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NON-COPRIME QUADRATIC SYSTEMS

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A recent result of Huang and Reyn on quadratic systems is reformulated and given a clearer proof.

In this work we study plane quadratic systems

$$x' = P(x, y) = \sum_{i+k=0}^{2} a_{ik}x^i y^k$$
$$y' = Q(x, y) = \sum_{i+k=0}^{2} b_{ik}x^i y^k,$$

where $' = \dfrac{d}{dt}$, whose second degree terms

$$P_2(x, y) = a_{20}x^2 + a_{11}xy + a_{02}y^2, \ Q_2(x, y) = b_{20}x^2 + b_{11}xy + b_{02}y^2$$

have a common linear factor. As a handy abbreviation, we shall call such systems *non-coprime*.

Hilbert's problem of determining an upper bound for the number of limit cycles of a polynomial system of given degree is at present intractable even for quadratic systems. Non-coprime quadratic systems are of interest in this connection since they are not totally intractable and since, unlike some other classes of quadratic systems which have been successfully studied, they may have more than one limit cycle. Žilevič [**5, 6**] gives examples of non-coprime quadratic systems with exactly two limit cycles, each surrounding a different critical point. A recent result of Huang and Reyn [4], which the authors kindly communicated to me, may be interpreted as saying that if a non-coprime quadratic system has limit cycles surrounding different critical points, then at least one critical point is surrounded by exactly one limit cycle. We give here a proof of this interesting result which appears to us to be clearer than the original one. Our papers [**2, 3**] may serve as covenient references for those properties of quadratic systems which we require.

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It is shown in [2] that a limit cycle of a quadratic system has a convex interior and surrounds a unique critical point, which is necessarily a focus. Thus the quadratic systems in which we are interested have two foci. It is also shown in [2] that a quadratic system has at most two foci, since any two foci are oppositely oriented. By Theorems A and B in [3], a quadratic system has at most one limit cycle if it has an invariant straight line or if the second degree terms are proportional. Thus such systems may be excluded from our consideration.

LEMMA 1. *If a non-coprime quadratic system has a focus or centre, if it has no invariant straight line, and if its second degree terms are not proportional, then by a non-singular affine transformation and a dilation of the time scale it can be brought to the form*

$$\begin{aligned} x' &= dx - y + \ell x^2 + mxy \\ y' &= x + ax^2 + bxy, \end{aligned} \tag{1}$$

where $-2 < d < 2$, $a \neq 0$ *and* $b\ell - am \neq 0$.

PROOF: The hypotheses are invariant under an arbitrary non-singular affine transformation and a dilation of the time scale. Consequently we may assume that an arbitrary focus or centre is located at the origin and that the quadratic system is in Ye's normal form

$$\begin{aligned} x' &= dx - y + \ell x^2 + mxy + ny^2 \\ y' &= x + ax^2 + bxy. \end{aligned}$$

Then $-2 < d < 2$ and $a \neq 0$. If $n = 0$, then $b\ell - am \neq 0$ and there is nothing more to do. If $n \neq 0$, then

$$\ell x^2 + mxy + ny^2 = (ax + by)(a'x + b'y).$$

Put

$$\xi = ax + by, \ \eta = y + \alpha\xi,$$

where $\alpha = -b/\left(a^2 + dab + b^2\right)$. Then

$$\begin{aligned} \xi' &= \alpha_{10}\xi + \alpha_{01}\eta + \alpha_{20}\xi^2 + \alpha_{11}\xi\eta \\ \eta' &= \beta_{10}\xi + \beta_{20}\xi^2 + \beta_{11}\xi\eta, \end{aligned}$$

where $\alpha_{01}\beta_{10} < 0$ and $\beta_{20} \neq 0$. By scaling we can bring this system to the form (1). □

By Theorem C of [3], the quadratic system (1) has at most one limit cycle if $m = 0$. Thus we now restrict attention to the system (1), where

$$-2 < d < 2, \ a \neq 0, \ b\ell - am \neq 0, \ m \neq 0.$$

If in (1) we make the scaling transformation $x \to \lambda x$, $y \to \varepsilon\lambda y$, $t \to \varepsilon\tau$, where $\lambda \neq 0$ and $\varepsilon = \pm 1$, we obtain a quadratic system of the same form:

$$\begin{aligned} x' &= \varepsilon dx - y + \varepsilon\lambda\ell x^2 + \lambda mxy \\ y' &= x + \lambda ax^2 + \varepsilon\lambda bxy. \end{aligned}$$

A limit cycle of (1) cannot intersect the line $x = 1/m$, since $x' = (d + \ell m)/m^2$ on this line. In particular, a limit cycle surrounding the origin must lie in the half-plane $1 - mx > 0$. A limit cycle surrounding the origin must also lie in the half-plane $1 + ax + by > 0$, since its interior is convex and it intersects the y-axis above and below the origin. If the system (1) has a critical point $(x_0, y_0) \neq (0, 0)$, then in addition a limit cycle surrounding the origin cannot intersect the line $x = x_0$, since it cannot surround (x_0, y_0) and since y' is of constant sign for $y > y_0$ and for $y < y_0$.

If we change the independent variable by setting $\dfrac{d\tau}{dt} = 1 - mx$, then (1) is replaced by the Liénard equation

$$\frac{d^2x}{d\tau^2} - f(x)\frac{dx}{d\tau} + g(x) = 0, \tag{2}$$

where $f(x) = f_1(x)/(1 - mx)^2$, $g(x) = xg_1(x)/(1 - mx)^2$ and

$$\begin{aligned} f_1(x) &= d + (b + 2\ell)x - (b + \ell)mx^2, \\ g_1(x) &= 1 + (a - m + bd)x + (b\ell - am)x^2. \end{aligned} \tag{3}$$

Thus $f_1(1/m) = (\ell + dm)/m$, $g_1(1/m) = b(\ell + dm)/m^2$.

The finite critical points of (1), other than the origin, are the points (x_0, y_0) such that

$$\begin{aligned} (d + \ell x_0)x_0 &= (1 - mx_0)y_0, \\ 1 + ax_0 + by_0 &= 0. \end{aligned}$$

It follows that $g_1(x_0) = 0$. At the critical point (x_0, y_0), the Jacobian $P_xQ_y - P_yQ_x$ has the value

$$\begin{aligned} D_0 &= (d + 2\ell x_0 + my_0)bx_0 + (1 - mx_0)ax_0 \\ &= (b\ell - am)x_0^2 - 1 \end{aligned}$$

and the divergence $P_x + Q_y$ has the value

$$T_0 = d + (b + 2\ell)x_0 + my_0.$$

Hence $(1-mx_0)T_0 = f_1(x_0)$ and

$$T_0^2 - 4D_0 = [d + (2\ell - b)x_0 + my_0]^2 - 4ax_0(1 - mx_0).$$

If we translate the critical point (x_0, y_0) to the origin by putting $x = x_0 + \xi$, $y = y_0 + \eta$, we obtain the system

$$\begin{aligned} \xi' &= \alpha\xi - \beta\eta + \ell\xi^2 + m\xi\eta \\ \eta' &= ax_0\xi + bx_0\eta + a\xi^2 + b\xi\eta, \end{aligned}$$

where

$$\alpha = d + 2\ell x_0 + my_0, \ \beta = 1 - mx_0.$$

If $\beta \neq 0$ and we put

$$\xi = \widetilde{x}, \ \eta = \widetilde{y} - \theta\widetilde{x},$$

where $\theta = bx_0/\beta$, this is transformed into the system

$$\begin{aligned} \widetilde{x}' &= T_0\widetilde{x} - \beta\widetilde{y} + (\ell - bmx_0/\beta)\widetilde{x}^2 + m\widetilde{x}\,\widetilde{y} \\ \widetilde{y}' &= D_0\widetilde{x}/\beta + \theta(\ell - b/\beta + a\beta/bx_0)\widetilde{x}^2 + (b/\beta)\widetilde{x}\,\widetilde{y}. \end{aligned}$$

If $D_0 > 0$ and we further put

$$\widetilde{x} = \nu\beta x, \ \widetilde{y} = y, \ t = \nu\tau,$$

where $\nu = D_0^{-1/2}$, we obtain a system of the original form (1):

$$\begin{aligned} \frac{dx}{d\tau} &= d_0x - y + \ell_0x^2 + m_0xy \\ \frac{dy}{d\tau} &= x + a_0x^2 + b_0xy, \end{aligned}$$

where

$$\begin{aligned} d_0 = \nu T_0, \ \ell_0 &= \nu^2[\ell(1 - mx_0) - bmx_0], \ m_0 = \nu m, \\ a_0 &= \nu^3[(a+m)(1 - mx_0) + (\ell - b + dm)bx_0], \ b_0 = \nu^2 b. \end{aligned}$$

Suppose the system (1) has a focus (x_+, y_+) distinct from the origin. Then $b\ell - am > 0$, since (with an obvious notation) $D_+ > 0$, and $ax_+(1 - mx_+) > 0$, since $T_+^2 - 4D_+ < 0$. Around the focus at the origin paths are described anticlockwise, since $Q_x = 1 > 0$. Since the two foci are oppositely oriented, at (x_+, y_+) we must have $Q_x = ax_+ < 0$ and hence $1 - mx_+ < 0$.

If $b = 0$ then $1 + m/a < 0$, since $x_+ = -1/a$. It follows that $y_+ = (\ell - ad)/a(a+m)$ and that the system (1) has no finite critical points besides the two foci. Since not only $g_1(x_+) = 0$ but also $g_1(1/m) = 0$, it will be convenient in the case $b = 0$ to put $x_- = 1/m$.

If $b \neq 0$ then the quadratic equation $g_1(x) = 0$ has a root $x_- \neq x_+$, since $D_+ > 0$. It follows that the system (1) has exactly one finite critical point (x_-, y_-) besides the two foci. Moreover x_+ and x_- have the same sign, since $x_+ x_- = (b\ell - am)^{-1}$. Hence (x_-, y_-) is a saddle, since

$$\begin{aligned} x_+ D_- + x_- D_+ &= x_+[(b\ell - am)x_-^2 - 1] + x_-[(b\ell - am)x_+^2 - 1] \\ &= (x_- - x_+) + (x_+ - x_-) = 0, \end{aligned}$$

and $|x_+| > |x_-|$, since

$$(b\ell - am)(x_+^2 - x_-^2) = D_+ - D_- > 0.$$

Suppose now that the system (1) has limit cycles surrounding two different critical points, which are necessarily the origin and the point (x_+, y_+). Since the Liénard equation (2) has no limit cycles in a simply-connected region in which f is of constant sign, the quadratic equation $f_1(x) = 0$ has roots ξ_+, ξ_- such that $m\xi_- < 1 < m\xi_+$. Moreover $m(x_- - \xi_-) > 0$, since a limit cycle of (1) which surrounds the origin cannot intersect the line $x = x_-$.

Since $f_1(x)$ has opposite signs for $x = 1/m$ and $x = \infty$, it follows that

$$(b + \ell)(\ell + dm) > 0. \tag{4}$$

In particular $b + \ell \neq 0$. We are going to show that also $d \neq 0$. The argument is derived from Chen and Wang [1].

Assume on the contrary that $d = 0$. Then $b + 2\ell \neq 0$, since f_1 has distinct roots. If we put

$$B(x, y) = |1 - mx|^{-1+\gamma b/m} e^{\gamma(bx - my)},$$

where $\gamma = (b + 2\ell)/m$, then

$$(BP)_x + (BQ)_y = [b\ell - am + \ell(m\gamma^{-1} - b)(1 - mx)^{-1}]\gamma x^2 B.$$

Since $b\ell - am > 0$, it follows from Dulac's criterion that there is no limit cycle in the half-plane $1 - mx > 0$ if $\ell(m\gamma^{-1} - b) \geqslant 0$, and no limit cycle in the half-plane $1 - mx < 0$ if $\ell(m\gamma^{-1} - b) \leqslant 0$. Since there are limit cycles in both half-planes, we have a contradiction.

It will now be shown that if $b = 0$ or $b(b+\ell) < 0$, then $d\ell m < 0$. By changing the signs of both y and t, if necessary, we may suppose that $b \leqslant 0 < b + \ell$. Then $\ell > 0$. Assume, contrary to the claim, that $dm > 0$. By changing the signs of both x and t, if necessary, we may suppose that $d > 0$, $m > 0$. Then $a < 0$, since $b\ell - am > 0$. The system

$$(1)_0 \qquad \begin{aligned} x' &= -y + \ell x^2 + mxy \\ y' &= x + ax^2 + bxy, \end{aligned}$$

obtained from (1) by setting $d = 0$, has an unstable weak focus at the origin, since $\ell m - a(b + 2\ell) > 0$. On the other hand, any path of $(1)_0$ which intersects a limit cycle L of (1) surrounding the origin crosses from the exterior of L to the interior, since

$$(P_0 - P)Q = -dx^2(1 + ax + by) \leqslant 0 \quad \text{on } L.$$

Consequently, by the Poincaré-Bendixson theorem, $(1)_0$ has a periodic orbit L_0 in the interior of L. Moreover L_0 surrounds the origin, since it lies in the half-plane $1 + ax + by > 0$. Hence, by the argument above, $\ell(m\gamma^{-1} - b) < 0$. Thus $m^2/(b + 2\ell) - b < 0$. Since $b + 2\ell > 0$, this is a contradiction.

PROPOSITION 2. *Suppose the quadratic system (1) has limit cycles surrounding the origin and another critical point. If $b = 0$, or $b(b+\ell) < 0$, then exactly one limit cycle surrounds the origin and its characteristic exponent is non-zero.*

PROOF: We restrict attention again to the case $b \leqslant 0 < b + \ell$. Then $\ell > 0$, $dm < 0$ and we may suppose $d > 0 > m$. Hence $a > 0$. Since $x_+ < 1/m$, $g_1(0) > 0$ and $g_1(1/m) \leqslant 0$, by (4), we must have $1/m \leqslant x_- < 0$. Any limit cycle of (2) which surrounds the origin must lie in the strip $x_- < x < \infty$. The unique root $\xi_- > 1/m$ of the quadratic equation $f_1(x) = 0$ satisfies not only $\xi_- > x_-$ but also $\xi_- < 0$, since $\xi_+\xi_- = -d/(b+\ell)m$.

Thus on the interval (x_-, ∞) we have $g(x) \gtrless 0$ according as $x \gtrless 0$ and $f(x) \gtrless 0$ according as $x \gtrless \xi_-$. To complete the proof of the proposition it is sufficient to show that f/g is a decreasing function on the interval (x_-, ξ_-) and on the interval $(0, \infty)$, since this will imply that all the hypotheses of Theorem 1 in [3] are satisfied.

Since $f/g = f_1/xg_1$, we need only show that $M(x) = xg_1f_1' - f_1(g_1 + xg_1')$ is negative on both intervals. Moreover, since the leading coefficients of f_1 and g_1 are positive, we may replace f_1 by $(x - \xi_+)(x - \xi_-)$ and g_1 by $(x - x_+)(x - x_-)$. Then

$$\begin{aligned} M(x) &= -(x - x_-)(x - \xi_+)[(x - \xi_-)^2 - \xi_-(\xi_- - x_+)] - x(x - x_+)(x - \xi_-)(x_- - \xi_+) \\ &< 0 \text{ for } x > 0 \text{ and for } x_- < x < \xi_-. \end{aligned}$$ ∎

We can now deduce without difficulty the main result.

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THEOREM 3. *Suppose a non-coprime quadratic system has limit cycles surrounding different critical points.*

If the system has more than two finite critical points, then at least one critical point is surrounded by exactly one limit cycle and its characteristic exponent is non-zero.

If the system has only two finite critical points, then each is surrounded by exactly one limit cycle. Moreover, their characteristic exponents are non-zero and of opposite signs.

PROOF: We may suppose that the quadratic system has the form (1). Then $b+\ell \neq 0$ and the system has two finite critical points if and only if $b = 0$. If $b = 0$ or $b(b+\ell) < 0$ then, by Proposition 2, exactly one limit cycle surrounds the origin and its characteristic exponent is non-zero. In fact the characteristic exponent has the opposite sign to d, since the limit cycle and the origin cannot be both stable or both unstable.

By translating the second focus (x_+, y_+) to the origin we can bring the system again to the form (1), with b replaced by $b_+ = \nu^2 b$ and $b(b+\ell)$ replaced by $b_+(b_+ + \ell_+) = \nu^4 b(b+\ell)(1 - mx_+)$, where $\nu = D_+^{-1/2}$. Since $b(b+\ell) > 0$ implies $b_+(b_+ + \ell_+) < 0$, this proves the theorem in the case of more than two critical points. It also shows that, in the case of two critical points, each is surrounded by exactly one limit cycle and its characteristic exponent is non-zero. Since we must have $d\ell m < 0$ and $d_+\ell_+ m_+ < 0$, the characteristic exponent of the limit cycle surrounding the origin has the sign of ℓm and the characteristic exponent of the limit cycle surrounding (x_+, y_+) has the sign of $\ell_+ m_+$. But $m_+ = \nu m$ has the sign of m and $\ell_+ = \nu^2 \ell(1 - mx_+)$ has the opposite sign to ℓ. ∎

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