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Rigidity of Lyapunov exponents for derived from Anosov diffeomorphisms

J. SANTANA COSTA† and A. TAHZIBI‡

† *DEMAT-UFMA, São Luis, MA, Brazil*
(e-mail: jsc.costa@ufma.br)
‡ *Departamento de Matemática, ICMC-USP, São Carlos, SP, Brazil*
(e-mail: tahzibi@icmc.usp.br)

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Abstract. For a class of volume-preserving partially hyperbolic diffeomorphisms (or non-uniformly Anosov) $f: \mathbb{T}^d \to \mathbb{T}^d$ homotopic to linear Anosov automorphism, we show that the sum of the positive (negative) Lyapunov exponents of f is bounded above (respectively below) by the sum of the positive (respectively negative) Lyapunov exponents of its linearization. We show this for some classes of derived from Anosov (DA) and non-uniformly hyperbolic systems with dominated splitting, in particular for examples described by Bonatti and Viana [SRB measures for partially hyperbolic systems whose central direction is mostly contracting. *Israel J. Math.* **115**(1) (2000), 157–193]. The results in this paper address a flexibility program by Bochi, Katok and Rodriguez Hertz [Flexibility of Lyapunov exponents. *Ergod. Th. & Dynam. Sys.* **42**(2) (2022), 554–591].

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1. *Introduction*

The Lyapunov exponents come from the study of differential equations in the thesis of Lyapunov [14]. They were systematically introduced into ergodic theory by works of Furstenberg and Kesten [11] and Oseledets [18]. They are directly related to the expansion rates of the system and also to the positive metric entropy, using for example the Margulis–Ruelle inequality and the Pesin entropy formula. Ya. Pesin explored the concept of Lyapunov exponents developing a rich theory of non-uniformly hyperbolic systems, which are the systems with non-zero Lyapunov exponents.

However, in general, these systems are not robust: they do not form an open set. In the works [3, 5], the authors show that if a non-uniformly hyperbolic system does not admit dominated splitting, it can be approximated by systems with zero Lyapunov exponents in C^1 topology. Usually, the presence of dominated decomposition guarantees higher regularity in C^1 topology, see for example [21]. In the non-invertible setting, [1] showed the existence of C^1 open sets without dominated decomposition such that integrated Lyapunov exponents vary continuously with the dynamics in the C^1 topology.

For a linear Anosov automorphism of the torus $A\colon \mathbb{T}^d \to \mathbb{T}^d$, the Lyapunov exponents are constant and indeed they are equal to the logarithm of the norm of eigenvalues of A. In general, Lyapunov exponents cannot be explicitly calculated. The regularity of Lyapunov exponents with respect to dynamics and invariant measures is a subtle question. Let us recall a flexibility conjecture by J. Bochi, A. Katok and F. Rodriguez Hertz. Let $\mathrm{Diff}^\infty_m(M)$ be the set of m-preserving diffeomorphisms (volume preserving) $f : M \to M$ of class C^∞.

Conjecture 1.1. [4] Given a connected component $C \subset \mathrm{Diff}^\infty_m(M)$ and any list of numbers $\xi_1 \geq \cdots \geq \xi_d$ with $\sum_{i=1}^d \xi_i = 0$, there exists an ergodic diffeomorphism $f \in C$ such that $\xi_i, i = 1, \ldots, d$ are the Lyapunov exponents with respect to m.

Moreover, in the setting of volume-preserving Anosov diffeomorphisms, they posed the following problem.

Problem 1. [4] (Strong flexibility) Let $A \in SL(d, \mathbb{Z})$ be a hyperbolic linear transformation inducing conservative Anosov diffeomorphism F_A on $\mathbb{T}^d$ with Lyapunov exponents: $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_u > 0 > \lambda_{u+1} \geq \cdots \geq \lambda_d$. Given any list of numbers $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_u \geq \xi_{u+1} \geq \cdots \geq \xi_d$ such that:
(1) $\sum_{i=1}^d \xi_i = 0$;
(2) $\sum_{i=1}^u \xi_i \leq \sum_{i=1}^u \lambda_i$,
does there exist a conservative Anosov diffeomorphism f homotopic to F_A such that $\{\xi_i\}$ is the list of all Lyapunov exponents with respect to volume measure?

In this work, we study a subset of transformations $f\colon \mathbb{T}^d \to \mathbb{T}^d$ homotopic to a linear Anosov automorphism $A\colon \mathbb{T}^d \to \mathbb{T}^d$, where f has some hyperbolicity (partial hyperbolicity or non-uniform hyperbolicity). See §2 for the definitions. In particular, we show that for a class of partially hyperbolic diffeomorphisms and homotopic to Anosov linear automorphism, not all lists of numbers can be realized as Lyapunov exponents. More precisely, in such a class of dynamics, we need the conditions (1) and (2) imposed in Problem 1.

This type of result also appears in [4, 10, 15–17, 22]. In [16], it has been proved that for any conservative partially hyperbolic systems in the torus $\mathbb{T}^3$, the stable or unstable Lyapunov exponent is bounded by the stable or unstable Lyapunov exponent of its linearization. It is well worth mentioning that Carrasco and Saghin [9] constructed a C^∞ and volume-preserving example on $\mathbb{T}^3$ that shows that the largest Lyapunov exponent of a diffeomorphism in the homotopy class of an Anosov linear automorphism of $\mathbb{T}^3$ may be larger than the largest Lyapunov exponent of A. However, in their example, f does not admit a three-bundle partially hyperbolic splitting.

We also mention that our results address (and give a negative answer in some special cases of derived from Anosov diffeomorhisms) a question in [9]: Does there exist a derived from Anosov diffeomorphism f such that the sum of the positive Lyapunov exponents of f is larger than the sum of the positive Lyapunov exponents of its linear part?

2. *Definitions and statements of results*

Definition 2.1. Let M be a closed manifold. A diffeomorphism $f: M \to M$ is called a partially hyperbolic diffeomorphism if there is a suitable norm $\|\cdot\|$ and the tangent bundle TM admits a Df-invariant decomposition $TM = E^s \oplus E^c \oplus E^u$ such that for all unitary vectors $v^\sigma \in E^\sigma_x$, $\sigma \in \{s, c, u\}$ and every $x \in M$, we have

$$\|D_x f v^s\| < \|D_x f v^c\| < \|D_x f v^u\|;$$

moreover,

$$\|D_x f v^s\| < 1 \quad \text{and} \quad \|D_x f v^u\| > 1.$$

Every diffeomorphism of the torus $\mathbb{T}^d$ induces an automorphism of the fundamental group and there exists a unique linear diffeomorphism f_* which induces the same automorphism on $\pi_1(\mathbb{T}^d)$. The diffeomorphism f_* is called linearization of f.

Anosov diffeomorphisms can be considered partially hyperbolic systems with $E^c = 0$. However, in this paper, whenever we consider an Anosov diffeomorphism as a partially hyperbolic system, we mean that there exists a non-trivial (centre bundle) partially hyperbolic decomposition.

Clearly, a partially hyperbolic diffeomorphism may have various partially hyperbolic invariant decompositions. We will consider several (non-necessarily disjoint) categories of partially hyperbolic diffeomorphisms indexed by the dimensions of invariant bundles.

More precisely, we say $f \in Ph_{(d_s,d_u)}(M)$ if f admits a partially hyperbolic decomposition with $\dim(E^\sigma) = d_\sigma$ for $\sigma \in \{s, u\}$. Clearly, $\dim(E^c) = \dim(M) - d_s - d_u$. For instance, consider the cat map A on $\mathbb{T}^2$ induced by matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, then $A \times A$ is a partially hyperbolic (in fact, Anosov) diffeomorphism and belongs to $Ph_{(1,1)} \cap Ph_{(2,1)} \cap Ph_{(1,2)}$.

Definition 2.2. Let $f: \mathbb{T}^d \to \mathbb{T}^d$ be a partially hyperbolic diffeomorphism, and f is called a derived from Anosov (DA) diffeomorphism if its linearization $f_*: \mathbb{T}^d \to \mathbb{T}^d$ is a linear Anosov automorphism.

Definition 2.3. We say that the diffeomorphism $f: M \to M$ admits a dominated splitting if there is an invariant (by Df) continuous decomposition $TM = E \oplus F$ and constants $0 < \nu < 1$, $C > 0$ such that

$$\frac{\|Df^n|_{E(x)}\|}{\|Df^{-n}|_{F(f^n(n))}\|^{-1}} \leq C\nu^n \quad \text{for all } x \in M, n > 0.$$

Let us recall a simple version of the Oseledets' theorem.

THEOREM 2.4. [18] *Let $f: M \to M$ be a C^1 diffeomorphism, then there is a full probability Borelian set $\mathcal{R}$ (this is, $\mu(\mathcal{R}) = 1$ for all f-invariant probability measure μ) such that for each $x \in \mathcal{R}$, there is a decomposition $T_x M = E_1 \oplus \cdots E_{k(x)}$ and constants $\lambda_1, \ldots, \lambda_{k(x)}$ such that*

$$\lim_{n\to\infty} \frac{1}{n}\|Df^n_x(v)\| = \lambda_i$$

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for all $v \in E_i$. *The* $\lambda_i(x)$ *is called a Lyapunov exponent. Moreover, for any* $1 \leq j \leq k(x)$,

$$\lim_{n\to\infty} \frac{1}{n} \det Df^n_x|_{F_j(x)} = \sum_{i=1}^{j} d_i \lambda_i,$$

where $F_j = E_1 \oplus \cdots \oplus E_j$ *and* $d_i = \dim(E_i)$.

In this paper, when m refers to a measure, it stands for a fixed Lebesgue measure on $\mathbb{T}^d$.

2.1. *Lyapunov exponents of partially hyperbolic diffeomorphisms.* In our first main theorem, we prove that linear Anosov diffeomorphisms maximize (respectively minimize) the sum of unstable (respectively stable) Lyapunov exponents, in any homotopy path totally inside partially hyperbolic diffeomorphisms.

The notion of metric entropy appears naturally when dealing with Lyapunov exponents. For a partially hyperbolic diffeomorphism, it is possible to define entropy of an invariant measure along unstable foliation and there is variational principle results for such entropy. A u-maximal entropy measure is a measure which attains the supremum of unstable entropy among all invariant measures. See §3.2.1 and references for more details.

THEOREM A. *Let* $f: \mathbb{T}^d \to \mathbb{T}^d$ *be a* C^2 *volume-preserving DA diffeomorphism in* Ph_{d_s,d_u} *with linearization* $A: \mathbb{T}^d \to \mathbb{T}^d$ *such that f and A are homotopic by a path fully contained in* $Ph_{d_s,d_u}(\mathbb{T}^d)$, *then*

$$\sum_{i=1}^{d_u} \lambda_i^u(f,x) \leq \sum_{i=1}^{d_u} \lambda_i^u(A) \quad \textit{and} \quad \sum_{i=1}^{d_s} \lambda_i^s(f,x) \geq \sum_{i=1}^{d_s} \lambda_i^s(A),$$

for m-almost every (a.e.) $x \in \mathbb{T}^d$ *and, the first (respectively second) inequality is strict unless m is a measure of u-maximal entropy (respectively u-maximal for* f^{-1}*).*

This is related to the result of [16]: let $f: \mathbb{T}^3 \to \mathbb{T}^3$ be a C^2 conservative derived from Anosov partially hyperbolic diffeomorphism with linearization A. Then, $\lambda^u(x) \leq \lambda^u(A)$ for almost every x. We recall that the authors used quasi-isometric property of unstable foliation for the three-dimensional derived from Anosov diffeomorphisms. They do not assume that f is in the same connected component of A.

THEOREM B. *Let* $f: \mathbb{T}^d \to \mathbb{T}^d$ *be a* C^2 *volume-preserving DA diffeomorphism in* Ph_{d_s,d_u} *with linearization* $A: \mathbb{T}^d \to \mathbb{T}^d$ *belonging to* Ph_{d_s,d_u}. *Suppose that for* $\sigma \in \{s, u\}$, *there is a* $(d - d_\sigma)$*-dimensional subspace* $P_\sigma \subset \mathbb{R}^d$, *such that* $\angle(E^\sigma_f(x), P_\sigma) > \alpha > 0$ *for all* $x \in \mathbb{R}^d$ *(*E^σ_f *stands for the lift of the bundles), then*

$$\sum_{i=1}^{d_u} \lambda_i^u(f,x) \leq \sum_{i=1}^{d_u} \lambda_i^u(A) \quad \textit{and} \quad \sum_{i=1}^{d_s} \lambda_i^s(f,x) \geq \sum_{i=1}^{d_s} \lambda_i^s(A),$$

for m-a.e. $x \in \mathbb{T}^d$.

Observe that the assumption on the existence of P_σ in the above theorem is a mild condition and is satisfied if the stable and unstable bundles of f do not vary too

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much. Indeed, we require that $\{E^{\sigma}_{f}(x), x \in \mathbb{R}^d\}$ is not dense in the Grassmannian of d_σ-dimensional subspaces.

2.2. *Lyapunov exponents of non-uniformly Anosov systems.* In this section, we deal with dynamics that are not necessarily partially hyperbolic. However, they enjoy some dominated splitting property: Oseledets' splitting (stable/unstable) is dominated.

Definition 2.5. A C^2-volume-preserving diffeomorphism f is called non-uniformly Anosov if it admits an f-invariant dominated decomposition $TM = E \oplus F$ such that $\lambda_i^F(x) > 0$ (all Lyapunov exponents along F) and $\lambda_i^E(x) < 0$ (all Lyapunov exponents along E) for μ-a.e x.

In the next theorem, we compare Lyapunov exponents of a non-uniformly Anosov diffeomorphism with those of linear Anosov automorphism if they are homotopic. Observe that by definition, the number of negative (positive) Lyapunov exponents of a non-uniformly Anosov diffeomorphism is constant almost everywhere and coincides with the dimension of the bundles in the dominated splitting. However, the Lyapunov exponents may depend on the orbit, as we do not assume ergodicity. In fact, it is interesting to know whether, in general, all topologically transitive non-uniformly Anosov diffeomorphisms are ergodic or not. We conjecture that all topologically transitive non-uniformly Anosov diffeomorphisms are ergodic.

Let $A \colon \mathbb{T}^d \to \mathbb{T}^d$ be a linear Anosov diffeomorphism such that $T\mathbb{T}^d = E^s_A \oplus E^u_A$. Denote by $d_s = \dim(E^s_A)$ and $d_u = \dim(E^u_A)$. After changing to an equivalent Riemannian norm, if necessary, there are $0 < \lambda < 1 < \gamma$ such that $\|A|_{E^s}\| \le \lambda$, $m(A|_{E^u}) \ge \gamma$ and E^s_A is orthogonal to E^u_A. Indeed, to use fewer constants in the proofs, we assume that E^u_i and E^u_j, $i \neq j$ are orthogonal, where E^u_i terms are generalized eigenspaces of A which coincide with the Oseledets decomposition. Recall that for a linear transformation T, $m(T) := \|T^{-1}\|^{-1}$. We refer to γ, λ as rates of hyperbolicity of A. Observe that Lyapunov exponents of any diffeomorphism are independent of the choice of the equivalent norm.

THEOREM C. *Let $f \colon \mathbb{T}^d \to \mathbb{T}^d$ be a C^2 conservative non-uniformly Anosov diffeomorphism with dominated decomposition $TM = E \oplus F$, homotopic to A such that:*

(1) $\dim(E) = d_s$ *and* $\dim(F) = d_u$;

(2) $E(x) \cap E^u_A = \{0\}$ *and* $F(x) \cap E^s_A = \{0\}$;

(3) $\|Df|_E\| < \gamma$ *and* $m(Df|_F) > \lambda$.

Suppose that the distributions E and F are integrable, then $\sum_{i=1}^{d_u} \lambda_i^F(f, x) \le \sum_{i=1}^{d_u} \lambda_i^u(A)$ *and* $\sum_{i=1}^{d_s} \lambda_i^E(f, x) \ge \sum_{i=1}^{d_s} \lambda_i^s(A)$ *for Lebesgue-a.e.* $x \in \mathbb{T}^d$.

Let us comment on the hypotheses: the first and second one ask for some compatibility of invariant bundles and the third one asks that any possible expansion in the dominated bundle E is less than the expansion rate of A and similarly any possible contraction in the dominating bundle F is weaker than the contraction rate of A.

Bonatti and Viana [6] constructed the first examples of robustly transitive diffeomorphisms that are not partially hyperbolic, which was later generalized in higher dimensions by [23]. Those classes of examples satisfy the hypotheses of the above theorem.

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THEOREM 2.6. [6, 8, 23] *There is an open set* $\mathcal{U} \subset \mathrm{Diff}^1_m(\mathbb{T}^d)$ *such that any* C^2*-diffeomorphism in* $\mathcal{U}$ *satisfies all hypotheses of Theorem C and the bundles E and F are integrable.*

In the above theorem, the integrability of invariant subbundles E and F is proved in [8].

2.3. *Regularity of foliations.* Any partially hyperbolic diffeomorphism admits invariant stable and unstable foliations. As the proofs of our main results are based on the regularity of such foliations, we need to recall some basic definitions and results in this subsection. Consider $\mathcal{F}$ a foliation in M, B a $\mathcal{F}$-foliated box and m the Lebesgue measure in M. Denote by Vol_{L_x} the Lebesgue measure of leaf L_x and m_{L_x} the disintegration of the Lebesgue measure m along the leaf L_x. The disintegrated measures are in fact a projective class of measures. However, whenever we fix a compact foliated box B, then we can use the Rohlin disintegration theorem for the normalized restriction of m on B to get probability conditional measures. So in what follows, after fixing a foliated box B by m_{L_x} and Vol_{L_x}, we understand probability measures whose support is inside the plaque of L_x which contains $x \in B$ and is inside B.

Definition 2.7. We say that the foliation $\mathcal{F}$ is *upper leafwise absolutely continuous* if for any foliated box B, $m_{L_x} \ll \mathrm{Vol}_{L_x}$ for m-a.e. $x \in B$. Equivalently, if given a set $Z \subset B$ such that $\mathrm{Vol}_{L_x}(Z \cap L_x) = 0$ for m-a.e. $x \in B$, then $m(Z) = 0$.

Definition 2.8. We say that the foliation $\mathcal{F}$ is *lower leafwise absolutely continuous* if $\mathrm{Vol}_{L_x} \ll m_{L_x}$ for m-a.e. $x \in B$. Equivalently, if given a set $Z \subset B$ such that $m(Z) = 0$, then $\mathrm{Vol}_{L_x}(Z \cap L_x) = 0$ for m-a.e. $x \in B$.

Definition 2.9. We say that the foliation $\mathcal{F}$ is *leafwise absolutely continuous* if $\mathrm{Vol}_{L_x} \sim m_{L_x}$ (this is, $m_{L_x} \ll \mathrm{Vol}_{L_x}$ and $\mathrm{Vol}_{L_x} \ll m_{L_x}$) for m-a.e. $x \in B$.

PROPOSITION 2.10. [7] *If* $f : M \to M$ *is a* C^2 *partially hyperbolic diffeomorphism, then the foliations* W^s *and* W^u *are leafwise absolutely continuous.*

We also mention the result of Ya. Pesin for non-uniformly hyperbolic systems, see [2, Theorem 4.3.1], which shows absolute continuity of local Pesin laminations.

3. *Proof of results*

Theorems A and B are obtained by the following result.

THEOREM 3.1. *Let* $f : \mathbb{T}^d \to \mathbb{T}^d$ *be a* C^2 *volume-preserving partially hyperbolic diffeomorphism such that there are closed non-degenerate* d_u*-forms and* d_s*-forms on* E^u *and* E^s*, respectively. Suppose that A, the linearization of f, is partially hyperbolic and* $\dim E^\sigma_f = \dim E^\sigma_A$, $\sigma \in \{s, c, u\}$*, then*

$$\sum_{i=1}^{d_u} \lambda_i^u(f, x) \leq \sum_{i=1}^{d_u} \lambda_i^u(A) \quad \text{and} \quad \sum_{i=1}^{d_s} \lambda_i^s(f, x) \geq \sum_{i=1}^{d_s} \lambda_i^s(A)$$

for m-a.e. $x \in \mathbb{T}^d$.

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For the proof of this theorem, we use a result of Saghin [20]. Let $f: M \to M$ be a diffeomorphism and W an f-invariant foliation on M, that is, $f(W(x)) = W(f(x))$, and $B_r(x, f)$ be the ball of the leaf $W(x)$ with radius r centred at x. We say that

$$\chi_W(f, x) = \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{Vol}(f^n(B_r(x, f)))$$

is the *volume growth rate of the foliation at* x and

$$\chi_W(f) = \sup_{x\in M} \chi_W(f, x)$$

is the *volume growth rate* of W. Here, Vol stands for the Vol_W which is the induced volume to the leaves of W. We use this notation throughout the paper except when it may create confusion.

When f is a partially hyperbolic diffeomorphism, we denote by $\chi_u(f)$ the volume growth of unstable foliation W^u.

THEOREM 3.2. [20] *Let $f: M \to M$ be a C^1 partially hyperbolic diffeomorphism such that there is a closed non-degenerate d_u-form on the unstable bundle E^u, then $\chi_u(f) =$* log $sp(f_{*,d_u})$*, where f_{*,d_u} is the induced map in the d_u-cohomology of De Rham.*

PROPOSITION 3.3. *Let $f: \mathbb{T}^d \to \mathbb{T}^d$ be a C^1 partially hyperbolic diffeomorphism admitting a closed non-degenerate d_u-form on the unstable bundle E^u. For fixed $x \in \mathbb{T}^d$, $r > 0$, the balls $B_r(x, f) \subset W^u(x, f)$ and $B_r(x, A) \subset W^u(x, A)$ satisfy the following. Given $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that if $n > n_0$, we have*

$$\mathrm{Vol}_{W^u(f)}(f^n(B_r(x, f))) \le (1+\varepsilon)^n \mathrm{Vol}_{W^u(A)}(A^n(B_r(x, A))).$$

Proof. By Theorem 3.2, we have that

$$\chi_u(f, x) \le \chi_u(f) = \log sp(f_{*,u}) = \log sp(A_{*,u}) = \chi_u(A) = \chi_u(A, x),$$

that is,

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathrm{Vol}_{W^u(f)}(f^n(B_r(x, f))) \le \lim_{n\to\infty} \frac{1}{n} \log \mathrm{Vol}_{W^u(A)}(A^n(B_r(x, A)));$$

therefore, given $\varepsilon > 0$, there is n_0 such that if $n > n_0$,

$$\frac{1}{n} \log \mathrm{Vol}_{W^u(f)}(f^n(B_r(x, f))) \le \frac{1}{n} \log \mathrm{Vol}_{W^u(A)}(A^n(B_r(x, A))) + \frac{1}{n} \log(1+\varepsilon)^n.$$

Then,

$$\mathrm{Vol}_{W^u(f)}(f^n(B_r(x, f))) \le (1+\varepsilon)^n \mathrm{Vol}_{W^u(A)}(A^n(B_r(x, A))). \qquad \square$$

Proof of Theorem 3.1. We prove the statement for the sum of unstable exponents. One may repeat the argument for f^{-1} to obtain the claim for stable exponents.

Suppose by contradiction that there is a positive volume set $Z \subset \mathcal{R} \subset \mathbb{T}^d$, (where $\mathcal{R}$ is the set of points satisfying Oseledets' theorem as stated in Theorem 2.4) such that for all $x \in Z$, we have $\sum_{i=1}^{d_u} \lambda_i^u(f, x) > \sum_{i=1}^{d_u} \lambda_i^u(A)$.

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For $q \in \mathbb{N} \setminus \{0\}$, we define the set

$$Z_q = \left\{ x \in Z; \sum_{i=1}^{d_u} \lambda_i^u(f, x) > \sum_{i=1}^{d_u} \lambda_i^u(A) + \log\left(1 + \frac{1}{q}\right) \right\}.$$

Since $\bigcup_{q=1}^{\infty} Z_q = Z$, there is a q such that $m(Z_q) > 0$. For any $x \in Z_q$, we have

$$\lim_{n\to\infty} \frac{1}{n} \log |\mathrm{Jac} f^n(x)|_{E^u}| > \sum_{i=1}^{d_u} \lambda_i^u(A) + \log\left(1 + \frac{1}{q}\right).$$

So, there is n_0 such that for $n \geq n_0$, we have

$$\begin{aligned} \frac{1}{n} \log |\mathrm{Jac} f^n(x)|_{E^u}| &> \sum_{i=1}^{d_u} \lambda_i^u(A) + \log\left(1 + \frac{1}{q}\right) \\ &> \frac{1}{n} \log e^{n \sum_{i=1}^{d_u} \lambda_i^u(A)} + \frac{1}{n} \log\left(1 + \frac{1}{q}\right)^n. \end{aligned}$$

So we get

$$|\mathrm{Jac} f^n(x)|_{E^u}| > \left(1 + \frac{1}{q}\right)^n e^{n \sum_{i=1}^{d_u} \lambda_i^u(A)}.$$

By this fact, for each $n > 0$, we define the set

$$Z_{q,n} = \left\{ x \in Z_q; |\mathrm{Jac} f^k(x)|_{E^u}| > \left(1 + \frac{1}{q}\right)^k e^{k \sum_{i=1}^{d_u} \lambda_i^u(A)} \text{ for all } k \geq n \right\}.$$

So, there is $N > 0$ with $m(Z_{q,N}) > 0$.

Now for any $x \in \mathbb{T}^d$, let B_x be a foliated box of W_f^u around x. By compactness, we can take a finite cover $\{B_{x_i}\}_{i=1}^{j}$ of $\mathbb{T}^d$. As W_f^u is absolutely continuous [7], there is i and $x \in B_{x_i}$ such that $\mathrm{Vol}_W^u(f)(B_{x_i} \cap W_f^u(x) \cap Z_{q,N}) > 0$.

Consider $B_r(x) \subset W_f^u(x)$ satisfying $\mathrm{Vol}_{W^u(f)}(B_r(x) \cap Z_{q,N}) > 0$. By Proposition 3.3, there is $K(\epsilon) \in \mathbb{N}$ such that for all $k \geq K(\epsilon)$,

$$\begin{aligned} \mathrm{Vol}_{W^u(f)}(f^k(B_r(x))) &\leq (1+\epsilon)^k \mathrm{Vol}_{W^u(A)}(A^k(B_r(x))) \\ &\leq (1+\epsilon)^k e^{k \sum \lambda_i^u(A)} \mathrm{Vol}_{W^u(A)}(B_r(x)). \end{aligned} \tag{3.1}$$

However, for $k \geq N$,

$$\begin{aligned} \mathrm{Vol}_{W^u(f)}(f^k(B_r(x))) &= \int_{B_r(x)} |\mathrm{Jac} f^k|_{E^u}| \, d\mathrm{Vol}_{W^u(f)} \\ &\geq \int_{B_r(x) \cap Z_{q,N}} |\mathrm{Jac} f^k|_{E^u}| \, d\mathrm{Vol}_{W^u(f)} \\ &> \int_{B_r(x) \cap Z_{q,N}} \left(1 + \frac{1}{q}\right)^k e^{k \sum_i \lambda_i^u(A)} \, d\mathrm{Vol}_{W^u(f)} \\ &= \left(1 + \frac{1}{q}\right)^k e^{k \sum_{i=1}^{d_u} \lambda_i^u(A)} \mathrm{Vol}_{W^u(f)}(B_r(x) \cap Z_{q,N}). \end{aligned} \tag{3.2}$$

Taking $\epsilon < 1/q$, for large enough k, the inequalities in equations (3.1) and (3.2) give a contradiction. □

3.1. *Proof of Theorem B.* We use Theorem 3.1 and the following proposition to conclude the proof of Theorem B.

PROPOSITION 3.4. *Let W be a foliation in $\mathbb{R}^d$ of dimension d_u. If there is a (d–d_u)-plane P in $\mathbb{R}^d$ such that $\angle(T_x W, P) > \alpha > 0$ for all $x \in \mathbb{R}^d$, then there is a d_u-form ω which is closed and non-degenerate on W.*

Proof. Let B be the orthogonal complement of P, there is always a closed and non-degenerate form in B, in fact, just take the volume form $\omega = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{d_u}$ which is non-degenerate in B and ω is closed. Note that ω is degenerate only in $B^\perp = P$ and by hypothesis, $\angle(T_x W, P) > \alpha > 0$. So, ω is closed and non-degenerate in $T_x W$ for all $x \in \mathbb{R}^d$. □

COROLLARY 3.5. *If $A : \mathbb{T}^d \to \mathbb{T}^d$ is a linear partially hyperbolic diffeomorphism and f is a C^2 conservative diffeomorphism which is a small C^1-perturbation of A, then $\sum_{i=1}^{d_u} \lambda_i^u(f, x) \le \sum_{i=1}^{d_u} \lambda_i^u(A)$ and $\sum_{i=1}^{d_s} \lambda_i^s(f, x) \ge \sum_{i=1}^{d_s} \lambda_i^s(A)$ for Lebesgue-a.e. $x \in \mathbb{T}^d$.*

3.2. *Proof of Theorem A.* From [19, Proposição 3.1.2], we have the following proposition.

PROPOSITION 3.6. [19] *Let $f : \mathbb{T}^d \to \mathbb{T}^d$ be a partially hyperbolic diffeomorphism homotopic to a linear Anosov diffeomorphism A such that:*

(a) *each element of the homotopic path is a partially hyperbolic diffeomorphism;*

(b) *if f_1 and f_2 are two elements of the homotopic path, then* $\dim E^\sigma(f_1) = \dim E^\sigma(f_2)$, $\sigma \in \{s, c, u\}$,

then there exists a closed non-degenerate d_u-form on $W^u(f)$.

By the proposition above, f has a d_u closed and non-degenerate form on W^u, using the inverse f^{-1}, we get a d_s closed and non-degenerate form on W^s. By Theorem 3.1, we conclude the first part of Theorem A. In the next subsection, we complete the proof.

3.2.1. *Maximizing measures.* In this section, we prove the last part of Theorem A. We write all proofs for the unstable bundle.

First, we recall the definition of topological and metric entropy along an expanding foliation. In [13], the authors define the notion of topological and metric entropy along unstable foliation and prove the variational principle.

Let $f : M \to M$ be a C^1-partially hyperbolic diffeomorphism and μ is an f-invariant probability measure. For a partition α of M, denote $\alpha_0^{n-1} = \bigvee_{i=0}^{n-1} f^{-i}\alpha$ and by $\alpha(x)$ the element of α containing x. Given $\varepsilon > 0$, let $\mathcal{P} = \mathcal{P}_\varepsilon$ denote the set of finite measurable partitions of M whose elements have diameters smaller than or equal to ε. For each $\alpha \in \mathcal{P}$, we define a partition η such that $\eta(x) = \alpha(x) \cap W^u_{\text{loc}}(x)$ for each $x \in M$, where $W^u_{\text{loc}}(x)$

denotes the local unstable manifold at x whose size is greater than the diameter ε of α. Let $\mathcal{P}^u = \mathcal{P}^u_\varepsilon$ denote the set of partitions η obtained in this way. We define the *conditional entropy of α given η with respect to μ* by

$$H_\mu(\alpha|\eta) := -\int_M \log \mu_x^\eta(\alpha(x))\, d\mu(x),$$

where μ_x^η refer to the disintegration of μ along η.

Definition 3.7. The *conditional entropy of f with respect to a measurable partition α given* $\eta \in \mathcal{P}^u$ is defined as

$$h_\mu(f, \alpha|\eta) = \limsup_{n\to\infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\eta).$$

The conditional entropy of f given $\eta \in \mathcal{P}^u$ is defined as

$$h_\mu(f|\eta) = \sup_{\alpha\in\mathcal{P}} h_\mu(f, \alpha|\eta),$$

and the unstable metric entropy of f is defined as

$$h^u_\mu(f) = \sup_{\eta\in\mathcal{P}^u} h_\mu(f|\eta).$$

Now, we go to define the unstable topological entropy.

We denote by ρ^u the metric induced by the Riemannian structure on the unstable manifold and let $\rho^u_n(x, y) = \max_{0\le j\le n-1} \rho^u(f^j(x), f^j(y))$. Let $W^u(x, \delta)$ be the open ball inside $W^u(x)$ centred at x of radius δ with respect to the metric ρ^u. Let $N^u(f, \epsilon, n, x, \delta)$ be the maximal number of points in $\overline{W^u(x, \delta)}$ with pairwise ρ^u_n-distances at least ϵ. We call such a set an (n, ϵ)*u-separated set* of $\overline{W^u(x, \delta)}$.

Definition 3.8. The unstable topological entropy of f on M is defined by

$$h^u_{\text{top}}(f) = \lim_{\delta\to 0} \sup_{x\in M} h^u_{\text{top}}(f, \overline{W^u(x, \delta)}),$$

where

$$h^u_{\text{top}}(f, \overline{W^u(x, \delta)}) = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log N^u(f, \epsilon, n, x, \delta).$$

Let $\mathcal{M}_f(M)$ and $\mathcal{M}^e_f(M)$ denote the set of all f-invariant and ergodic probability measures on M, respectively.

THEOREM 3.9. [13] *Let $f : M \to M$ be a C^1-partially hyperbolic diffeomorphism. Then,*

$$h^u_{\text{top}}(f) = \sup\{h^u_\mu(f) : \mu \in \mathcal{M}_f(M)\}.$$

Moreover,

$$h^u_{\text{top}}(f) = \sup\{h^u_\nu(f) : \nu \in \mathcal{M}^e_f(M)\}.$$

Furthermore, the authors proved that unstable topological entropy coincides with unstable volume growth.

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THEOREM 3.10. [13] *Unstable topological entropy coincides with unstable volume growth, i.e.* $h^u_{\mathrm{top}}(f) = \chi_u(f)$.

As in the hypotheses of Theorem A, the diffeomorphisms f and A are homotopic, so using Theorem 3.2, we have

$$h^u_{\mathrm{top}}(f) = \chi_u(f) = \log sp(f_*, u) = \log sp(A_*, u) = \chi_u(A) = h^u_{\mathrm{top}}(A).$$

Thus,

$$\sum_{i=1}^{d_u} \lambda^u_i(f, x) = h^u_m(f) \le h^u_{\mathrm{top}}(f) = h^u_{\mathrm{top}}(A) = \sum_{i=1}^{d_u} \lambda^u_i(A).$$

Then we conclude that $h^u_m(f) = h^u_{\mathrm{top}}(f)$.

3.3. *Proof of Theorem C.* We remember that $A\colon \mathbb{T}^d \to \mathbb{T}^d$ is a linear Anosov diffeomorphism and $0 < \lambda < 1 < \gamma$ are rates for hyperbolicity, $d_s = \dim(E^s_A)$ and $d_u = \dim(E^u_A)$. Let W^s_A, W^u_A be the stable and unstable foliation of A and by $\widetilde{W}^s_A$, $\widetilde{W}^u_A$, we denote their lifts to the universal cover $\mathbb{R}^d$. These foliations are stable and unstable foliations of $\widetilde{A}$ which is a lift of A. We use '$\sim$' for objects in the universal cover. The norm and distance on $\mathbb{R}^d$ are the lift of adapted norm and corresponding distance where the hyperbolicity conditions of A are satisfied, see §2.2 before the announcement of Theorem C.

Similar to [12, Proposition 2.5 and Corollary 2.6], we show the following proposition.

PROPOSITION 3.11. *Let* $f\colon \mathbb{T}^d \to \mathbb{T}^d$ *be a diffeomorphism that admits a dominated splitting* $TM = E \oplus F$ *with* $\|Df|_E\| \le \hat{\gamma} < \gamma$ *and* $m(Df|_F) \ge \hat{\lambda} > \lambda$, *homotopic to* A *such that* $\dim(E) = d_s$ *and* $\dim(F) = d_u$. *Suppose that the distributions* E *and* F *are integrable, and denote by* $\mathcal{E}$ *and* $\mathcal{F}$ *their respective tangent foliations, then there is* $R > 0$ *such that:*

- $\mathcal{E}(x) \subset B_R(\widetilde{W}^s_A(x))$;
- $\mathcal{F}(x) \subset B_R(\widetilde{W}^u_A(x))$,

where $B_R(\widetilde{W}^s_A(x)) \subset \mathbb{R}^d$ *is the set of points which are at distance* R *from* $W^s_A(x)$. *Use a similar definition for* $B_R(\widetilde{W}^u_A(x))$.

COROLLARY 3.12. *Fixing* $x \in \mathbb{R}^n$, *if* $\|x - y\| \to \infty$ *and* $y \in \mathcal{E}(x)$, *then* $(x - y)/\|x - y\| \to \widetilde{E}^s_A(x)$ *uniformly. More precisely, for* $\varepsilon > 0$, *there is* $M > 0$ *such that if* $x \in \mathbb{R}^d$, $y \in \mathcal{E}(x)$ *and* $\|x - y\| > M$, *then*

$$\|\pi^u_A(x - y)\| < \varepsilon \|\pi^s_A(x - y)\|,$$

where π^s_A *is the orthogonal projection on the subspace* E^s_A *along* E^u_A, *and* π^u_A *is the projection on the subspace* E^u_A *along* E^s_A.

Analogous statements hold for $\mathcal{F}$. The following proposition is a topological remark (see [12]) and comes from the fact that f and A are homotopic and A is not singular.

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PROPOSITION 3.13. *Let $f: \mathbb{T}^d \to \mathbb{T}^d$ be a homeomorphism with linearization A, then for each $k \in \mathbb{Z}$ and $C > 1$, there is an $M > 0$ such that for all $x, y \in \mathbb{R}^d$,*

$$\|x - y\| > M \Rightarrow \frac{1}{C} < \frac{\|\tilde{f}^k(x) - \tilde{f}^k(y)\|}{\|\tilde{A}^k(x) - \tilde{A}^k(y)\|} < C.$$

More generally, for each $k \in \mathbb{Z}$, $C > 1$ and any linear projection $\pi: \mathbb{R}^d \to \mathbb{R}^d$, with A-invariant image, there is $M > 0$ such that for $x, y \in \mathbb{R}^d$, with $\|x - y\| > M$,

$$\frac{1}{C} < \frac{\|\pi(\tilde{f}^k(x) - \tilde{f}^k(y))\|}{\|\pi(\tilde{A}^k(x) - \tilde{A}^k(y))\|} < C.$$

Proof of Proposition 3.11. To prove the first claim (the second one is similar), it is enough to show that $\|\pi_A^u(x - y)\|$ is uniformly bounded for all $y \in \mathcal{E}(x)$. Let $C > 1$ close enough to 1 such that $C\hat{\gamma} < \gamma$ and $1 < \gamma/C$. Put $k = 1$ and C chosen as above in Proposition 3.13, and get appropriate M. By contradiction, suppose that $\|\pi_A^u(x - y)\|$ is not bounded. So, there is $y \in \mathcal{E}(x)$ with $\|\pi_A^u(x - y)\| > M$ and consequently,

$$\|\pi_A^u(\tilde{f}(x) - \tilde{f}(y))\| > \frac{1}{C}\|\pi_A^u(\tilde{A}(x - y))\| = \frac{1}{C}\|(\tilde{A}(\pi_A^u(x - y))\| \geq \frac{\gamma}{C}\|\pi_A^u(x - y)\| > M,$$

which implies that we can use induction and for any $n \geq 1$, obtain

$$\|\pi_A^u(\tilde{f}^n(x) - \tilde{f}^n(y))\| > \frac{\gamma^n}{C^n}M.$$

Finally, there is a constant $\eta > 0$ such that

$$\|\tilde{f}^n(x) - \tilde{f}^n(y)\| > \eta\frac{\gamma^n}{C^n}M. \tag{3.3}$$

Now consider a smooth curve $\alpha: [a, b] \to \mathcal{E}(x)$ with $\alpha(a) = x$ and $\alpha(b) = y$ whose length is $d_{\mathcal{E}}(x, y)$. Then,

$$\|\tilde{f}^n(x) - \tilde{f}^n(y)\| \leq d_{\mathcal{E}}(\tilde{f}^n(x), \tilde{f}^n(y)) \leq l(\tilde{f}^n(\alpha(t))) = \int_a^b \left\|\frac{d}{dt}(\tilde{f}^n(\alpha(t))\right\| dt$$
$$\leq \int_a^b \|D\tilde{f}^n|_{W^{cs}}\| \cdot \|(\alpha'(t))\| \, dt < \int_a^b \hat{\gamma}^n\|\alpha'(t)\| \, dt,$$

implies that

$$\|\tilde{f}^n(x) - \tilde{f}^n(y)\| < \hat{\gamma}^n \, d_{\mathcal{E}}(x, y). \tag{3.4}$$

Equations (3.3) and (3.4) and $\gamma/C > \hat{\gamma}$ give us a contradiction when n is large enough. □

PROPOSITION 3.14. *Let $f: M \to M$ be a C^2 conservative non-uniformly Anosov diffeomorphism with $TM = E \oplus F$. If the distribution E is integrable, then the respective foliation $\mathcal{E}$ is upper leafwise absolutely continuous.*

Let $\mathcal{R}$ be the regular set in Pesin sense such that $m(\mathcal{R}) = 1$ and $\mathcal{R} = \bigcup_{l=1}^{\infty} \mathcal{R}_l$ is the union of Pesin's block where $m(\mathcal{R}_l) \to 1$, $\mathcal{R}_l \subset \mathcal{R}_{l+1}$. Moreover, the size of Pesin stable manifolds of points in $\mathcal{R}_l$ is larger than some positive constant r_l. In general, $r_l \to 0$

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when $l \to \infty$. We use an absolute continuity result due to Pesin [**2**, Theorem 4.3.1] which essentially shows the absolute continuity of Pesin's stable laminations in each block.

Proof of Proposition 3.14. Suppose by contraction that $\mathcal{E}$ is not upper leafwise absolutely continuous. So, there exist a foliated box B and a set $Z \subset B$ with $\mathrm{Vol}_{\mathcal{E}}(Z \cap \mathcal{E}(x)) = 0$ for m-a.e. $x \in B$ and $m(Z) > 0$. There exists $l \geq 1$ such that $m(Z \cap \mathcal{R}_l) > 0$. Take a Lebesgue density point $z \in Z \cap \mathcal{R}_l$ such that for V a small neighbourhood of z, we have $m(Z \cap \mathcal{R}_l \cap V) > 0$.

As Pesin stable manifolds are contained in the leaves of $\mathcal{E}$, our assumption $\mathrm{Vol}_{\mathcal{E}}(Z \cap \mathcal{E}(x)) = 0$ and upper leafwise absolute continuity of stable manifolds imply that Z has zero measure with respect to conditional measures of m along Pesin stable manifolds in $\mathcal{R}_l$.) Hence, $m(Z \cap \mathcal{R}_l \cap V) = 0$, which is a contradiction. □

Fix $x \in \mathbb{R}^n$ and denote by $U_r \subset \widetilde{W}^u_A(x)$ the ball of radius r with centre x.

PROPOSITION 3.15. *Let $f : \mathbb{T}^d \to \mathbb{T}^d$ as in Theorem C, then given $\varepsilon > 0$, there is $r_0 > 0$ and a constant $C_0 > 0$ such that*

$$\mathrm{Vol}_{\mathcal{F}}(\tilde{f}^n \pi_x^{-1}(U_r)) \leq C_0 (1+\varepsilon)^{nd_u} e^{n \sum_{i=1}^{d_u} \lambda_i^u(A)} \mathrm{Vol}_{\widetilde{W}^u_A}(U_r)$$

for all $n > 0$ and $r \geq r_0$, where π_z is the orthogonal projection from $\mathcal{F}(z)$ to $\widetilde{W}^u_A(z)$ (along $\widetilde{E}^s_A$) for any $z \in \mathbb{R}^n$.

Proof of Proposition 3.15. We prove the following claims.

CLAIM 1. *For $z \in \mathbb{R}^d$, the orthogonal projection $\pi_z : \mathcal{F}(z) \to \widetilde{W}^u_A(z)$ is a uniform bi-Lipschitz diffeomorphism.*

Proof of Claim 1. By item (2) of Theorem C and continuity of E and F, there is $\alpha > 0$ such that the angle $\angle(E, E^u_A) > \alpha$ and $\angle(F, E^s_A) > \alpha$. So $\widetilde{W}^s_A$ is uniformly transversal to the foliation $\mathcal{F}$ and there is $\beta > 0$ such that

$$\|d\pi_z(x)(v)\| \geq \beta \|v\| \tag{3.5}$$

for any $x \in \mathcal{F}(z)$ and $v \in T_x\mathcal{F}(z)$. This implies that $\mathrm{Jac}\pi_z(x) \neq 0$. By the inverse function theorem, for each $x \in \mathcal{F}(z)$, there is a ball $B(x, \delta) \subset \mathcal{F}(z)$ such that $\pi_z|_{B(x,\delta)}$ is a diffeomorphism. In fact, from the proof of the inverse function theorem and equation (3.5), δ can be taken independent of x. Again, by equation (3.5), there is $\varepsilon > 0$, independent of x such that $B(\pi_z(x), \varepsilon) \subset \pi_z(B(x, \delta))$. To prove that π_z is surjective, we show that $\pi_z(\mathcal{F}(z))$ is an open and closed subset. As π_z is a local homeomorphism, then $\pi_z(\mathcal{F}(z))$ is open. To verify that it is also closed, let $y_n \in \pi_z(\mathcal{F}(z))$ be a sequence converging to y, and hence there is n_0 large enough such that $y \in B(\varepsilon, y_{n_0})$ and, therefore, $y \in \pi_z(\mathcal{F}(z))$. So, π_z is surjective. Moreover, π_z is a covering map and injectivity follows from the fact that any covering map from a path-connected space to a simply connected space is a homeomorphism.

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Let us prove that π_z is bi-Lipschitz. In fact,

$$\|\pi_z(x) - \pi_z(y)\| \leq \|x - y\| \leq d_{\mathcal{F}}(x, y).$$

Let us show that π_z^{-1} is also Lipschitz. This is immediate from equation (3.5). Indeed, let $[x, y]$ be the line segment in $\widetilde{W}_A^u(z)$ connecting x to y, so the set $\pi_z^{-1}([x, y]) = \Gamma$ is a smooth curve connecting the points $\pi_z^{-1}(x)$ and $\pi_z^{-1}(y)$ in $\mathcal{F}(z)$. So,

$$d_{\mathcal{F}}(\pi_z^{-1}(x), \pi_z^{-1}(y)) \leq \text{length } (\Gamma) = \int_{[x,y]} |d\pi_z^{-1}(t)|\, dt \leq \frac{1}{\beta}\|x - y\|. \qquad \square$$

CLAIM 2. *Given* $\varepsilon > 0$, *there is* $r_0 > 0$ *such that* $\tilde{f}^n(\pi_x^{-1}(U_r)) \subset \pi_{\tilde{f}^n(x)}^{-1}(U_{r,n})$ *for every* $r \geq r_0$ *and* $n \geq 1$, *where* $U_{r,n} \subset \widetilde{W}_A^u(\tilde{f}^n(x))$ *is a* d_u*-dimensional ellipsoid with volume bounded above by* $(1+\epsilon)^{nd_u}\mathrm{Vol}_{\tilde{W}_A^u}(A^n(U_r))$.

Proof. Let $r \geq r_0 := (2R + 2K)/\varepsilon m(A|_{E^u})$, where R comes from Proposition 3.11 and $\|\tilde{f} - \tilde{A}\|_\infty \leq K$. We prove that

$$\tilde{f}(\pi_x^{-1}(U_r)) \subset \pi_{\tilde{f}(x)}^{-1}(U_{r,1}), \tag{3.6}$$

where $U_{r,1} \subset \widetilde{W}_A^u(\tilde{f}(x))$ is obtained from $\tilde{A}(U_r)$ by first applying a homothety of ratio $(1+\epsilon)$ centred at $\tilde{A}(x)$ and then translating by $\tilde{f}(x) - \tilde{A}(x)$. Observe that $U_{r,1}$ is an ellipsoid inside the affine d_u dimensional subspace passing through $\tilde{f}(x)$. Take any y on the boundary of U_r. Let $z := \pi_{\tilde{f}(x)}(\tilde{f}(\pi_x^{-1}(y))) \in \widetilde{W}_A^u(\tilde{f}(x))$. On the one hand, we have

$$\|z - (\tilde{A}(y) + \tilde{f}(x) - \tilde{A}(x))\| \leq \|\tilde{f}(x) - \tilde{A}(x)\| + \|z - \tilde{A}(y)\| \leq K + \|z - \tilde{A}(y)\|. \tag{3.7}$$

On the other hand,

$$\begin{aligned} \|z - \tilde{A}(y)\| &\leq \|z - \tilde{f}(\pi_x^{-1}(y))\| + \|\tilde{f}(\pi_x^{-1}(y)) - \tilde{A}(\pi_x^{-1}(y))\| \\ &\quad + \|\tilde{A}(\pi_x^{-1}(y)) - \tilde{A}(y)\| \\ &\leq 2R + K. \end{aligned} \tag{3.8}$$

In the above inequalities, we have used Proposition 3.11 two times to get

$$\|z - \tilde{f}(\pi_x^{-1}(y)\| \leq R,$$

$$\|\tilde{A}(\pi_x^{-1}(y)) - \tilde{A}(y)\| \leq \|y - \pi_x^{-1}(y)\| \leq R,$$

(observe that $y - \pi_x^{-1}(y)$ belongs to the stable subspace of A) and finally

$$\|\tilde{f}(\pi_x^{-1}(y))) - \tilde{A}(\pi_x^{-1}(y))\| \leq \|\tilde{f} - \tilde{A}\|_\infty \leq K.$$

Now, putting the inequalities in equations (3.7) and (3.8) together, we get

$$\|z - (\tilde{A}(y) + \tilde{f}(x) - \tilde{A}(x))\| \leq 2K + 2R.$$

Observe that $\tilde{A}(y) + \tilde{f}(x) - \tilde{A}(x)$ is the translation of the $\tilde{A}(y)$ and belongs to $\widetilde{W}_A^u(\tilde{f}(x))$. Indeed, $\tilde{A}(y) - \tilde{A}(x)$ is a vector which belongs to the unstable bundle of $\tilde{A}$ and we identify the unstable bundle at $\tilde{f}(x)$ with the corresponding affine subspace of $\mathbb{R}^n$ passing through

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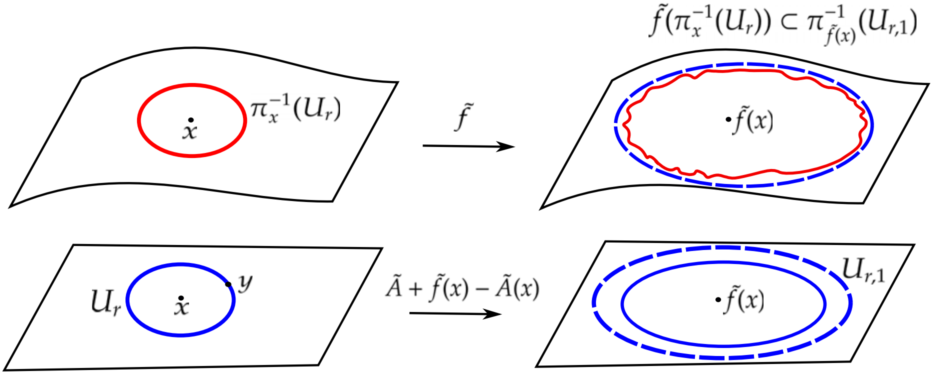

FIGURE 1. Volume comparison.

$\tilde f(x)$. Now, as the distance between $(1+\epsilon)\tilde A(U_r)$ and $\tilde A(U_r)$ is $\epsilon r m(\tilde A|_{E^u}) \geq 2R + 2K$ (by the choice of r), we conclude that z belongs to the ellipsoid $U_{r,1} := (\tilde f(x) - \tilde A(x)) + (1+\epsilon)\tilde A(U_r)$ which proves equation (3.6). Moreover, observe that $\mathrm{Vol}_{\widetilde W^u_A}(U_{r,1}) = (1+\epsilon)^{d_u}\mathrm{Vol}_{\widetilde W^u_A}(\tilde A(U_r)) = (1+\epsilon)^{d_u} e^{\sum_{i=1}^{d_u}\lambda_i^u(A)}\mathrm{Vol}_{\widetilde W^u_A}(U_r)$.

Now we apply again $\tilde f$ and obtain

$$\tilde f^2(\pi_x^{-1}(U_r)) \subset \tilde f(\pi_{\tilde f(x)}^{-1}(U_{r,1})).$$

As the distance between $\tilde A(U_{r,1})$ and $(1+\epsilon)\tilde A(U_{r,1})$ is larger than $\varepsilon r m(\tilde A|_{E^u})$, similarly as above, we obtain

$$\tilde f(\pi_{\tilde f(x)}^{-1}(U_{r,1})) \subset \pi_{\tilde f^2(x)}^{-1}(U_{r,2}),$$

where $U_{r,2}$ is a translation of $(1+\varepsilon)^2\tilde A^2(U_r)$. In fact, inductively, we obtain

$$\tilde f^{n+1}(\pi_x^{-1}(U_r)) \subset \tilde f(\tilde f^n(\pi_x^{-1}(U_r))) \subset \tilde f(\pi_{\tilde f^n(x)}^{-1}(U_{r,n})) \subset \pi_{\tilde f^{n+1}(x)}^{-1}(U_{r,n+1}), \tag{3.9}$$

where $U_{r,n+1}$ is an ellipsoid with volume less than $(1+\varepsilon)^{(n+1)d_u} e^{(n+1)\sum_{i=1}^{d_u}\lambda_i^u(A)}\mathrm{Vol}_{\widetilde W^u_A}(U_r)$. The last inclusion in equation (3.9) follows using the same arguments as above to prove equation (3.6) substituting the ball U_r by the ellipsoid $U_{r,n}$. □

By Claim 1, for all $x \in M$, we have $|\mathrm{Jac}\,\pi_x^{-1}|$ is uniformly bounded and, consequently, there is a constant $C_0 > 0$ such that

$$\mathrm{Vol}_{\mathcal F}(\tilde f^n \pi_x^{-1}(U_r)) \leq \mathrm{Vol}_{\mathcal F}(\pi_{\tilde f^n(x)}^{-1}(U_{r,n}) \leq C_0(1+\varepsilon)^{nd_u} e^{n\sum_{i=1}^{d_u}\lambda_i^u(A)}\mathrm{Vol}_{\widetilde W^u_A}(U_r).$$

It concludes the proof of the Proposition 3.15 (see Figure 1). □

Proof of Theorem C. Suppose by contradiction that there is a positive volume set $Z \subset \mathbb{T}^d$ such that $\sum_{i=1}^{d_u}\lambda_i^{cu}(f,x) > \sum_{i=1}^{d_u}\lambda_i^u(A)$ for any $x \in Z$.

Let $P\colon \mathbb{R}^d \to \mathbb{T}^d$ be the covering map, $D \subset \mathbb{R}^d$ a fundamental domain and $\widetilde Z = P^{-1}(Z) \cap D$. We have $\mathrm{Vol}(\widetilde Z) > 0$.

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For each $q \in \mathbb{N} \setminus \{0\}$, we define the set

$$Z_q = \left\{ x \in \widetilde{Z}; \sum_{i=1}^{d_u} \lambda_i^{cu}(\tilde{f}, x) > \sum_{i=1}^{d_u} \lambda_i^u(A) + \log\left(1 + \frac{1}{q}\right) \right\}.$$

We have $\bigcup_{q=1}^{\infty} Z_q = \widetilde{Z}$, and thus there is q such that $m(Z_q) > 0$. For each $x \in Z_q$, it follows that

$$\lim_{n\to\infty} \frac{1}{n} \log |\mathrm{Jac}\, \tilde{f}^n(x)|_F| > \sum_{i=1}^{d_u} \lambda_i^u(A) + \log\left(1 + \frac{1}{q}\right).$$

So there is n_0 such that for $n \geq n_0$, we have

$$\begin{aligned} \frac{1}{n} \log |\mathrm{Jac}\, \tilde{f}^n(x)|_F| &> \sum_{i=1}^{d_u} \lambda_i^u(A) + \log\left(1 + \frac{1}{q}\right) \\ &> \frac{1}{n} \log e^{n \sum_{i=1}^{d_u} \lambda_i^u(A)} + \frac{1}{n} \log \left(1 + \frac{1}{q}\right)^n. \end{aligned}$$

This implies that

$$|\mathrm{Jac}\, \tilde{f}^n(x)|_F| > \left(1 + \frac{1}{q}\right)^n e^{n \sum_{i=1}^{d_u} \lambda_i^u(A)}.$$

For every $n > 0$, we define

$$Z_{q,n} = \left\{ x \in Z_q; |\mathrm{Jac}\, \tilde{f}^k(x)|_F| > \left(1 + \frac{1}{q}\right)^k e^{k \sum_{i=1}^{d_u} \lambda_i^u(A)} \text{ for all } k \geq n \right\}.$$

There is $N > 0$ with $\mathrm{Vol}(Z_{q,N}) > 0$.

For each $x \in D$, consider $B_x \subset \mathbb{R}^d$ a foliated box of $\mathcal{F}$. By compactness, there is finite cover $\{B_{x_i}\}_{i=1}^{j}$ covering $\overline{D}$. Since W_f^{cu} is absolutely continuous and the covering map is smooth, then $\mathcal{F}$ is absolutely continuous, and thus there is some i and $p \in B_{x_i}$ such that $\mathrm{Vol}_{\mathcal{F}}(B_{x_i} \cap \mathcal{F}(p) \cap Z_{q,N}) > 0$.

There is a set $\pi_p^{-1}(U_r) \subset \mathcal{F}(p)$, where $\pi_p^{-1}(U_r)$ is as in Proposition 3.15 containing p and r is large enough such that $\mathrm{Vol}_{\mathcal{F}}(\pi_p^{-1}(U_r) \cap Z_{q,N}) > 0$. Let $\alpha > 0$ be such that $\mathrm{Vol}_{\mathcal{F}}(\pi_p^{-1}(U_r) \cap Z_{q,N}) = \alpha \mathrm{Vol}_{\mathcal{F}}(\pi_p^{-1}(U_r))$, and by Proposition 3.15, we have

$$\mathrm{Vol}_{\mathcal{F}}(\tilde{f}^n(\pi_p^{-1}(U_r))) \leq C(1+\varepsilon)^d e^{n \sum \lambda_i^u(A)} \mathrm{Vol}_{\tilde{E}_A^u}(U_r). \tag{3.10}$$

However,

$$\begin{aligned} \mathrm{Vol}_{\mathcal{F}}(\tilde{f}^n(\pi_p^{-1}(U_r))) &= \int_{\pi_p^{-1}(U_r)} |\mathrm{Jac}\, \tilde{f}^n(x)|_F| \, d\mathrm{Vol}_{\mathcal{F}} \\ &\geq \int_{\pi_p^{-1}(U_r) \cap Z_{q,N}} |\mathrm{Jac}\, \tilde{f}^n(x)|_F| \, d\mathrm{Vol}_{\mathcal{F}} \\ &> \int_{\pi_p^{-1}(U_r) \cap Z_{q,N}} \left(1 + \frac{1}{q}\right)^n e^{n \sum_i \lambda_i^u(A)} \, d\mathrm{Vol}_{\mathcal{F}} \\ &> \left(1 + \frac{1}{q}\right)^n e^{n \sum_i \lambda_i^u(A)} \mathrm{Vol}_{\mathcal{F}}(\pi_p^{-1}(U_r) \cap Z_{q,N}) \end{aligned}$$

$$> \left(1+\frac{1}{q}\right)^n e^{n\sum_i \lambda_i^u(A)} \alpha \mathrm{Vol}_{\mathcal{F}}(\pi_p^{-1}(U_r))$$
$$> \left(1+\frac{1}{q}\right)^n e^{n\sum_i \lambda_i^u(A)} \alpha \mathrm{Vol}_{\tilde{E}_A^u}(U_r). \tag{3.11}$$

Equations (3.10) and (3.11) give us a contradiction when n is large enough, and thus prove Theorem C. □

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