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HOLOMORPHIC AUTOMORPHISMS AND PROPER HOLOMORPHIC SELF-MAPPINGS OF A TYPE OF GENERALISED MINIMAL BALL

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Abstract

In this paper, we first give a description of the holomorphic automorphism group of a convex domain which is a simple case of the so-called generalised minimal ball. As an application, we show that any proper holomorphic self-mapping on this type of domain is biholomorphic.

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1. Introduction

Since there is no Riemann mapping theorem in several complex variables, the study of various specific domains is an important and interesting problem. The aim of this article is to study a type of generalised minimal ball.

For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, consider the norm

$$N_*(z) = \sqrt{\frac{\|z\|^2 + \|z \bullet z\|}{2}},$$

where $z \bullet w = \sum_{i=1}^n z_i w_i$ and $\|z\|^2 = z \bullet \bar{z}$. The norm N_* is the smallest norm in $\mathbb{C}^n$ that coincides with the Euclidean norm in $\mathbb{R}^n$, and satisfies some restrictions [3]. The minimal ball $B_* := \{z \in \mathbb{C}^n : N_*(z) < 1\}$ is the first known bounded domain in $\mathbb{C}^n$ which is neither Reinhardt nor homogeneous. For studies of it, see, for example, [5, 7, 8, 10, 15].

The *generalised minimal ball* $\Omega_{d,k,\ell,a}$, introduced in [14], can be regarded as an interpolation between the minimal ball and the Euclidean ball. Fix $d \in \mathbb{N}$ and two d-tuples $k = (k_1, \ldots, k_d) \in \mathbb{N}^d$ and $\ell = (\ell_1, \ldots, \ell_d) \in \mathbb{N}^d$. Let $a = (a_1, \ldots, a_d) \in [1, \infty)^d$.

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Then $\Omega_{d,k,\ell,a}$ is given by

$$\Omega_{d,k,\ell,a} := \Big\{ Z = (Z(1), \ldots, Z(d)) \in M_{k_1,\ell_1}(\mathbb{C}) \times \cdots \times M_{k_d,\ell_d}(\mathbb{C}) : \sum_{j=1}^{d} \|Z(j)\|_*^{2a_j} < 1 \Big\},$$

where $M_{p,q}(\mathbb{C})$ denotes the space of all $p \times q$-matrices with complex entries, and

$$\|M\|_* := \Big(\sum_{j=1}^{p} \Big(\sum_{s=1}^{q} \|z_{js}\|^2 + \Big\| \sum_{s=1}^{q} z_{js}^2 \Big\| \Big) \Big)^{1/2}, \quad M = (z_{js}) \in M_{p,q}(\mathbb{C}).$$

Note that $\Omega_{1,1,n,1}$ is the minimal ball in $\mathbb{C}^n$, up to scaling. If $\ell = (1, \ldots, 1)$ and $a = (1, \ldots, 1)$, then $\Omega_{d,k,\ell,a}$ is the unit ball in $\mathbb{C}^{k_1+\cdots+k_d}$. So $\Omega_{d,k,\ell,a}$ represents many important classes of domains in several complex variables.

In this article, we study the holomorphic automorphism group of a circular bounded convex domain which is a simple case of $\Omega_{d,k,\ell,a}$ and defined by

$$\Omega := \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^m : \|z\|^2 + \|w\|^2 + \|w \bullet w\| < 1\}, \tag{1.1}$$

where $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $w = (w_1, \ldots, w_m) \in \mathbb{C}^m$ with $m \geq 3$. In fact, Ω is just $\Omega_{d,k,\ell,a}$ with $d = 2$, $k = (n, 1)$, $\ell = (1, m)$ and $a = (1, 1)$.

Our first main result describes the holomorphic automorphisms and the second is an Alexander-type theorem for Ω.

Theorem 1.1. *For $m \geq 3$, the holomorphic automorphism group of the domain Ω given by (1.1) comprises transformations $\varphi : (z, w) \mapsto (\widetilde{z}, \widetilde{w})$ of the form*

$$\widetilde{z} = \phi(z), \quad \widetilde{w} = e^{i\theta}\gamma(z)Bw,$$

where

(1) $\phi \in \mathrm{Aut}(\mathbb{B}^n)$;

(2) *$\gamma(z)$ is a holomorphic function defined on $\mathbb{B}^n$ by*

$$\gamma(z) = \frac{\sqrt{1 - \|a\|^2}}{1 - \langle z, a\rangle}, \quad a = \phi^{-1}(0) \in \mathbb{B}^n,$$

where $\langle \cdot\,, \cdot \rangle$ denotes the standard Hermitian inner product on $\mathbb{C}^n$;

(3) *$B \in O(m, \mathbb{R})$, the group of real orthogonal matrices with rank m, and $\theta \in \mathbb{R}$ is a real number.*

Here we require $m \geq 3$ because if $m = 1$ then Ω is biholomorphic to the unit ball and if $m = 2$ then Ω is biholomorphic to $\{(z, w) \in \mathbb{C}^{n+2} : \|z\|^2 + \|w_1\| + \|w_2\| < 1\}$, for which the automorphism group and proper holomorphic self-mappings have been studied in [6] and [2], respectively.

Theorem 1.2. *Any proper holomorphic self-mapping of Ω is a holomorphic automorphism.*

Section 2, contains some preparations and the proof of Theorem 1.1. The proof of Theorem 1.2 is given in Section 3.

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2. The holomorphic automorphism group

Denote by H the set of all the transformations given in Theorem 1.1. One readily checks that H is a subgroup of $\mathrm{Aut}(\Omega)$. We want to show that $H = \mathrm{Aut}(\Omega)$.

Let

$$Q := \{(z, w) \in \mathbb{C}^{n+m} : w \bullet w = 0\}.$$

Then it is easy to see that the nonsmooth boundary points of Ω are exactly $Q \cap \partial\Omega$. A boundary point p is called *complex extreme* if there is no nonconstant holomorphic map $\varphi : \Delta \to \overline{\Omega}$ such that $\varphi(0) = p$. If $p \in \partial\Omega$ is a local holomorphic peak point, then by the maximum principle it is complex extreme.

Lemma 2.1. *The smooth part of the boundary of Ω is strongly pseudoconvex and all boundary points of Ω are complex extreme.*

Proof. Let $p = (z_0, w_0) = (z_1^0, \dots, z_n^0, w_1^0, \dots, w_m^0) \in \partial\Omega$ be a smooth boundary point of Ω, that is, $\|w_0 \bullet w_0\| \neq 0$. Let

$$\rho(z, w) = \|z\|^2 + \|w\|^2 + \|w \bullet w\| - 1$$

be the defining function of Ω. Then

$$\frac{\partial^2 \rho}{\partial z_i \partial \overline{z}_j}(p) = \delta_{ij} \quad \text{for } i, j = 1, \dots, n,$$

$$\frac{\partial^2 \rho}{\partial z_i \partial \overline{w}_j}(p) = 0 \quad \text{for } i = 1, \dots, n,\ j = 1, \dots, m,$$

$$\frac{\partial^2 \rho}{\partial w_i \partial \overline{w}_j}(p) = \delta_{ij} + \frac{w_i^0 \overline{w}_j^0}{\|w_0 \bullet w_0\|} \quad \text{for } i, j = 1, \dots, m.$$

Thus for any nonzero vector $V = (x, y) \in \mathbb{C}^{n+m}$,

$$\sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial z_i \partial \overline{z}_j}(p) x_i \overline{x}_j + \sum_{i,j=1}^{m} \frac{\partial^2 \rho}{\partial w_i \partial \overline{w}_j}(p) y_i \overline{y}_j = \|x\|^2 + \|y\|^2 + \frac{\| \sum_{i=1}^m w_i^0 y_i \|^2}{\|w_0 \bullet w_0\|} \geq \|V\|.$$

So p is strongly pseudoconvex and hence complex extreme.

Now let $p = (z_0, w_0)$ be a nonsmooth boundary point of Ω, that is $w_0 \bullet w_0 = 0$. By definition, we see that p is in the intersection of the closures of $\mathbb{B}^{n+m}$ and Ω. So, there is a local holomorphic peak function at p and hence p is complex extreme. □

For our further study, we need the notion of *complex geodesic*. A complex geodesic of Ω is a holomorphic isometric embedding of the unit disc into Ω. Here the unit disc is equipped with the Poincaré metric and Ω is equipped with the Kobayashi metric (see, for example, [4]). Note that the images of a complex geodesic under holomorphic automorphisms of Ω are also complex geodesics of Ω. In the following, we apply the properties of complex geodesics of the minimal ball (developed by Zwonek [15]), to study the automorphism group of the generalised minimal ball.

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Lemma 2.2. *If* $f \in \mathrm{Aut}(\Omega)$*, then* $f(0,0) \in Q$.

Proof. If $f(0,0) = (0,0)$, then since $(0,0) \in Q$ we are done. Thus, we assume that $f(0,0) = p \neq (0,0)$.

Denote by $\mathbb{C}p$ the complex line through p and $(0,0)$. Since Ω is a balanced convex domain and the boundary of Ω is complex extreme, by [15, Lemma 4], for any $q \in \mathbb{C}p \cap \Omega$ there is a $g \in \mathrm{Aut}(\Omega)$ such that $g(0,0) = q$. If p is not in Q, then it is easy to see that there exists $g_j \in \mathrm{Aut}(\Omega)$ such that $g_j(0,0) \to \widetilde{q} \in \mathbb{C}q \cap \partial\Omega$. By Lemma 2.1, the point $\widetilde{q}$ is strongly pseudoconvex. Thus, by the well-known Wong–Rosay theorem [11, 13], Ω is biholomorphic to the unit ball $\mathbb{B}^{n+m} \subset \mathbb{C}^{n+m}$. But this is a contradiction (since Ω does not even have a smooth boundary). Therefore, we must have $f(0,0) \in Q$. □

Set $\Omega_0 := \{(z,0) \in \mathbb{C}^n \times \mathbb{C}^m \ : \|z\| < 1\} \subset \Omega$.

Lemma 2.3. *If* $f \in \mathrm{Aut}(\Omega)$*, then* $f(\Omega_0) \subset \Omega_0$.

Proof. Assume that there exists a $(z,0) \in \Omega_0$ such that $f(z,0) = (a,b)$ with $b \neq 0$. Choose an automorphism g as given in Theorem 1.1 such that $g(0,0) = (z,0)$. Set $F := f \circ g$. Then, $F(0,0) = f \circ g(0,0) = (a,b)$. By Lemma 2.2, $(a,b) \in Q$ and thus $b \bullet b = 0$.

Choose $(z_0,w_0) \in \Omega$ such that $F(z_0,w_0) = (\overline{a},\overline{b})$. Set $\widetilde{F}(z,w) := \overline{F(\overline{z},\overline{w})}$. Since $F^{-1} \circ \widetilde{F}(0,0) = (z_0,w_0)$, $(z_0,w_0) \in Q$. Since the boundary of Ω is complex extreme, the complex line

$$L := \mathbb{C}(z_0,w_0) \cap \Omega$$

through $(0,0)$ and (z_0,w_0) is a complex geodesic. Since F is an automorphism of Ω, $F(L)$ is a complex geodesic through (a,b) and $(\overline{a},\overline{b})$.

For any $q \in F(L)$, there is a point $p \in L$ such that $q = F(p)$. By [15, Lemma 4], there is a $\varphi \in \mathrm{Aut}(\Omega)$ such that $\varphi(0,0) = p$. Then, $q = F \circ \varphi(0,0) \in Q$ by Lemma 2.2. Since $q \in F(L)$ is arbitrary, we see that $F(L) \subset Q$.

On the other hand, by [15, Lemma 1], the complex geodesic $F(L)$ is contained in the set $\{\lambda(a,b) + \mu(\overline{a},\overline{b}) : \lambda \in \mathbb{C}, \mu \in \mathbb{C}\} \cap Q$. Since $F(L) \subset Q$, for $\lambda_0, \mu_0 \in \mathbb{C}$ giving a point $(\lambda_0 a + \mu_o \overline{a}, \lambda_0 b + \mu_0 \overline{b}) \in F(L)$, we must have

$$(\lambda_0 b + \mu_0 \overline{b}) \bullet (\lambda_0 b + \mu_0 \overline{b}) = 0.$$

Thus, if $b = (b_1, \ldots, b_m)$,

$$(\lambda_0 b + \mu_0 \overline{b}) \bullet (\lambda_0 b + \mu_0 \overline{b}) = \lambda_0^2 b \bullet b + 2\lambda_0\mu_0 \|b\|^2 + \mu_0^2 \overline{b} \bullet \overline{b} = 2\lambda_0\mu_0\|b\|^2 = 0.$$

Since λ_0 and μ_0 are arbitrary, $\|b\|^2 = 0$ which gives $b = 0$. This contradiction shows that $f(\Omega_0) \subset \Omega_0$. □

By the above lemmas and the theorem of Cartan, we see that every element of $g \in \mathrm{Aut}(\Omega)$ can be expressed as $g = \psi_g \circ L_g$, where $\psi_g \in H$ and L_g is a linear automorphism of Ω fixing the origin. We can now prove Theorem 1.1.

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PROOF OF THEOREM 1.1. Let L be a linear automorphism of Ω fixing the origin. Then L is an automorphism of $\mathbb{C}^{n+m}$. We will show that $L : (z, w) \mapsto (\widetilde{z}, \widetilde{w})$ is of the form

$$\widetilde{z} = Uz, \quad \widetilde{w} = Bw,$$

where $U \in U(n)$ is a unitary matrix and $B \in \mathbb{S}^1 \cdot O(m, \mathbb{R})$.

First, note that L preserves

$$Q \cap \partial\Omega = \{(z, w) \in \mathbb{C}^{n+m} : \|z\|^2 + \|w\|^2 = 1, w \bullet w = 0\},$$

since $Q \cap \partial\Omega$ is the nonsmooth part of $\partial\Omega$. Thus, L is a unitary matrix of rank $n + m$. Combining this with the fact that Q is invariant under the dilations

$$d_r : (z, w) \mapsto (rz, rw), \quad r > 0,$$

on $\mathbb{C}^{n+m}$ and $L(d_r(z, w)) = d_r(L(z, w))$ for every $(z, w) \in \mathbb{C}^{n+m}$, shows that $L(Q) = Q$.

Set $L = (L_1, L_2)$, where

$$L_1 : \mathbb{C}^{n+m} \to \mathbb{C}^n, \quad L_2 : \mathbb{C}^{n+m} \to \mathbb{C}^m$$

are linear. Lemma 2.2 implies $L(\Omega_0) = \Omega_0$, so that L_2 does not depend on z. Moreover,

$$L_1|_{\mathbb{C}^n \times \{0\}} : \Omega_0 \to \Omega_0$$

is an automorphism of the unit ball $\mathbb{B}^n$ preserving the origin, which shows that L_1 is a unitary matrix. Therefore, we may assume that L has the form:

$$L\begin{pmatrix} z^t \\ w^t \end{pmatrix} = \begin{pmatrix} U & A \\ 0_{m\times n} & B \end{pmatrix}\begin{pmatrix} z^t \\ w^t \end{pmatrix}$$

with U unitary. Since L preserves Q,

$$(wB^t) \bullet (wB^t) = wB^tBw^t = 0, \tag{2.1}$$

for any $w = (w_1, \ldots, w_m)$ with $w \bullet w = 0$. As in [5, Lemma 4], we can take

$$w = (0, \ldots, 1, \ldots, \pm\sqrt{-1}, \ldots, 0)$$

where the ith component is 1 and the jth component is $\pm\sqrt{-1}$. Set $B = (b_{ij})_{m\times m}$. Then (2.1) is equivalent to

$$\sum_{k=1}^{m}(b_{ik} + \sqrt{-1}b_{jk})^2 = \sum_{k=1}^{m} b_{ik}^2 + \sum_{k=1}^{m} 2\sqrt{-1}b_{ik}b_{jk} - \sum_{k=1}^{m} b_{jk}^2 = 0, \tag{2.2}$$

and

$$\sum_{k=1}^{m}(b_{ik} - \sqrt{-1}b_{jk})^2 = \sum_{k=1}^{m} b_{ik}^2 - \sum_{k=1}^{m} 2\sqrt{-1}b_{ik}b_{jk} - \sum_{k=1}^{m} b_{jk}^2 = 0. \tag{2.3}$$

From (2.2) minus (2.3),

$$\sum_{k=1}^{m} b_{ik}b_{jk} = 0 \quad \text{for } i \neq j,$$

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which, combined with (2.2), gives

$$\sum_{k=1}^{m} b_{ik}b_{ik} = \sum_{k=1}^{m} b_{jk}b_{jk} \quad \text{for } i \neq j.$$

Hence,

$$B^t B = \lambda I_{m\times m} \tag{2.4}$$

for some $\lambda \in \mathbb{C}$. Since L is unitary,

$$\begin{pmatrix} I_{n\times n} & 0 \\ 0 & I_{m\times m} \end{pmatrix} = L\overline{L}^t = \begin{pmatrix} U & A \\ 0_{m\times n} & B \end{pmatrix}\begin{pmatrix} \overline{U}^t & 0_{n\times m} \\ \overline{A}^t & \overline{B}^t \end{pmatrix} = \begin{pmatrix} U\overline{U}^t + A\overline{A}^t & A\overline{B}^t \\ B\overline{A}^t & B\overline{B}^t \end{pmatrix}.$$

Thus,

$$\|\lambda\|^m = \|\det(BB^t)\| = \|\det(B\overline{B}^t)\| = 1.$$

So $\lambda \in \mathbb{S}^1$ and, by (2.4), $B \in \mathbb{S}^1 \cdot O(m, \mathbb{R})$.

Next, we show that $A = 0$. Since L preserves the boundary of Ω,

$$\|Uz + Aw\|^2 + \|Bw\|^2 + \|(Bw) \bullet (Bw)\| = 1 \quad \text{for all } (z, w) \in \partial\Omega.$$

Since $\|Bw\| = \|w\|$ and $\|(Bw) \bullet (Bw)\| = \|w \bullet w\|$, the above equation can be written as

$$\|Uz + Aw\|^2 + \|w\|^2 + \|w \bullet w\| = 1 \quad \text{for all } (z, w) \in \partial\Omega.$$

By taking $\widehat{z} = U_1 z$ with U_1 a unitary matrix, such that $\mathrm{Re}\,\langle \widehat{z}, Aw\rangle = 0$,

$$\|U\widehat{z}\|^2 + \|Aw\|^2 + \|w\|^2 + \|w \bullet w\| = 1 \quad \text{for } (\widehat{z}, w) \in \partial\Omega.$$

Since $\|U\widehat{z}\| = \|z\|$ and $\|z\|^2 + \|w\|^2 + \|w \bullet w\| = 1$, $\|Aw\| = 0$. This implies that $A = 0$. □

3. Proper holomorphic self-mappings

In this section, we show that any proper holomorphic self-mapping of Ω is biholomorphic.

Let us first recall some basic definitions. A nonempty subset $E \subset \mathbb{C}^{n+m}$ is said to be *affine* if whenever $x_1, \ldots, x_k \in E$, then $\sum_{i=1}^k c_i x_i \in E$ for all $c_1, \ldots, c_k \in \mathbb{C}$ with $\sum_{i=1}^k c_i = 1$. It is easy to see that E is affine if and only if $E = V + p$ for some $p \in \mathbb{C}^{n+m}$ and some vector space $V \subset \mathbb{C}^{n+m}$. A subset of Ω is called an affine subset if it is the intersection of Ω with some affine subset of $\mathbb{C}^{n+m}$. By [12, Proposition 2.4.2], if $\psi \in \mathrm{Aut}(\mathbb{B}^{n+m})$ and E is an affine subset of $\mathbb{B}^{n+m}$, then $\psi(E)$ is also an affine subset of $\mathbb{B}^{n+m}$.

First, we prove the following lemma.

Lemma 3.1. *Let $f \in \mathrm{Aut}(\Omega)$. If f fixes a point of Ω, then the set of fixed points is an affine subset of Ω, and in particular it is smooth. Furthermore, f stabilises $Q \cap \Omega$, that is, $f(Q \cap \Omega) \subset Q \cap \Omega$.*

PROOF. By [12, Theorem 2.4.4], if $\psi \in \mathrm{Aut}(\mathbb{B}^{n+m})$ fixes a point of $\mathbb{B}^{n+m}$, then the set of fixed points of ψ is affine. By Theorem 1.1, $\mathrm{Aut}(\Omega)$ is a subgroup of $\mathrm{Aut}(\mathbb{B}^{n+m})$. So if $f \in \mathrm{Aut}(\Omega)$ then f also maps affine subsets to affine subsets. Thus, if f fixes a point of Ω, then the set of fixed points of f is affine. The second assertion is clear by the explicit expression of f, as given in Theorem 1.1. □

In the sequel, we also need the following results.

COROLLARY 3.2 [9, Corollary 3]. *Let $D \subset \mathbb{C}^{m_1n_1+\cdots+m_dn_d}$ be any bounded circular domain which contains the origin.*

(1) *If $f : \Omega_{d,k,\ell,a} \to D$ is a proper holomorphic mapping, then f extends holomorphically to a neighbourhood of $\overline{\Omega}_{d,k,\ell,a}$.*
(2) *If $\partial\Omega_{d,k,\ell,a}$ is not smooth and ∂D is smooth, then there is no proper holomorphic mapping from $\Omega_{d,k,\ell,a}$ to D.*

Now, let $f : \Omega \to \Omega$ be proper. Then f is said to be *factored by automorphisms* if there is a finite subgroup $\Gamma \subset \mathrm{Aut}(\Omega)$ such that, for all $p \in \Omega$,

$$f^{-1} \circ f(p) = \{\gamma(p) : \gamma \in \Gamma\}.$$

THEOREM 3.3 [8, Theorem 2]. *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$, D' a domain in $\mathbb{C}^n$ and $f : D \to D'$ a proper holomorphic mapping with branch locus V_f. Denote by F the restriction of f to $D \setminus f^{-1}(f(V_f))$. Then the following statements are equivalent:*

(1) *there exist $b \in D' \setminus f(V_f)$ and $x \in f^{-1}(b)$ such that $F_*(\pi_1(D \setminus f^{-1}(f(V_f)), x))$ is a normal subgroup of $\pi_1(D' \setminus f(V_f), b)$;*
(2) *f is factored by automorphisms.*

This theorem implies that the branch locus of f is given by

$$V_f = \bigcup_{\{\gamma\in\Gamma:\gamma\neq \mathrm{id}\}} \{z \in D : \gamma(z) = z\}. \tag{3.1}$$

Set $z = (z_1, \ldots, z_n, z_{n+1}, \ldots, z_{n+m}) = (z_1, \ldots, z_n, w_1, \ldots, w_m)$. Then Ω takes the form

$$\Omega = \{(z_1, \ldots, z_{n+m}) \in \mathbb{C}^{n+m} : \rho(z) < 0\},$$

where

$$\rho(z) := \|z_1\|^2 + \cdots + \|z_n\|^2 + \|z_{n+1}\|^2 + \cdots + \|z_{n+m}\|^2 + \|z_{n+1}^2 + \cdots + z_{n+m}^2\| - 1.$$

Let $f : \Omega \to \Omega$ be a proper holomorphic self-mapping. We show that f is biholomorphic by following the steps listed below.

We begin by assuming that f is branched and denote the branch locus of f by V_f.

Step 1. Let $Q = \{z \in \mathbb{C}^{n+m} : z_{n+1}^2 + \cdots + z_{n+m}^2 = 0\}$. Then we see that $f(V_f) = Q \cap \Omega$.

Step 2. Since $\pi_1(\Omega \setminus Q) = \mathbb{Z}$ (the proof of this fact is almost exactly as in [8, Lemma 6], thus we omit it), f can be factored by a finite subgroup Γ of $\mathrm{Aut}(\Omega)$. In particular, $f^{-1} \circ f(V_f) = V_f$ by (3.1). By Step 1, $f^{-1}(Q \cap \Omega) = V_f$.

Step 3. Let $f^2 := f \circ f$ and let V_{f^2} be the branch locus of f^2. By definition, $V_{f^2} = V_f \cup f^{-1}(V_f)$. Then $f^2(V_{f^2}) = f \circ f(V_f) \cup f(V_f) = f(Q \cap \Omega) \cup (Q \cap \Omega)$. At the same time, by Step 1, $f^2(V_{f^2}) = Q \cap \Omega$. Thus, $Q \cap \Omega = f(Q \cap \Omega) \cup (Q \cap \Omega)$. Since f is proper and $Q \cap \Omega$ is an analytic subset, $f(Q \cap \Omega)$ is also an analytic subset. By the irreducibility of $Q \cap \Omega$, we conclude that $f(Q \cap \Omega) = Q \cap \Omega$.

Step 4. From $f(Q \cap \Omega) = Q \cap \Omega$, it follows that $f^{-1} \circ f(Q \cap \Omega) = f^{-1}(Q \cap \Omega)$. By Theorem 3.3,

$$f^{-1} \circ f(Q \cap \Omega) = \bigcup_{\gamma \in \Gamma} \{\gamma(z) : z \in Q \cap \Omega\}.$$

From Lemma 3.1, $\gamma(z) \in Q \cap \Omega$ for all $z \in Q \cap \Omega$. Thus, $f^{-1} \circ f(Q \cap \Omega) \subset Q \cap \Omega$. Again, by the irreducibility of $Q \cap \Omega$, we have $f^{-1} \circ f(Q \cap \Omega) = Q \cap \Omega$. This implies that $Q \cap \Omega = f^{-1}(Q \cap \Omega)$.

Step 5. By Steps 2 and 4, $V_f = Q \cap \Omega$. Since $Q \cap \Omega$ is irreducible and f can be factored by automorphisms, there is a $\gamma \in \Gamma$ such that $V_f = \{z \in \Omega : \gamma(z) = z\}$. Thus, by Lemma 3.1, $Q \cap \Omega = V_f$ is smooth. But this is impossible since $Q \cap \Omega$ is singular at 0. This shows that the initial assumption is not true. Since Ω is simply connected, this implies that f is biholomorphic and the proof of this claim is complete.

Remark 3.4. The above analysis shows that to prove Theorem 1.2 we only need to show that $f(V_f) = Q \cap \Omega$. By Corollary 3.2, f can be holomorphically extended to a mapping on a domain larger than Ω, which we will continue to denote by f. Let $r(z) = z_{n+1}^2 + \cdots + z_{n+m}^2$. Then $Q = \{z \in \mathbb{C}^{n+m} : r(z) = 0\}$. If we can show that $f(\overline{V}_f \cap \partial\Omega) \subset Q \cap \partial\Omega$, then $r \circ f|_{\overline{V}_f \cap \partial\Omega} = 0$. By the maximal principle, $r \circ f|_{V_f} = 0$. Hence, $f(V_f) \subset Q \cap \Omega$. By Remmert's proper mapping theorem and the irreducibility of the analytic set $Q \cap \Omega$, we see that $f(V_f) = Q \cap \Omega$.

Now we show that $f(\overline{V}_f \cap \partial\Omega) \subset Q \cap \partial\Omega$, by contradiction. Let $p \in \overline{V}_f \cap \partial\Omega$ and $q := f(p)$. Assume that $q \in \partial\Omega$ is a smooth boundary point.

Note that by the strong pseudoconvexity of the smooth part of the boundary of Ω, we have $\overline{V}_f \cap \partial\Omega \subset \partial\Omega \cap Q$. Thus, without loss of generality (composing f with automorphisms of Ω if necessary), we can assume that

$$p = (\underbrace{0, \ldots, 0, a}_{n}, \underbrace{0, \ldots, 0, ib, b}_{m}),$$

where $a > 0$, $b > 0$ and $\|a\|^2 + \|ib\|^2 + \|b\|^2 = 1$. (One can also have $a = 0$, but the proof in that case, which we omit, would be similar but simpler.) We can get a contradiction by using scaling techniques as follows.

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Step 1: Localisation at p.

The domain $\Omega - p$ is represented by

$$\Omega - p = \Big\{z \in \mathbb{C}^{n+m} : \sigma_1(z) := \sum_{i=1}^{n} \|z_i\|^2 + 2a\,\mathrm{Re}\, z_n + \sum_{j=n+1}^{n+m} \|z_j\|^2 + 2b\,\mathrm{Re}\,(z_{n+m} - iz_{n+m-1})$$
$$+ \|z_{n+1}^2 + \cdots + z_{n+m}^2 + 2b(z_{n+m} + iz_{n+m-1})\| < 0\Big\}.$$

Under a linear transformation g_1 defined by

$$g_1(z)_j = \zeta_j = \begin{cases} z_j & \text{for } j = 1, \ldots, n+m-2, \\ z_{n+m} + iz_{n+m-1} & \text{for } j = n+m-1, \\ z_{n+m} - iz_{n+m-1} & \text{for } j = n+m, \end{cases}$$

we have $g_1(\Omega - p) = \{\zeta \in \mathbb{C}^{n+m} : \sigma_2(\zeta) < 0\}$, where

$$\sigma_2(\zeta) := \sum_{i=1}^{n} \|\zeta_i\|^2 + 2a\,\mathrm{Re}\,\zeta_n + 2b\,\mathrm{Re}\,\zeta_{n+m} + \frac{1}{2}(\|\zeta_{n+m}\|^2 + \|\zeta_{n+m-1}\|^2)$$
$$+ \sum_{j=n+1}^{n+m-2} \|\zeta_j\|^2 + \|\zeta_{n+1}^2 + \cdots + \zeta_{n+m-2}^2 + 2b\zeta_{n+m-1} + \zeta_{n+m-1}\zeta_{n+m}\|.$$

Define the linear transformation g_2 by

$$g_2(s)_j = s_j = \begin{cases} \zeta_j & \text{for } j = 1, \ldots, n+m-1, \\ a\zeta_n + b\zeta_{n+m} & \text{for } j = n+m, \end{cases}$$

and set $g = g_2 \circ g_1$. Then $G := g(\Omega) = \{s \in \mathbb{C}^{n+m} : \sigma_3(s) < 0\}$, where

$$\sigma_3(s) := 2\,\mathrm{Re}\, s_{n+m} + \sum_{j=1}^{n} \|s_j\|^2 + \sum_{j=n+1}^{n+m-2} \|s_j\|^2 + \frac{\|as_n - s_{n+m}\|^2}{2b^2} + \frac{\|s_{n+m-1}\|^2}{2}$$
$$+ \left\| s_{n+1}^2 + \cdots + s_{n+m-2}^2 + 2bs_{n+m-1} + \frac{1}{b} s_{n+m-1}(s_{n+m} - as_n) \right\|.$$

Step 2: Centring. Write $z = (z', z_{n+m}) \in \mathbb{C}^{n+m}$. Now take $t^k = (0', -\delta_k) \in g(\Omega)$, $k = 1, 2, \ldots$, where δ_k is a sequence of positive numbers converging to 0. Since f is continuous, the sequence $\{q^k\}$, $q^k := f \circ g^{-1}(t^k)$, converges to $q = f \circ g(0)$. Let $V \subset \mathbb{C}^{n+m}$ be a neighbourhood of q such that $V \cap \partial\Omega$ is strongly pseudoconvex. Let $\xi^k = (\xi_1^k, \ldots, \xi_{n+m}^k)$ be the projection of $q^k = (q_1^k, \ldots, q_{n+m}^k)$ on the boundary of Ω. Then ξ^k is unique if V is small enough. Now consider the centring mapping:

$$h_k(z) = \begin{cases} z_j^* = \dfrac{\partial\rho}{\partial\bar{z}_{n+m}}(\xi^k)(z_j - \xi_j^k) - \dfrac{\partial\rho}{\partial\bar{z}_j}(\xi^k)(z_{n+m} - \xi_{n+m}^k), & j = 1, \ldots, n+m-1, \\ z_{n+m}^* = \displaystyle\sum_{j=1}^{n+m} \frac{\partial\rho}{\partial\bar{z}_j}(\xi^k)(z_j - \xi_j^k). \end{cases}$$

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The mapping h_k maps ξ^k to 0 and the real normal vector to $\partial\Omega$ at ξ^k into the line $\{z_1 = \cdots = z_{n+m-1} = 0, \operatorname{Im} z_{n+m} = 0\}$. Let $D_k := h_k(\Omega) = \{z^* \in \mathbb{C}^{n+m} : \rho_k(z^*) < 0\}$ where $\rho_k := \rho \circ h_k^{-1}$. Set $\gamma_k := \delta(h_k(q^k), D_k)$, where $\delta(h_k(q^k), D_k)$ denotes the Euclidean distance between $h_k(q^k)$ and ∂D_k. Then, $h_k(q^k) = (0', -\gamma_k)$. The sequence of proper holomorphic mappings defined by

$$f_k = h_k \circ f \circ g^{-1} : G \to D_k$$

satisfies $f_k(0', -\delta_k) = (0', -\gamma_k)$ for all k.

Step 3: *Stretching*. For the stretching of G, consider the sequence of inhomogeneous dilations of coordinates

$$\Lambda_k(s) = \left(\frac{s_1}{\sqrt{\delta_k}}, \ldots, \frac{s_{n+m-2}}{\sqrt{\delta_k}}, \frac{s_{n+m-1}}{\delta_k}, \frac{s_{n+m}}{\delta_k}\right).$$

Then, we get a sequence of domains $\widehat{G}_k$ with defining functions

$$\begin{aligned}\widehat{\varphi}_k(s) &:= \frac{1}{\delta_k}\sigma_3 \circ \Lambda_k^{-1}(s)\\ &= 2\operatorname{Re} s_{n+m} + \sum_{j=1}^{n}\|s_j\|^2 + \sum_{j=n+1}^{n+m-2}\|s_j\|^2 + \frac{\|as_n - \sqrt{\delta_k}s_{n+m}\|^2}{2b^2} + \delta_k\frac{\|s_{n+m-1}\|^2}{2}\\ &\quad + \left\|s_{n+1}^2 + \cdots + s_{n+m-2}^2 + 2bs_{n+m-1} + \frac{1}{b}s_{n+m-1}(\delta_k s_{n+m} - a\sqrt{\delta_k}s_n)\right\|.\end{aligned}$$

Let $\delta_k \to 0$. Then $\widehat{\varphi}_k$ converges uniformly on compact sets of $\mathbb{C}^{n+m}$ to

$$\begin{aligned}\widehat{\varphi}(s) := 2\operatorname{Re} s_{n+m} + \sum_{j=1}^{n-1}\|s_j\|^2 + \left(1 + \frac{a^2}{2b^2}\right)\|s_n\|^2 + \sum_{j=n+1}^{n+m-2}\|s_j\|^2\\ + \|s_{n+1}^2 + \cdots + s_{n+m-2}^2 + 2bs_{n+m-1}\|.\end{aligned}$$

Set $\widehat{G} := \{s \in \mathbb{C}^{n+m} : \widehat{\varphi}(s) < 0\}$. By the linear transformation

$$\eta_j = \begin{cases} s_j & \text{for } j = 1, \ldots, n-1, n+1, \ldots, n+m-2,\\ \sqrt{1 + \dfrac{a^2}{2b^2}}\, s_n & \text{for } j = n,\\ s_{n+1}^2 + \cdots + s_{n+m-2}^2 + 2bs_{n+m-1} & \text{for } j = n+m-1,\\ s_{n+m} & \text{for } j = n+m,\end{cases}$$

we see that $\widehat{G}$ is biholomorphic to

$$E_1 := \{\eta \in \mathbb{C}^{n+m} : 2\operatorname{Re}\eta_{n+m} + \|\eta_1\|^2 + \cdots + \|\eta_{n+m-2}\|^2 + \|\eta_{n+m-1}\| < 0\}.$$

The fractional transformation

$$(\eta_1, \ldots, \eta_{n+m-2}, \eta_{n+m-1}, \eta_{n+m}) \mapsto \left(\frac{\sqrt{2}\eta_1}{\eta_{n+m} - 1}, \ldots, \frac{\sqrt{2}\eta_{n+m-2}}{\eta_{n+m} - 1}, \frac{2\eta_{n+m-1}}{(\eta_{n+m} - 1)^2}, \frac{\eta_{n+m} + 1}{\eta_{n+m} - 1}\right).$$

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maps E_1 biholomorphically onto the domain

$$E_2 := \{\eta \in \mathbb{C}^{n+m} : \|\eta_1\|^2 + \cdots + \|\eta_{n+m-2}\|^2 + \|\eta_{n+m-1}\| + \|\eta_{n+m}\|^2 < 1\}.$$

For the stretching of $\widehat{G}$, consider the inhomogeneous dilation of coordinates

$$L_k(z^*) = \Big(\frac{z_1^*}{\sqrt{\gamma_k}}, \dots, \frac{z_{n+m-1}^*}{\sqrt{\gamma_k}}, \frac{z_{n+m}^*}{\gamma_k}\Big).$$

Then, D_k corresponds to $\widehat{D}_k$ with defining function $\widehat{\rho}_k(z^*) = (1/\gamma_k)\rho_k \circ L_k^{-1}(z^*)$. We can see that $\widehat{\rho}_k(z^*)$ converges uniformly on compact sets of $\mathbb{C}^{n+m}$ to

$$\widehat{\rho}(z^*) = 2\,\mathrm{Re}\, z_{n+m}^* + \|z_1^*\|^2 + \cdots + \|z_{n+m-1}^*\|^2.$$

Set $\widehat{D} := \{z^* \in \mathbb{C}^{n+m} : \widehat{\rho}(z^*) < 0\}$. Then $\widehat{D}$ is biholomorphic to the unit ball $\mathbb{B}^{n+m}$ by Cayley's transformation.

Step 4: Construction of proper holomorphic mappings. What we have done can be summarised by the sequence

$$\widehat{G}_k = \Lambda_k(G) \xrightarrow{\Lambda_k^{-1}} G = g(\Omega) \xrightarrow{g^{-1}} \Omega \xrightarrow{f} \Omega \xrightarrow{h_k} D_k = h_k(\Omega) \xrightarrow{L_k} \widehat{D}_k = L_k(D_k)$$

where

(1) $\widehat{G}_k = \{\widehat{\varphi}_k < 0\}$, $\widehat{\varphi}_k$ converges uniformly on compact subsets to $\widehat{\varphi}$ and $\widehat{G} = \{\widehat{\varphi} < 0\}$; by step 3 the domain $\widehat{G}$ is biholomorphic to E_2.
(2) $t^k = (0', -\delta_k) \in G$, where $\delta_k > 0$ tends to 0.
(3) $q^k = f \circ g^{-1}(t^k)$, where f is proper and g is continuous, so that q^k converges to $f \circ g^{-1}(0) = q \in \partial\Omega$; by assumption, Ω is strongly pseudoconvex near q.
(4) $\xi^k \in \partial\Omega$, with $\|q^k - \xi^k\| = \delta(q^k, \Omega)$ satisfying $h_k(\xi^k) = 0$, and $h_k(q^k) = (0', -\gamma_k)$, where $\gamma_k = \delta(h_k(q^k), D_k)$; if $f_k = h_k \circ f \circ g^{-1}$, then $f_k(0', -\delta_k) = (0', \gamma_k)$.
(5) $\widehat{D}_k = \{\widehat{\rho}_k < 0\}$, $\widehat{\rho}_k$ converges uniformly to $\widehat{\rho}$ and $\widehat{D} = \{\widehat{\rho} < 0\}$, which is biholomorphic to $\mathbb{B}^{n+m}$.

The sequence of proper holomorphic mappings is defined by

$$\widehat{f}_k(z) = L_k \circ h_k \circ f \circ g^{-1} \circ \Lambda_k^{-1}(z)$$

and it satisfies $\widehat{f}_k(0', -1) = (0', -1)$. Fix any compact set $K \subset \widehat{G}$. Then $\widehat{f}_k$ is well defined on K for large k. By exhausting $\widehat{G}$ with an increasing sequence of compact sets, we may assume that $\{\widehat{f}_k\}$ converges to a holomorphic mapping $\widehat{f} : \widehat{G} \to \overline{\widehat{D}}$. Since $\widehat{D}$ is strongly pseudoconvex, thus taut, and $\widehat{f}(0', -1) = (0', -1) \in \widehat{D}$, it follows that $\widehat{f}(\widehat{G}) \subset \widehat{D}$. Finally, the limit mapping $\widehat{f}$ is proper. (The proof of this is almost exactly as in [8, Lemmas 4 and 5] and thus we omit it.)

Now we give the proof of Theorem 1.2.

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PROOF OF THEOREM 1.2. By the above steps, we have a proper holomorphic mapping $\widehat{f}: \widehat{G} \to \widehat{D}$, where $\widehat{G}$ is biholomorphic to E_2 and $\widehat{D}$ is biholomorphic to $\mathbb{B}^{n+m}$. So we obtain a proper holomorphic mapping $\widetilde{f}: E_2 \to \mathbb{B}^{n+m}$. Define $\phi : \mathbb{B}^{n+m} \to E_2$ by

$$\phi(z_1, \dots, z_{n+m-2}, z_{n+m-1}, z_{n+m}) = (z_1, \dots, z_{n+m-2}, z_{n+m-1}^2, z_{n+m}).$$

Then ϕ is proper but not biholomorphic. Observe that

$$\tilde{f} \circ \phi : \mathbb{B}^{n+m} \to \mathbb{B}^{n+m}$$

is proper. By [1], $\tilde{f} \circ \phi$ is biholomorphic. This is impossible since ϕ is branched. Therefore, our assumption $p = f(q) \notin \partial\Omega \cap Q$ must be false. By Remark 3.4, f is biholomorphic. This completes the proof. □

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