ON CONTINUOUS FUNCTIONS WHICH ARE PARTIALLY DIFFERENTIABLE ALMOST EVERYWHERE

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In the theory of surface area one meets situations where a function $z = f(x, y)$ which is defined and continuous on a closed rectangle E, is partially differentiable on E except on a subset of E of Lebesgue measure zero.

With f_x and f_y existing at $P \in E$, it is not clear when $f(x, y)$ has a differential at P. This lack of information is due to the fact that there might not exist a neighborhood $N(P, \delta)$ at every point of which both f_x and f_y exist.* The extant presentations of the existence of the differential make essential use of the law of the mean for functions of one variable. The law of the mean requires that the function be differentiable at every point of an open interval.

In this paper we give, among other things, a sufficient condition for the existence of the differential at a point P, where there does not exist a neighborhood $N(P, \delta)$ at each point of which both f_x and f_y exist.

Let $z = f(x, y)$ be defined and continuous on a closed rectangle E and be partially differentiable on E except on a subset of E of Lebesgue measure zero. We say that f_x and f_y are continuous on their common domain if for each $\varepsilon > 0$ and each $P \in E$ at which both f_x and f_y exist, there exists $\delta > 0$ such that if both f_x and f_y exist at $Q \in E$ and the distance between P and Q is less than δ, then $|f_x(P) - f_x(Q)| < \varepsilon$ and $|f_y(P) - f_y(Q)| < \varepsilon$. For verbal economy, let us say that $f \in L$ if it satisfies the foregoing conditions.

THEOREM. <u>Let</u> $f \in L$. <u>Let</u> $P: (x_o, y_o) \in E$ <u>and let</u> $(x_o + \Delta x, y_o + \Delta y) \in E$. <u>Then for each</u> $\varepsilon > 0$ <u>there exists</u> $\tau > 0$ <u>such that if</u> $(x_o + \Delta x, y_o + \Delta y) \in E$ <u>and if</u> $|\Delta x| < \tau$ <u>and</u> $|\Delta y| < \tau$, <u>then</u>

$$\Delta z = f(x_o + \Delta x, y_o + \Delta y) - f(x_o, y_o)$$

$$= f_x(x_o, y_o)\Delta x + f_y(x_o, y_o)\Delta y + \eta\Delta x + \mu\Delta y + \overline{\Delta x^2 \Delta y^2},$$ <u>where</u> $|\eta| < \varepsilon$ and $|\mu| < \varepsilon$.

*Such functions are easy to construct. One takes a continuously partially differentiable function and merely "flattens" (i.e., replaces by a plane) a suitable sequence of subsets of S in a neighborhood of $(P, f(P))$.

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Proof. We take up the case where $P: (x_o, y_o)$ is interior to E.

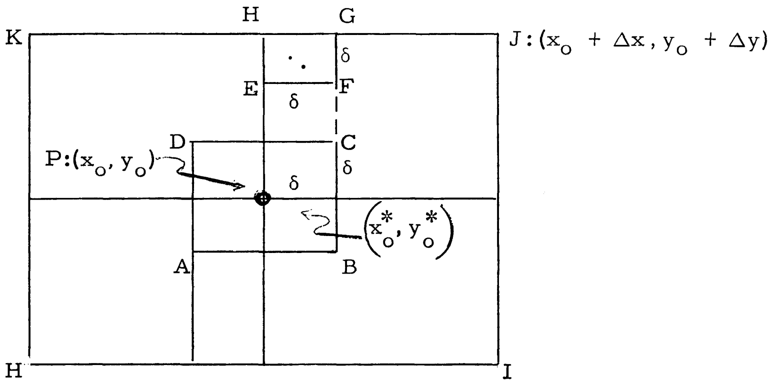

Figure I

For given Δx and Δy with $|\Delta x| < 1$ and $|\Delta y| < 1$ such that the rectangle HIJK with center at $P: (x_o, y_o)$ is a subset of E, the function $f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)$, as a function of x and y, is continuous at P. Hence, there exists $\delta > 0$ such that, if $Q: (x_1, y_1)$ is inside the square ABCD, then $f(x_o + \Delta x, y_o + \Delta y) - f(x_o, y_o + \Delta y)$ $= f(x_1 + \Delta x, y_1 + \Delta y) - f(x_1, y_1 + \Delta y) + \lambda$, where $|\lambda| < \overline{\Delta x}^2 \, \overline{\Delta y}^2$.

Inside the square EFGH (see Figure I) there exists a point $(x_o^*, y_o^* + \Delta y)$ at which f_x and f_y exist. $f(x_o + \Delta x, y_0 + \Delta y)$ $- f(x_o, y_o + \Delta y) = f(x_o^* + \Delta x, y_o^* + \Delta y) - f(x_o^*, y_o^* + \Delta y) + \lambda$, where $|\lambda| < \overline{\Delta x}^2 \, \overline{\Delta y}^2$. Since f_x exists at $(x_o^*, y_o^* + \Delta y)$, for each $\varepsilon > 0$, there exists $\alpha > 0$ such that, if $|\Delta x| < \alpha$, then $f(x_o^* + \Delta x, y_o^* + \Delta y) - f(x_o^*, y_o^* + \Delta y)$ $= f_x(x_o^*, y_o^* + \Delta y)\Delta x + \mu\Delta x$, where $|\mu| < \varepsilon$. Thus, $f(x_o + \Delta x, y_o + \Delta y)$ $- f(x_o, y_o + \Delta y) = f_x(x_o^*, y_o^* + \Delta y)\Delta x + \mu\Delta x + \lambda$.

Since f_x is continuous on its domain, there exists $\beta > 0$ such that, if $|\Delta x| < \alpha$ and $|\Delta x| < \beta$, then $f_x(x_o^*, y_o^* + \Delta y) = f_x(x_o^*, y_o^*) + \theta$, where $|\theta| < \frac{\varepsilon}{2}$. Thus, $f(x_o + \Delta x, y_o + \Delta y) - f(x_o, y_o + \Delta y)$ $= f_x(x_o, y_o)\Delta x + \theta\Delta x + \mu\Delta x + \lambda$.

Since f_y exists at (x_o, y_o), there exists $\gamma > 0$ such that, if

$|\Delta y| < \gamma$, then $f(x_o, y_o + \Delta y) - f(x_o, y_o) = f_y(x_o, y_o)\Delta y + \phi\Delta y$, where $|\phi| < \frac{\varepsilon}{2}$. Hence if $|\Delta x|$ and $|\Delta y|$ are each less than τ, the least of α, β, and λ, then

$$f(x_o + \Delta x, y_o + \Delta y) - f(x_o, y_o) = f_x(x_o, y_o)\Delta x + f_y(x_o, y_o)\Delta y + \\ + (\theta + \mu)\Delta x + \phi\Delta y + \lambda,$$

where

$$|\theta| < \frac{\varepsilon}{2}, \quad |\mu| < \frac{\varepsilon}{2}, \quad |\phi| < \varepsilon \text{ and } |\lambda| < \overline{\Delta x}^2\, \overline{\Delta y}^2.$$

To complete the proof when P is interior to E, take $\eta = \theta + \mu$, and $\mu = \phi$. The case when P is on the boundary of E is handled in the obvious manner.

COROLLARY 1. Let $f \in L$. If f_x and f_y both exist at $(x_o, y_o) \in E$, then $f(x, y)$ has a differential at (x_o, y_o).

We wish to show that, for each $\varepsilon > 0$, there exists $\tau > 0$ such that, if $|\Delta x| < \tau$ and $|\Delta y| < \tau$, then $|f((x_o + \Delta x), y_o + \Delta y) - f(x_o, y_o) - f_x(x_o, y_o)\Delta x - f_y(x_o, y_o)\Delta y| < \varepsilon\sqrt{\overline{\Delta x}^2 + \overline{\Delta y}^2}$.

Proof. Take τ such that if $|\Delta x| < \tau$ and $|\Delta y| < \tau$, then $|\eta|, |\mu| < \frac{\varepsilon}{3}$ and $\overline{\Delta x}^2\, \overline{\Delta y}^2 < \frac{\varepsilon}{3}\sqrt{\overline{\Delta x}^2 + \overline{\Delta y}^2}$. Then $|f(x_o + \Delta x, y_o + \Delta y) - f(x_o, y_o) - f_x(x_o, y_o)\Delta x - f_y(x_o, y_o)\Delta y| < \frac{2}{3}\varepsilon\sqrt{\overline{\Delta x}^2 + \overline{\Delta y}^2} + \frac{\varepsilon}{3}\sqrt{\overline{\Delta x}^2 + \overline{\Delta y}^2} = \varepsilon\sqrt{\overline{\Delta x}^2 + \overline{\Delta y}^2}$.

Thus, a sufficient condition that $f(x, y)$ have a differential at $P: (x_o, y_o)$ is that

1) f is continuous in some neighborhood $N(P, \delta)$,

2) f is partially differentiable at (x_o, y_o),

3) f is partially differentiable on $N(P, \delta)$ except on a subset of $N(P, \delta)$ of measure zero, and

4) f_x and f_y are continuous on their domain in $N(P, \delta)$.

COROLLARY 2 (The Chain Rule). Let $f \in L$. Let f_x and f_y exist at $P: (x_o, y_o) \in E$. Let $x = g(s, t)$ and $y = h(s, t)$ be defined on a neighborhood $N((s_o, t_o), \delta)$, where $x_o = g(s_o, t_o)$ and $y_o = h(s_o, t_o)$. Finally, let $g(s, t)$ and $h(s, t)$ be partially differentiable at (s_o, t_o). Then, f, as a function of s and t, is partially differentiable at (s_o, t_o) and

$$\left.\frac{\partial f}{\partial s}\right]_{s_o, t_o} = \left.\frac{\partial f}{\partial x}\right]_{x_o, y_o} \left.\frac{\partial x}{\partial s}\right]_{s_o, t_o} + \left.\frac{\partial f}{\partial y}\right]_{x_o, y_o} \left.\frac{\partial y}{\partial s}\right]_{s_o, t_o}$$

and

$$\left.\frac{\partial f}{\partial t}\right]_{s_o, t_o} = \left.\frac{\partial f}{\partial x}\right]_{x_o, y_o} \left.\frac{\partial x}{\partial t}\right]_{s_o, t_o} + \left.\frac{\partial f}{\partial y}\right]_{x_o, y_o} \left.\frac{\partial y}{\partial t}\right]_{s_o, t_o} .$$

The proof is immediate.

COROLLARY 3. Let $f \in L$. Let f_x and f_y exist at $P: (x_o, y_o) \in E$. Then, for each direction θ, the directional derivative of f at (x_o, y_o) exists and is given by

$$f_x(x_o, y_o) \cos\theta + f_y(x_o, y_o) \sin\theta .$$

The proof is immediate.

COROLLARY 4. Let $f \in L$. Let f_x and f_y exist at $P: (x_o, y_o) \in E$. Then the surface $S = f(E)$ has a tangent plane at $(x_o, y_o, f(x_o, y_o))$.

Proof. There certainly exists a continuously partially differentiable surface $S^* = f^*(E)$ such that $(x_o, y_o, f(x_o, y_o)) \in S^*$ and

$$\left.\frac{\partial f^*}{\partial x}\right]_{x_o, y_o} = \left.\frac{\partial f}{\partial x}\right]_{x_o, y_o} \quad \text{and} \quad \left.\frac{\partial f^*}{\partial y}\right]_{x_o, y_o} = \left.\frac{\partial f}{\partial y}\right]_{x_o, y_o} .$$

The function f^* has a directional derivative at (x_o, y_o) in every direction θ. Moreover, this directional derivative of f^* is equal to that of f in the same direction θ. Since S^* has a normal line at $(x_o, y_o, f(x_o))$ so does S. Thus, S has a tangent plane at $(x_o, y_o, f(x_o))$.

The foregoing discussion was confined to functions of two variables. It is clear, however, that the definitions, the propositions and the proofs carry over to the general case of a function of n real variables.

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