}{2}\int_0^{t-u} g(Z_{t-s}, \underline{z})\,\mathrm{d}s\biggr] \biggm| Z_t = z,\ \underline{Z}_t = \underline{z},\ \rho_t = u\biggr),
\end{aligned}$$

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and we can rewrite the last expectation in terms of the two aforementioned independent Bessel-3 bridges if we wish. It follows that

$$\begin{aligned}\psi_t(z,\underline{z}) &= \tilde{\mathbb{E}}\bigg(\exp\bigg[-\frac{1}{2}\int_0^t g(Z_s,\underline{Z}_s)\,\mathrm{d}s\bigg]\,\bigg|\, Z_t=z,\ \underline{Z}_t=\underline{z}\bigg)\\ &= \int_0^t \kappa_t(z,\underline{z},u)\tilde{\mathbb{P}}(\rho_t\in\mathrm{d}u \mid Z_t=z,\ \underline{Z}_t=\underline{z})\,\mathrm{d}u\\ &= \int_0^t \kappa_t(z,\underline{z},u)\frac{\chi_t(z,\underline{z},u)}{\phi_t(z,\underline{z})}\,\mathrm{d}u.\end{aligned}$$

7. A generalized forward Kolmogorov equation

In this section we assume that $m=x=x_0$, so $X_0=\underline{X}_0=x_0>0$ and we use $\mathbb{E}$ as shorthand for $\mathbb{E}_{x_0,x_0}$. We further assume that $\int_0^\varepsilon\int_0^u \tilde{m}(u,v)\,\mathrm{d}v\,\mathrm{d}u=\infty$, so $\mathbb{P}_{x,x}(H_0<\infty)=0$, i.e. X cannot hit 0 a.s., and, for simplicity, we assume that σ is bounded (we can easily relax this assumption by working in log space as in the previous section, but in the interests of clarity and succinctness, we do not do this here). Let $\mathcal{O}=\{(x,y)\in\mathbb{R}^+\times\mathbb{R}^+ : x\geq y\}$ denote the support of $(X_t,\underline{X}_t)$.

Theorem 7.1. *$(X_t,\underline{X}_t)$ satisfies the following forward equation*

$$\frac{\partial}{\partial t}\mathbb{E}(f(X_t,\underline{X}_t,t))=\mathbb{E}\bigg(f_t(X_t,\underline{X}_t,t)+\frac{1}{2}f_{xx}(X_t,\underline{X}_t,t)\sigma(X_t,\underline{X}_t)^2\bigg) \tag{7.1}$$

for all test functions $f\in C_b^{2,1,1}(\mathcal{O}\times\mathbb{R}^+)$ satisfying $f_y(y,y,t)=0$.

Proof. See Appendix B.

Remark 7.1. If $f\in C_c^\infty(\mathcal{O}\times\mathbb{R}^+)$ (C_c^∞ means smooth with compact support), re-writing (7.1) in terms of integrals and integrating from $t=0$ to ∞, and using the fact that $f(t,X_t,\underline{X}_t)=0$ a.s. for sufficiently large t, we see that $p(t,\,\mathrm{d}x,\,\mathrm{d}y)=\mathbb{P}(X_t\in\mathrm{d}x,\,\underline{X}_t\in\mathrm{d}y)$ satisfies

$$\int_{t=0}^\infty\int_{\mathcal{O}}\bigg(f_t+\frac{1}{2}\sigma(x,y)^2f_{xx}\bigg)p(t,\,\mathrm{d}x,\,\mathrm{d}y)\,\mathrm{d}t=0. \tag{7.2}$$

Remark 7.2. If $p(t,\,\mathrm{d}x,\,\mathrm{d}y)$ admits a density so that $p(t,\,\mathrm{d}x,\,\mathrm{d}y)=p(t,x,y)\,\mathrm{d}x\,\mathrm{d}y$, and p and σ are twice continuously differentiable in x and p is once differentiable in t, then integrating (7.2) by parts we have

$$\int_{t=0}^\infty\int_{\mathcal{O}} f(x,y,t)\bigg[-\partial_t p+\partial_{xx}^2\bigg(\frac{1}{2}\sigma(x,y)^2p\bigg)\bigg]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t=0,$$

and, thus (by the arbitraryness of f), $p(t,x,y)$ is a classical solution to the family of forward Kolmogorov equations

$$\partial_t p=\partial_{xx}^2\big(\tfrac{1}{2}\sigma(x,y)^2p\big),\qquad x\neq y,$$

for all $y\leq x$ (see [21, p. 252], Theorem 3.2.6 of [22], and [9] for similar results and weak formulations for a standard diffusion process).

7.1. A forward equation for down-and-out call options

Proposition 7.1. *Assume that* $k > 0$ *and* $0 < b < x_0$. *Then*

$$\mathbb{E}((X_t - k)^+ \mathbf{1}_{\{\underline{X}_t > b\}}) - (X_0 - k)^+ = \tfrac{1}{2}\mathbb{E}(L^k_{t\wedge H_b}) - (b-k)^+\mathbb{P}(\underline{X}_t \le b), \tag{7.3}$$

where L^a_t *is the semimartingale local time of* X *at* a *as defined in, e.g. Theorem 3.7.1 of [12] and* $H_b = \inf\{s\colon X_s = y\}$, *subject to the following boundary condition at* $x = y$:

$$\mathbb{E}((X_t - b)^+ \mathbf{1}_{\{\underline{X}_t > b\}}) = \mathbb{E}((X_t - b)\,\mathbf{1}_{\{\underline{X}_t > b\}}) = x_0 - b. \tag{7.4}$$

Remark 7.3. Equation (7.3) is a forward equation for a down-and-out call option on X_t with strike x, which knocks out if X hits y before time t. Specifically (assuming zero interest rates and dividends), the left-hand side is the fair price of the down-and-out call, and the $\mathbb{P}(\underline{X}_t \le y)$ term on the right-hand side is the price of a one-touch option on $\underline{X}_t$ which pays 1 if X hits y before t.

Remark 7.4. Equation (7.4) is the same condition that appears in [20], and if $\underline{X}_t$ has no atom at y, we can differentiate (7.4) with respect to y to obtain the condition in Theorem 3.1 of [19].

Remark 7.5. The financial interpretation of (7.4) is the well-known result that (for zero dividends and interest rates) we can *semistatically hedge* a down-and-out call option with barrier b equal to the strike k, by buying one unit of stock and holding $-b$ dollars, and unwinding the position if/when the barrier is struck (see, e.g. Appendix A of [7]).

Proof of Proposition 7.1. From the generalized Itô formula given in, e.g. Theorem 3.7.1 of [12], we obtain

$$\mathrm{d}(X_t - k)^+ = \mathbf{1}_{\{X_t > k\}}\,\mathrm{d}X_t + \tfrac{1}{2}\,\mathrm{d}L^k_t.$$

Integrating from time 0 to $t \wedge H_b$ we obtain

$$\begin{aligned}(X_{t\wedge H_b} - k)^+ - (X_0 - k)^+ &= (X_t - k)^+ \mathbf{1}_{\{H_b > t\}} + (b-k)^+ \mathbf{1}_{\{H_b \le t\}} - (X_0 - k)^+\\ &= \int_0^{t\wedge H_b} \mathbf{1}_{\{X_s > x\}}\,\mathrm{d}X_s + \frac{1}{2}L^k_{t\wedge H_b}.\end{aligned}$$

Taking expectations and simplifying, we obtain (7.3).

To obtain the boundary condition in (7.4), we use the optional sampling theorem for the bounded stopping time $t \wedge H_b$ to obtain

$$\begin{aligned}\mathbb{E}(X_{t\wedge H_b}) &= x_0\\ &= \mathbb{E}(X_t\,\mathbf{1}_{\{\underline{X}_t > b\}}) + \mathbb{E}(X_{H_b}\,\mathbf{1}_{\{H_b \le t\}})\\ &= \mathbb{E}(X_t\,\mathbf{1}_{\{\underline{X}_t > b\}}) + b\mathbb{P}(\underline{X}_t \le b)\\ &= \mathbb{E}((X_t - b)\,\mathbf{1}_{\{\underline{X}_t > b\}}) + b\mathbb{E}(\mathbf{1}_{\{\underline{X}_t > b\}}) + b\mathbb{E}(\mathbf{1}_{\{\underline{X}_t \le b\}})\\ &= \mathbb{E}((X_t - b)\,\mathbf{1}_{\{\underline{X}_t > b\}}) + b\\ &= \mathbb{E}((X_t - b)^+\,\mathbf{1}_{\{\underline{X}_t > b\}}) + b,\end{aligned}$$

where the last equality follows because $X_t > b$ on $\{\underline{X}_t > b\}$, i.e. if X does not hit b before time t.

Appendix A. Proof of Theorem 5.1

The process (X^n_t) defined in (5.1) is just a regular one-dimensional diffusion process for $t \in [H^n_{(k+1)/n}, H^n_{k/n})$ for each $k = 0, \ldots, [x_0 n] - 1$. Using the standard occupation time formula for $t \in [H^n_{(k+1)/n}, H^n_{k/n})$ for each k (see Theorem 49.1 of [21]), we have

$$\int_{H^n_{(k+1)/n}\wedge t}^{H^n_{k/n}\wedge t} f_n(X^n_s, \underline{X}^n_s)\,\mathrm{d}s = \int_\delta^\infty f\Big(x, \frac{k}{n}\Big)\tilde{m}\Big(x, \frac{k}{n}\Big) l^{x,(k/n,(k+1)/n]}_{A^n_t\wedge\tau_\delta}\,\mathrm{d}x$$
$$= \int_\delta^\infty \sum_{k/n<m\le(k+1)/n} f_n(x,m)\tilde{m}_n(x,m) l^{x,m}_{A^n_t\wedge\tau_\delta}\,\mathrm{d}x,$$

where $f_n(x,m) = f(x,[nm]/n)$, $l^{x,(a,b]}_t = \int_0^t \mathbf{1}_{\{\underline{B}_s\in(a,b]\}}\,\mathrm{d}l^x_s$ is the local time that B has accrued at x at time t while $\underline{B} \in (a,b]$, and we are summing over (a.s. countable) m-values in $(k/n, (k+1)/n]$ for which there is a nonzero upward excursion from a minimum at m.

Summing over k until time $t \wedge H^n_\delta$ and taking the finite sum inside the integral on the right-hand side, we obtain

$$\begin{aligned}\int_0^{t\wedge H^n_\delta} f(X^n_s, \underline{X}^n_s)\,\mathrm{d}s &= \int_0^t f(X^n_s, \underline{X}^n_s)\,\mathbf{1}_{\{s<H^n_\delta\}}\,\mathrm{d}s\\ &= \sum_{k=0}^{[x_0 n]-1}\int_\delta^\infty \sum_{k/n<m\le(k+1)/n} f_n(x,m)\tilde{m}_n(x,m) l^{x,m}_{A^n_t\wedge\tau_\delta}\,\mathrm{d}x\\ &= \int_\delta^\infty \Big[\sum_{\delta<m\le x} f_n(x,m)\tilde{m}_n(x,m) l^{x,m}_{A^n_t\wedge\tau_\delta}\Big]\,\mathrm{d}x\\ &= \int_\delta^{\sup_{0\le s\le\tau_\delta} B_s}\Big[\sum_{\delta<m\le x} f_n(x,m)\tilde{m}_n(x,m) l^{x,m}_{A^n_t\wedge\tau_\delta}\Big]\,\mathrm{d}x. \end{aligned}\tag{A.1}$$

For the left-hand integral, from Proposition 5.1, we know that $H^n_\delta \to H_\delta$ a.s. and $X^n_{t\wedge H^n_\delta} \to X_{t\wedge H_\delta}$ a.s., so $f(X^n_s, \underline{X}^n_s)\,\mathbf{1}_{\{s<H^n_\delta\}} \to f(X_s, \underline{X}_s)\,\mathbf{1}_{\{s<H_\delta\}}$ Lebesgue almost everywhere on $[0,t]$, a.s. Thus, by the dominated convergence theorem, we have

$$\int_0^t \mathbf{1}_{\{s\le H^n_\delta\}f(X^n_s,\underline{X}^n_s)\,\mathrm{d}s} \to \int_0^t \mathbf{1}_{\{s\le H_\delta\}}\, f(X_s, \underline{X}_s)\,\mathrm{d}s = \int_0^{t\wedge H_\delta} f(X_s,\underline{X}_s)\,\mathrm{d}s \quad \text{a.s.}$$

For the integrand on the right-hand side, we have the upper bound

$$\sum_{\delta<m\le x} f_n(x,m)\tilde{m}_n(x,m) l^{x,m}_{A^n_t\wedge\tau_\delta} \le f_{\max}\tilde{m}_{\max}(\delta,\omega) l^x_{A^n_t\wedge\tau_\delta} < \infty \quad \text{a.s.,}$$

where $\tilde{m}_{\max}(\delta,\omega) = \sup_{0\le s\le\tau_\delta}\tilde{m}(B_s, \underline{B}_s) < \infty$ a.s. Thus, letting $n\to\infty$ on both sides of (A.1), and applying the dominated convergence theorem to the right-hand side as well, and then applying Fubini's theorem, we obtain (5.4).

Appendix B. Proof of Theorem 7.1

Let $\sigma_t = \sigma(X_t, \underline{X}_t)$; X_t and $\underline{X}_t$ are continuous semimartingales, so we can apply Itô's formula to the test function $f \in C^{2,1,1}_b(\mathcal{O}\times\mathbb{R}^+)$:

$$\begin{aligned}\mathrm{d}f(X_t,\underline{X}_t,t) &= f_x(X_t,\underline{X}_t,t)\,\mathrm{d}X_t + \tfrac{1}{2} f_{xx}(X_t,\underline{X}_t,t)\sigma_t^2\,\mathrm{d}t + f_y(\underline{X}_t,\underline{X}_t,t)\,\mathrm{d}\underline{X}_t,\\ &= f_x(X_t,\underline{X}_t,t)\,\mathrm{d}X_t + \tfrac{1}{2} f_{xx}(X_t,\underline{X}_t,t)\sigma_t^2\,\mathrm{d}t.\end{aligned}$$

Here we have used the fact that $X_t = \underline{X}_t$ on the growth set of $\underline{X}_t$ in the final term (by growth set we mean the support of the random measure induced by the process Y on $[0, T]$, i.e. the complement of the largest open set of zero measure). (Recall that $\psi_y(y, y, t) = 0$.) Integrating we obtain

$$f(X_t, \underline{X}_t, t) - f(x_0, x_0, 0) = \int_0^t f_x(X_s, \underline{X}_s, s)\,\mathrm{d}X_s + \int_0^t \frac{1}{2} f_{xx}(X_s, \underline{X}_s, s)\sigma_s^2\,\mathrm{d}s.$$

Taking expectations, and applying Fubini's theorem yields

$$\mathbb{E}(f(X_t, \underline{X}_t, t)) - f(x_0, x_0, 0) = \int_0^t \frac{1}{2}\mathbb{E}(f_{xx}(X_s, \underline{X}_s, s)\sigma_s^2)\,\mathrm{d}s. \qquad \text{(B.1)}$$

Since X_t and $\underline{X}_t$ are continuous in t a.s. and $\sigma(\cdot, \cdot)$ is continuous, $\sigma_t = \sigma(X_t, \underline{X}_t, t)$ is also continuous in t a.s. Moreover, $f \in C_b^{2,1,1}$, so $f_{xx}(\cdot, \cdot)$ is bounded and continuous, and $f_{xx}(X_u, \underline{X}_u, u)\sigma_u^2 \to f_{xx}(X_s, \underline{X}_s, s)\sigma_s^2$ a.s. as $u \to s$. As σ is also bounded, from the dominated convergence theorem we have

$$\lim_{u\to s} \mathbb{E}(f_{xx}(X_u, \underline{X}_u, u)\sigma_u^2) = \mathbb{E}(f_{xx}(X_s, \underline{X}_s, s)\sigma_s^2),$$

so the integrand $\mathbb{E}(f_{xx}(X_s, \underline{X}_s, s)\sigma_s^2)$ in (B.1) is continuous in s for all s. Thus, using the fundamental theorem of calculus, we can differentiate (B.1) everywhere with respect to t to get

$$\frac{\partial}{\partial t}\mathbb{E}(f(X_t, \underline{X}_t, t)) = \mathbb{E}\bigg(f_t(X_t, \underline{X}_t, t) + \frac{1}{2} f_{xx}(X_t, \underline{X}_t, t)\sigma(X_t, \underline{X}_t)^2\bigg).$$

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