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Weighted composition operators on weighted Bergman and Dirichlet spaces

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Abstract. We study the boundedness and compactness of weighted composition operators acting on weighted Bergman spaces and weighted Dirichlet spaces by using the corresponding Carleson measures. We give an estimate for the norm and the essential norm of weighted composition operators between weighted Bergman spaces as well as the composition operators between weighted Hilbert spaces.

1 Introduction and preliminaries

In this paper, we consider weighted composition operators acting on the weighted Hilbert spaces of analytic functions on the unit disk. Let φ be an analytic map of the unit disk $\mathbb{D}$ of the complex plane into itself, and let $u \in \mathcal{H}(\mathbb{D})$, where $\mathcal{H}(\mathbb{D})$ is the space of analytic functions on $\mathbb{D}$. We define the weighted composition operator uC_φ by

$$uC_\varphi(f)(z) = u(z) \cdot f \circ \varphi(z), \qquad z \in \mathbb{D}.$$

Given a positive integrable function $\omega \in C^2[0,1)$, we extend it by $\omega(z) = \omega(|z|)$ for $z \in \mathbb{D}$, and call such ω a weight function. The *weighted Bergman space* $\mathcal{A}^2_\omega$ is the space of analytic functions on $\mathbb{D}$ such that

$$\|f\|^2_{\mathcal{A}^2_\omega} := \int_{\mathbb{D}} |f(z)|^2 \omega(z) dA(z) < \infty,$$

where $dA(z) = dxdy/\pi$ stands for the normalized area measure in $\mathbb{D}$.

The *weighted Hilbert space* $\mathcal{D}^2_\omega$ is the space of analytic functions f on $\mathbb{D}$ such that $f' \in \mathcal{A}^2_\omega$. The space $\mathcal{D}^2_\omega$ is endowed with the norm

$$\|f\|^2_{\mathcal{D}^2_\omega} := |f(0)|^2 + \|f'\|^2_{\mathcal{A}^2_\omega}, \qquad f \in \mathcal{D}^2_\omega.$$

Let $\omega_\alpha = (1-|z|)^\alpha$, $\alpha > -1$. The Hardy space H^2 corresponds to $\mathcal{D}^2_{\omega_1}$, the classical Dirichlet space $\mathcal{D}$ is precisely $\mathcal{D}^2_{\omega_0}$, and finally the standard Bergman spaces $\mathcal{A}^2_\alpha$ can be identified with $\mathcal{A}^2_{\omega_\alpha}$. The boundedness and compactness of composition operators

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and Toeplitz operators between Bergman spaces $\mathcal{A}^2_{\omega_\alpha}$ and between Dirichlet spaces $\mathcal{D}^2_{\omega_\alpha}$ have been investigated by several authors (see, for example, [1, 4–6, 8, 12–14, 16]). Constantin in [2] extended the results of Luecking in [13] to weighted Bergman spaces with Békollé weights and studied composition operators on these spaces. Kriete and MacCluer in [11] studied the same problem on weighted Bergman spaces with fast and regular weights. Kellay and Lefèvre in [10] gave a characterization for bounded and compact composition operators between weighted Hilbert spaces with admissible weights in terms of generalized Nevanlinna counting functions. In this paper, we give characterization for boundedness and compactness of weighted composition operators between weighted Bergman spaces and between weighted Hilbert spaces. More precisely, throughout the paper, we consider a weight ω will satisfy the following properties: $\omega(0) = 1$, ω is nonincreasing, $\omega(r)(1-r)^{-(1+\delta)}$ is nondecreasing for some $\delta > 0$, and $\lim_{r\to 1^-} \omega(r) = 0$. Such a weight function is called *almost standard*. We study bounded and compact weighted composition operators on $\mathcal{A}^2_\omega$ and $\mathcal{D}^2_\omega$. We give a direct method to calculate the norm and the essential norm of $uC_\varphi : \mathcal{A}^2_\omega \to \mathcal{A}^2_\omega$ in terms of u and the norm of φ^n, the nth power of φ by using the corresponding Carleson measures. We prove that

- If $\sup_{n\geq 0} \dfrac{n\|u\varphi^n\|^2_{\mathcal{A}^2_\omega}}{\|z^n\|^2_{\mathcal{A}^2_\omega}} < \infty$, then $uC_\varphi : \mathcal{A}^2_\omega \to \mathcal{A}^2_\omega$ is bounded.
- If $\sup_{n\geq 0} \dfrac{n\|u'\varphi^n\|^2_{\mathcal{A}^2_\omega}}{\|z^n\|^2_{\mathcal{A}^2_\omega}} < \infty$ and $\sup_{n\geq 0} \dfrac{n\|u\varphi'\varphi^n\|^2_{\mathcal{A}^2_\omega}}{\|z^n\|^2_{\mathcal{A}^2_\omega}} < \infty$, then $uC_\varphi : \mathcal{D}^2_\omega \to \mathcal{D}^2_\omega$ is bounded.

Throughout this paper, we use the following notations:

- $A \lesssim B$ means that there is an absolute constant C such that $A \leq CB$.
- $A \asymp B$ if both $A \lesssim B$ and $B \lesssim A$.

2 Carleson measures for weighted Bergman spaces

A positive Borel measure μ on $\mathbb{D}$ is a Carleson measure for $\mathcal{A}^2_\omega$ if the inclusion map

$$I_\mu : \mathcal{A}^2_\omega \to L^2(\mathbb{D}, \mu)$$

is bounded. It means, for some positive constant C,

$$\int_{\mathbb{D}} |f(z)|^2 d\mu(z) \leq C\|f\|^2_{\mathcal{A}^2_\omega}, \qquad f \in \mathcal{A}^2_\omega.$$

Denote by $D(z,r)$ the disk of radius r centered at z, $D_r = D(0,r)$. The pseudo-hyperbolic distance which is defined via the Möbius transform ϕ_a is given by

$$\rho(a,z) = |\phi_a(z)| = \left|\frac{a-z}{1-\bar{a}z}\right|, \qquad a, z \in \mathbb{D}.$$

We associate the pseudo-hyperbolic disk of radius r centered at a with

$$E(a,r) = \{z \in \mathbb{D} : \rho(a,z) < r\}, \qquad a \in \mathbb{D},\ 0 < r < 1.$$

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For $h > 0$ and $\theta \in \mathbb{R}$, consider the Carleson square given by

$$S(h,\theta) = \{z = re^{it} : 1 - h \le r < 1, |t - \theta| \le h\}.$$

The following theorem of Stevic–Sharma [15, Theorem 1] characterizes the Carleson measures for $\mathcal{A}^2_\omega$. For the case of $\mathcal{A}^2_{\omega_\alpha}$, $\alpha > -1$, the result was proved by Hastings [9] (see also [5, 17]).

Theorem 2.1 *Let ω be an almost standard weight, and let μ be a positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent:*

(i) *μ is a Carleson measure for $\mathcal{A}^2_\omega$ and* $\int_{\mathbb{D}} |f|^2 d\mu \lesssim \|\mu\|_\omega \|f\|^2_{\mathcal{A}^2_\omega}$.

(ii) $\|\mu\|_\omega = \sup_{a\in\mathbb{D}} \dfrac{\mu(E(a,r))}{(1-|a|^2)^2\omega(a)} < \infty$, $r \in (0,1)$.

(iii) $\sup_{a\in\mathbb{D}} \dfrac{(1-|a|^2)^\delta}{\omega(a)} \int_{\mathbb{D}} |\phi_a'(z)|^{2+\delta} d\mu(z) < \infty$.

(iv) $\sup_{0<h<1} \dfrac{\mu(S(h,\theta))}{h^2\omega(1-h)} < \infty$.

Note that (iv) is not announced in [15, Theorem 1]. By similar arguments given in [7, p. 65] and from Theorem 2.1, we have

$$\|\mu\|_\omega = \sup_{a\in\mathbb{D}} \frac{\mu(E(a,r))}{(1-|a|^2)^2\omega(a)} \asymp \sup_{0<h<1} \frac{\mu(S(h,\theta))}{h^2\omega(1-h)}$$
$$\asymp \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^\delta}{\omega(a)} \int_{\mathbb{D}} |\phi_a'(z)|^{2+\delta} d\mu(z).$$

For obtaining our main results, we need some auxiliary results.

Lemma 2.2 *Let ω be an almost standard weight, then*

$$|f(z)|^2 \le C \frac{\|f\|^2_{\mathcal{A}^2_\omega}}{(1-|z|^2)^2\omega(z)}, \qquad z \in \mathbb{D}, \ f \in \mathcal{A}^2_\omega,$$

for some positive constant C.

Proof Since $|f|^2$ is subharmonic,

$$|f(z)|^2 \le \frac{4}{\pi(1-|z|^2)^2} \int_{D(z,(1-|z|^2)/2)} |f(\xi)|^2 dA(\xi).$$

Let $D_1 = \{\xi \in D(z,(1-|z|^2)/2) : |\xi| \le |z|\}$ and $D_2 = \{\xi \in D(z,(1-|z|^2)/2) : |\xi| > |z|\}$. For $z \in D_1$, we have $\omega(z) \le \omega(\xi)$, and for $z \in D_2$, $(1-|\xi|)^{1+\delta}/\omega(\xi) \le (1-|z|)^{1+\delta}/\omega(z)$.

Thus,

$$
\begin{aligned}
|f(z)|^2 &\lesssim \frac{1}{(1-|z|^2)^2}\int_{D_1}|f(\xi)|^2\frac{\omega(\xi)}{\omega(\xi)}dA(\xi)\\
&\quad+\frac{1}{(1-|z|^2)^2}\int_{D_2}|f(\xi)|^2\frac{(1-|\xi|)^{1+\delta}}{\omega(\xi)}\frac{\omega(\xi)}{(1-|\xi|)^{1+\delta}}dA(\xi)\\
&\lesssim \frac{1}{(1-|z|^2)^2\omega(z)}\int_{D_1}|f(\xi)|^2\omega(\xi)dA(\xi)\\
&\quad+\frac{1}{(1-|z|^2)^2}\frac{(1-|z|)^{1+\delta}}{\omega(z)}\frac{1}{(1-|z|)^{1+\delta}}\int_{D_2}|f(\xi)|^2\omega(\xi)dA(\xi)\\
&\lesssim \frac{1}{(1-|z|^2)^2\omega(z)}\int_{D(z,(1-|z|^2)/2)}|f(\xi)|^2\omega(\xi)dA(\xi)\\
&\le \frac{\|f\|^2_{\mathcal{A}^2_\omega}}{(1-|z|^2)^2\omega(z)}.
\end{aligned}
$$

■

Lemma 2.3 [10, Lemma 2.4] *Let ω be an almost standard weight. Then,*

$$\int_{\mathbb{D}}\frac{\omega(z)dA(z)}{|1-\overline{a}z|^{4+2\delta}}\approx\frac{\omega(a)}{(1-|a|^2)^{2+2\delta}}.$$

For $r\in(0,1)$, let $\mu_r=\mu\,|_{\mathbb{D}\setminus D_r}$, and let

$$N_r=\sup_{0<h\le 1-r}\frac{\mu(S(h,\theta))}{h^2\omega(1-h)}.$$

Analogous to [5, Lemma 1], we have the following result.

Lemma 2.4 *If μ is a Carleson measure for $\mathcal{A}^2_\omega$, then so is μ_r and*

$$\|\mu_r\|_\omega\lesssim N_r\lesssim\sup_{|a|\ge r}\frac{(1-|a|^2)^\delta}{\omega(a)}\int_{\mathbb{D}}|\phi_a'(z)|^{2+\delta}d\mu(z). \tag{2.1}$$

Proof As in the proof of [5, Lemma 1], we get $\sup_{0<h<1}\frac{\mu(S(h,\theta))}{h^2\omega(1-h)}\lesssim N_r$ and

$$\|\mu_r\|_\omega=\sup_{a\in\mathbb{D}}\frac{\mu_r(E(a,1/2))}{(1-|a|^2)^2\omega(a)}\lesssim\sup_{0<h<1}\frac{\mu_r(S(h,\theta))}{h^2\omega(1-h)}.$$

This implies that

$$\|\mu_r\|_\omega\lesssim N_r=\sup_{0<h\le 1-r}\frac{\mu(S(h,\theta))}{h^2\omega(1-h)}. \tag{2.2}$$

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Now, let $0 < h \le 1 - r$ and $a = (1-h)e^{ih}$. Then, $|a| \ge r$, and for each $z \in S(h,\theta)$, $|\varphi_a'(z)| \gtrsim \frac{1}{h}$. Therefore,

$$\frac{\mu(S(h,\theta))}{h^2\omega(1-h)} = \frac{h^\delta \mu(S(h,\theta))}{h^{2+\delta}\omega(1-h)} \lesssim \frac{(1-|a|^2)^\delta}{\omega(a)} \int_{S(h,\theta)} |\phi_a'(z)|^{2+\delta} d\mu(z)$$
$$\le \frac{(1-|a|^2)^\delta}{\omega(a)} \int_{\mathbb{D}} |\phi_a'(z)|^{2+\delta} d\mu(z).$$

Applying (2.2) ensures that

$$\|\mu_r\|_\omega \lesssim \sup_{|a|\ge r} \frac{(1-|a|^2)^\delta}{\omega(a)} \int_{\mathbb{D}} |\phi_a'(z)|^{2+\delta} d\mu(z).$$ ■

3 Boundedness properties of weighted composition operators between weighted Bergman and Dirichlet spaces

Let $u \in \mathcal{H}(\mathbb{D})$ and $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic self-map. For every Borel set $E \subseteq \mathbb{D}$, we define the pullback measure

$$\mu_{\varphi,u}^\omega(E) := \int_{\varphi^{-1}(E)} |u(z)|^2 \omega(z) dA(z).$$

It can be easily seen that

$$\|u \cdot C_\varphi f\|_{\mathcal{A}_\omega^2}^2 = \int_{\mathbb{D}} |f(z)|^2 d\mu_{\varphi,u}^\omega(z), \qquad f \in \mathcal{A}_\omega^2. \tag{3.1}$$

For $a \in \mathbb{D}$, let

$$f_a^\omega(z) = \frac{(1-|a|^2)^{1+\delta}}{\sqrt{\omega(a)}(1-\bar{a}z)^{2+\delta}}, \qquad z \in \mathbb{D}. \tag{3.2}$$

By Lemma 2.3, $f_a^\omega \in \mathcal{A}_\omega^2$ and $\sup_{a\in\mathbb{D}} \|f_a^\omega\|_{\mathcal{A}_\omega^2} < \infty$. We have the following result.

Theorem 3.1 *Let ω be an almost standard weight, let $u \in \mathcal{A}_\omega^2$, and let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic. The following statements are equivalent:*

(i) *$uC_\varphi : \mathcal{A}_\omega^2 \to \mathcal{A}_\omega^2$ is bounded.*
(ii) *$\mu_{\varphi,u}^\omega$ is a Carleson measure for $\mathcal{A}_\omega^2$.*
(iii) *$\sup_{a\in\mathbb{D}} \|u \cdot C_\varphi f_a^\omega\|_{\mathcal{A}_\omega^2}^2 < \infty$.*

Furthermore, let f_a^ω be the function given in (3.2), then

(3.3)

$$\|uC_\varphi\|_{\mathcal{A}_\omega^2 \to \mathcal{A}_\omega^2}^2 \approx \sup_{a\in\mathbb{D}} \|uC_\varphi f_a^\omega\|_{\mathcal{A}_\omega^2}^2 \approx \|\mu_{\varphi,u}^\omega\|_\omega \approx \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^\delta}{\omega(a)} \int_{\mathbb{D}} |\phi_a'(z)|^{2+\delta} d\mu_{\varphi,u}^\omega.$$

Proof The equality (3.1) implies that the boundedness of $u \cdot C_\varphi$ is equivalent to the measure $d\mu^\omega_{\varphi,u}$ is a Carleson measure for $\mathcal{A}^2_\omega$. Therefore,

$$\begin{aligned}\sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^\delta}{\omega(a)} \int_{\mathbb{D}} |\phi_a'(z)|^{2+\delta} d\mu^\omega_{\varphi,u}(z) &= \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^{2+2\delta}}{\omega(a)} \int_{\mathbb{D}} \frac{d\mu^\omega_{\varphi,u}(z)}{|1-\overline{a}z|^{4+2\delta}} \\ &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f^\omega_a(z)|^2 d\mu^\omega_{\varphi,u}(z) \\ &\lesssim \|\mu^\omega_{\varphi,u}\|_\omega \sup_{a\in\mathbb{D}} \|f^\omega_a\|^2_{\mathcal{A}^2_\omega} < \infty.\end{aligned}$$

Again by Theorem 2.1, we get

$$\begin{aligned}\|\mu^\omega_{\varphi,u}\|_\omega &\approx \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^{2+2\delta}}{\omega(a)} \frac{\mu^\omega_{\varphi,u}(E(a,r))}{(1-|a|^2)^{4+2\delta}} \\ &\lesssim \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^\delta}{\omega(a)} \int_{E(a,r)} \frac{(1-|a|^2)^{2+\delta}}{|1-\overline{a}z|^{4+2\delta}} d\mu^\omega_{\varphi,u}(z) \\ &\leq \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^\delta}{\omega(a)} \int_{\mathbb{D}} |\phi_a'(z)|^{2+\delta} d\mu^\omega_{\varphi,u}(z) < \infty.\end{aligned}$$
■

Let

$$\omega_n := \|z^n\|^2_{\mathcal{A}^2_\omega} = 2\int_0^1 r^{2n+1}\omega(r)dr.$$

We have the following lemma.

Lemma 3.2 *Let ω be an almost standard weight. Then,*

$$\frac{\omega(a)}{(1-|a|^2)^{2\delta+2}} \approx \sum_{n=0}^\infty n^{2\delta+2}|a|^{2n}\omega_n.$$

Proof Let $\lambda = \delta + 2$. By Lemma 2.3, and since $\omega(z) = \omega(|z|)$, we have

$$\begin{aligned}\frac{\omega(a)}{(1-|a|^2)^{2\delta+2}} &\approx \int_{\mathbb{D}} \frac{\omega(z)dA(z)}{|1-\overline{a}z|^{2\delta+4}} \\ &\approx \sum_{m,n=0}^\infty \frac{\Gamma(m+\lambda)\Gamma(n+\lambda)}{m!n!\Gamma(\lambda)^2} \overline{a}^m a^n \int_0^1 r^{n+m+1}\omega(r)dr \int_0^{2\pi} e^{i(n-m)\theta}\frac{d\theta}{2\pi} \\ &\approx \sum_{n=0}^\infty n^{2\lambda-2}|a|^{2n}\omega_n.\end{aligned}$$
■

In the following theorem, we estimate the norm of $uC_\varphi : \mathcal{A}^2_\omega \to \mathcal{A}^2_\omega$ in terms of u and φ^n.

Theorem 3.3 *Let ω be an almost standard weight, let $u \in \mathcal{A}^2_\omega$, and let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic.*

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(i) *If* $\sup_{n\geq 0} \frac{n\|u\varphi^n\|^2_{\mathcal{A}^2_\omega}}{\|z^n\|^2_{\mathcal{A}^2_\omega}} < \infty$, *then* uC_φ *is bounded.*

(ii) *Conversely, if* uC_φ *is bounded, then* $\sup_{n\geq 0} \frac{\|u\varphi^n\|^2_{\mathcal{A}^2_\omega}}{\|z^n\|^2_{\mathcal{A}^2_\omega}} < \infty$.

Proof By (3.3) and applying Lemma 3.2, we get

$$\begin{aligned}
\|uC_\varphi\|^2_{\mathcal{A}^2_\omega\to\mathcal{A}^2_\omega} &\approx \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^\delta}{\omega(a)} \int_{\mathbb{D}} |\phi'_a(z)|^{2+\delta} d\mu^\omega_{\varphi,u}(z)\\
&= \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^{2\delta+2}}{\omega(a)} \int_{\mathbb{D}} \frac{|u(z)|^2\omega(z)dA(z)}{|1-\overline{a}\varphi(z)|^{2\delta+4}}\\
&\lesssim \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^{2\delta+2}}{\omega(a)} \sum_{n=0}^{\infty} n^{2\delta+3}|a|^{2n} \int_{\mathbb{D}} |\varphi(z)|^{2n}|u(z)|^2\omega(z)dA(z)\\
&\approx \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^{2\delta+2}}{\omega(a)} \sum_{n=0}^{\infty} n^{2\delta+3}|a|^{2n}\|u\varphi^n\|^2_{\mathcal{A}^2_\omega}\\
&\lesssim \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^{2\delta+2}}{\omega(a)} \sum_{n=0}^{\infty} n^{2\delta+2}|a|^{2n}\omega_n < +\infty.
\end{aligned}$$

Conversely, let $uC_\varphi : \mathcal{A}^2_\omega \to \mathcal{A}^2_\omega$ be bounded. Then,

$$\sup_{n\geq 0} \frac{\|u\varphi^n\|^2_{\mathcal{A}^2_\omega}}{\|z^n\|^2_{\mathcal{A}^2_\omega}} = \sup_{n\geq 0} \left\| uC_\varphi\left(\frac{z^n}{\|z^n\|_{\mathcal{A}^2_\omega}}\right)\right\|^2_{\mathcal{A}^2_\omega} \leq \|uC_\varphi\|^2_{\mathcal{A}^2_\omega\to\mathcal{A}^2_\omega} < \infty. \qquad \blacksquare$$

Remark 3.4 Note by (3.3)

$$\sup_{n\geq 0} \frac{\|u\varphi^n\|^2_{\mathcal{A}^2_\omega}}{\|z^n\|^2_{\mathcal{A}^2_\omega}} \lesssim \|uC_\varphi\|^2_{\mathcal{A}^2_\omega\to\mathcal{A}^2_\omega} \lesssim \sup_{n\geq 0} \frac{n\|u\varphi^n\|^2_{\mathcal{A}^2_\omega}}{\|z^n\|^2_{\mathcal{A}^2_\omega}}.$$

Now, we study the composition operator in the weighted Dirichlet spaces. We need the following lemma.

Lemma 3.5 *Let* ω *be an almost standard weight, then for some positive constant* C *and for each* $f \in \mathcal{D}^2_\omega$ *and* $z \in \mathbb{D}$,

$$|f(z)|^2 \leq C\frac{\|f\|^2_{\mathcal{D}^2_\omega}}{(1-|z|^2)^2\omega(z)}. \tag{3.4}$$

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Proof Let $f \in \mathcal{D}^2_\omega$, and we have

$$|f(z)-f(0)|^2 \le \int_0^1 |f'(zt)|^2|z|^2 dt \lesssim \|f'\|^2_{\mathcal{A}^2_\omega} \int_0^1 \frac{|z|^2 dt}{(1-|z|t)^2\omega(|z|t)}$$

$$\lesssim \frac{|z|^2\|f'\|^2_{\mathcal{A}^2_\omega}}{(1-|z|^2)^2\omega(z)}. \tag{3.5}$$

Since $(1-|z|^2)^2\omega(z) \le \omega(0) = 1$,

$$|f(0)|^2 \le \frac{|f(0)|^2}{(1-|z|^2)^2\omega(z)}. \tag{3.6}$$

It then follows by (3.5) and (3.6) that

$$|f(z)|^2 \lesssim |f(z)-f(0)|^2 + |f(0)|^2 \lesssim \frac{\|f\|^2_{\mathcal{D}^2_\omega}}{(1-|z|^2)^2\omega(z)}.$$ ■

Theorem 3.6 *Let ω be an almost standard weight, let $u \in \mathcal{D}^2_\omega$, and let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic. If $\mu^\omega_{\varphi,u\varphi'}$ and $\mu^\omega_{\varphi,u'}$ are $\mathcal{A}^2_\omega$-Carleson measures, then $uC_\varphi : \mathcal{D}^2_\omega \to \mathcal{D}^2_\omega$ is bounded.*

Proof For $f \in \mathcal{D}^2_\omega$, by (3.4),

$$|u(0)f(\varphi(0))|^2 \lesssim \frac{|u(0)|^2}{\omega(\varphi(0))(1-|\varphi(0)|^2)^2}\|f\|^2_{\mathcal{D}^2_\omega} = C\|f\|^2_{\mathcal{D}^2_\omega},$$

where C depends only on $\omega(\varphi(0))$, $\varphi(0)$, and $u(0)$. Thus,

$$\|uC_\varphi f\|^2_{\mathcal{D}^2_\omega} = |u(0)f(\varphi(0))|^2 + \int_{\mathbb{D}} |(uC_\varphi f)'(z)|^2\omega(z)dA(z)$$

$$\lesssim \|f\|^2_{\mathcal{D}^2_\omega} + \int_{\mathbb{D}} |u'(z)|^2|f(\varphi(z))|^2\omega(z)dA(z)$$

$$+ \int_{\mathbb{D}} |u(z)|^2|f'(\varphi(z))|^2|\varphi'(z)|^2\omega(z)dA(z). \tag{3.7}$$

Since $\mu^\omega_{\varphi,u\varphi'}$ is $\mathcal{A}^2_\omega$-Carleson measure, we have

$$\int_{\mathbb{D}} |u(z)|^2|f'(\varphi(z))|^2|\varphi'(z)|^2\omega(z)dA(z) = \int_{\mathbb{D}} |f'(z)|^2 d\mu^\omega_{\varphi,u\varphi'}(z) \lesssim \|f'\|^2_{\mathcal{A}^2_\omega}. \tag{3.8}$$

On the other hand,

$$\int_{\mathbb{D}} |u'(z)|^2|f(\varphi(z))|^2\omega(z)dA(z)$$

$$\le \int_{\mathbb{D}} \left(|f(0)| + \int_0^1 |f'(t\varphi(z))||\varphi(z)|dt\right)^2 |u'(z)|^2\omega(z)dA(z)$$

$$\le 2\int_{\mathbb{D}} |f(0)|^2|u'(z)|^2\omega(z)dA(z)$$

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$$+2\int_{\mathbb{D}}\left(\int_0^1|f'(t\varphi(z))||\varphi(z)|dt\right)^2|u'(z)|^2\omega(z)dA(z)$$

$$\lesssim \|f\|^2_{\mathcal{D}^2_\omega}\mu^\omega_{\varphi,u'}(\mathbb{D})+\int_0^1\left(\int_{\mathbb{D}}|f'(tz)|^2d\mu^\omega_{\varphi,u'}(z)\right)dt.$$

For $0<t<1$, let $g_t(z)=f'(tz)$. Then,

$$\|g_t\|^2_{\mathcal{A}^2_\omega}=\int_{\mathbb{D}}|f'(tz)|^2\omega(z)dA(z)=\int_{\mathbb{D}}|f'(tz)|^2\omega(tz)\frac{\omega(z)}{\omega(tz)}dA(z)\le\|f\|^2_{\mathcal{D}^2_\omega},$$

which guaranties that $g_t\in\mathcal{A}^2_\omega$. Therefore,

$$\int_0^1\left(\int_{\mathbb{D}}|f'(tz)|^2d\mu^\omega_{\varphi,u'}(z)\right)dt=\int_0^1\left(\int_{\mathbb{D}}|g_t(z)|^2d\mu^\omega_{\varphi,u'}(z)\right)dt$$

$$\lesssim\|\mu^\omega_{\varphi,u'}\|_\omega\int_0^1\|g_t\|^2_{\mathcal{A}^2_\omega}dt$$

$$\le\|\mu^\omega_{\varphi,u'}\|_\omega\|f\|^2_{\mathcal{D}^2_\omega}.$$

We conclude that

$$\int_{\mathbb{D}}|u'(z)|^2|f(\varphi(z))|^2\omega(z)dA(z)\lesssim\|\mu^\omega_{\varphi,u'}\|_\omega\|f\|^2_{\mathcal{D}^2_\omega}. \tag{3.9}$$

By (3.7)–(3.9), we deduce that $uC_\varphi:\mathcal{D}^2_\omega\to\mathcal{D}^2_\omega$ is bounded. ∎

Corollary 3.7. *Let ω be an almost standard weight, let $u\in\mathcal{D}^2_\omega$, and let $\varphi:\mathbb{D}\to\mathbb{D}$ be analytic. If*

$$\sup_{n\ge0}\frac{n\|u'\varphi^n\|^2_{\mathcal{A}^2_\omega}}{\|z^n\|^2_{\mathcal{A}^2_\omega}}<\infty\qquad\text{and}\qquad\sup_{n\ge0}\frac{n\|u\varphi'\varphi^n\|^2_{\mathcal{A}^2_\omega}}{\|z^n\|^2_{\mathcal{A}^2_\omega}}<\infty,$$

then $uC_\varphi:\mathcal{D}^2_\omega\to\mathcal{D}^2_\omega$ is bounded.

Proof By Remark 3.4,

$$\|\mu^\omega_{\varphi,u\varphi'}\|_\omega\lesssim\sup_{n\ge0}\frac{n\|u\varphi'\varphi^n\|^2_{\mathcal{A}^2_\omega}}{\omega_n}<\infty,\qquad\text{and}\qquad\|\mu^\omega_{\varphi,u'}\|_\omega\lesssim\sup_{n\ge0}\frac{n\|u'\varphi^n\|^2_{\mathcal{A}^2_\omega}}{\omega_n}<\infty.$$

Applying Theorem 2.1, we deduce that $\mu^\omega_{\varphi,u\varphi'}$ and $\mu^\omega_{\varphi,u'}$ are $\mathcal{A}^2_\omega$-Carleson measures, and by Theorem 3.6, we deduce that $uC_\varphi:\mathcal{D}^2_\omega\to\mathcal{D}^2_\omega$ is bounded. ∎

Corollary 3.8. *Let ω be an almost standard weight, then*

$$\sup_{n\ge0}\frac{\|\varphi'\varphi^n\|^2_{\mathcal{A}^2_\omega}}{\|z^n\|^2_{\mathcal{A}^2_\omega}}\lesssim\|C_\varphi\|^2_{\mathcal{D}^2_\omega\to\mathcal{D}^2_\omega}\lesssim\sup_{n\ge0}\frac{n\|\varphi'\varphi^n\|^2_{\mathcal{A}^2_\omega}}{\|z^n\|^2_{\mathcal{A}^2_\omega}}.$$

Proof Let $\tilde{\mathcal{D}}_\omega=\{f\in\mathcal{D}^2_\omega:\ f(0)=0\}$. The boundedness of $C_\varphi|_{\tilde{\mathcal{D}}_\omega}$ implies that $\varphi'C_\varphi:\mathcal{A}^2_\omega\to\mathcal{A}^2_\omega$ is bounded and $\|\varphi'C_\varphi\|_{\mathcal{A}^2_\omega\to\mathcal{A}^2_\omega}\le\|C_\varphi\|_{\mathcal{D}^2_\omega\to\mathcal{D}^2_\omega}$. Conversely, if $\varphi'C_\varphi:$

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$\mathcal{A}^2_\omega \to \mathcal{A}^2_\omega$ is bounded, then by Theorems 3.1 and 3.6 and (3.8), we deduce that $C_\varphi : \mathcal{D}^2_\omega \to \mathcal{D}^2_\omega$ is bounded and $\|C_\varphi\|_{\mathcal{D}^2_\omega \to \mathcal{D}^2_\omega} \lesssim \|\varphi' C_\varphi\|_{\mathcal{A}^2_\omega \to \mathcal{A}^2_\omega}$. Then, by Remark 3.4, we get the result. ∎

4 The essential norm of weighted composition operators between weighted Bergman and Dirichlet spaces

In this section, we characterize the essential norm of bounded weighted composition operators between weighted Bergman and Dirichlet spaces.

Let X be Banach spaces. The essential norm of a bounded linear operator $T : X \to X$ is its distance to the set of compact operators K mapping on X, that is,

$$\|T\|_{e,X\to X} = \inf\{\|T - K\|_{X\to X} : K \text{ is compact}\},$$

where $\|\cdot\|_{X\to X}$ is the operator norm. By representation theorem for bounded linear functionals on a Hilbert space, to each $z \in \mathbb{D}$, there is unique element K^ω_z in $\mathcal{A}^2_\omega$, such that

$$f(z) = \langle f, K^\omega_z \rangle, \qquad f \in \mathcal{A}^2_\omega.$$

The function $K^\omega_z(\xi)$, with $(\xi, z) \in \mathbb{D}\times\mathbb{D}$, is called the reproducing kernel for $\mathcal{A}^2_\omega$. Let

$$\omega_n := \|z^n\|^2_{\mathcal{A}^2_\omega} = 2\int_0^1 r^{2n+1}\omega(r)dr.$$

A simple computation shows that $K^\omega_z(\xi) = \sum_{n=0}^\infty \frac{\bar{z}^n \xi^n}{\omega_n}$. In the following lemma, we provide an estimate of the reproducing kernel of $\mathcal{A}^2_\omega$ on the diagonal.

Proposition 4.1. *Let ω be an almost standard weight, then*

$$\|K^\omega_z\|^2_{\mathcal{A}^2_\omega} \approx \frac{1}{(1-|z|^2)^2\omega(z)}, \qquad \in \mathbb{D}. \tag{4.1}$$

Proof Fix $z \in \mathbb{D}$. By Lemma 2.2,

$$|K^\omega_z(z)|^2 \lesssim \frac{\|K^\omega_z\|^2_{\mathcal{A}^2_\omega}}{(1-|z|^2)^2\omega(z)} = \frac{K^\omega_z(z)}{(1-|z|^2)^2\omega(z)},$$

which means that

$$\|K^\omega_z\|^2_{\mathcal{A}^2_\omega} = K^\omega_z(z) \lesssim \frac{1}{(1-|z|^2)^2\omega(z)}.$$

For the other side, for $a \in \mathbb{D}$, define $g_a(z) = \frac{1}{(1-\bar{a}z)^{\delta+2}}$. By Lemma 2.3, $\|g_a\|^2_{\mathcal{A}^2_\omega} \approx \frac{\omega(a)}{(1-|a|^2)^{2\delta+2}}$. Thus,

$$\frac{1}{(1-|a|^2)^{2\delta+4}} = g_a^2(a) = \langle g_a, K^\omega_a\rangle^2 \le \|g_a\|^2_{\mathcal{A}^2_\omega}\|K^\omega_a\|^2_{\mathcal{A}^2_\omega} \lesssim \frac{\omega(a)}{(1-|a|^2)^{2\delta+2}}\|K^\omega_a\|^2_{\mathcal{A}^2_\omega},$$

which ensures that

$$\|K_a^\omega\|^2_{\mathcal{A}^2_\omega} \gtrsim \frac{1}{(1-|a|^2)^2\omega(a)}. \qquad \blacksquare$$

Theorem 4.2 *Let ω be an almost standard weight. Let $uC_\varphi : \mathcal{A}^2_\omega \to \mathcal{A}^2_\omega$ be bounded. Then,*

$$\|uC_\varphi\|^2_{e;\mathcal{A}^2_\omega\to\mathcal{A}^2_\omega} \approx \limsup_{|a|\to 1^-} \frac{\mu(E(a,1/2))}{(1-|a|^2)^2\omega(a)}$$

$$\approx \limsup_{|a|\to 1^-} \frac{(1-|a|^2)^\delta}{\omega(a)} \int_{\mathbb{D}} |\phi_a'(z)|^{2+\delta} d\mu^\omega_{\varphi,u}(z). \tag{4.2}$$

Proof Since $uC_\varphi : \mathcal{A}^2_\omega \to \mathcal{A}^2_\omega$ is bounded by Theorem 3.1, $\mu^\omega_{\varphi,u}$ is a Carleson measure for $\mathcal{A}^2_\omega$ and $u \in \mathcal{A}^2_\omega$. Considering the compact operator $S_n : \mathcal{A}^2_\omega \to \mathcal{A}^2_\omega$ by $S_n f = \sum_{k=0}^n \hat{f}(k)z^k$ and letting $R_n = I - S_n$, we have

$$\|uC_\varphi\|_{e;\mathcal{A}^2_\omega\to\mathcal{A}^2_\omega} \le \|uC_\varphi\circ S_n\|_{e;\mathcal{A}^2_\omega\to\mathcal{A}^2_\omega} + \|uC_\varphi\circ R_n\|_{e;\mathcal{A}^2_\omega\to\mathcal{A}^2_\omega} = \|uC_\varphi\circ R_n\|_{e;\mathcal{A}^2_\omega\to\mathcal{A}^2_\omega}.$$

Thus,

$$\|uC_\varphi\|_{e;\mathcal{A}^2_\omega\to\mathcal{A}^2_\omega} \le \liminf_{n\to\infty} \|uC_\varphi\circ R_n\|_{e;\mathcal{A}^2_\omega\to\mathcal{A}^2_\omega} \le \liminf_{n\to\infty} \|uC_\varphi\circ R_n\|_{\mathcal{A}^2_\omega\to\mathcal{A}^2_\omega}. \tag{4.3}$$

Fixing $f \in \mathcal{A}^2_\omega$ and $r \in (0,1)$, let $D_r = D(0,r)$. We get

$$\begin{aligned}
\|uC_\varphi\circ R_n f\|^2_{\mathcal{A}^2_\omega} &= \int_{\mathbb{D}} |u(z)|^2 |R_n f(\varphi(z))|^2 \omega(z)\,dA(z)\\
&= \int_{\mathbb{D}} |R_n f(z)|^2 d\mu^\omega_{\varphi,u}(z)\\
&= \int_{D_r} |R_n f(z)|^2 d\mu^\omega_{\varphi,u}(z) + \int_{\mathbb{D}\setminus D_r} |R_n f(z)|^2 d\mu^\omega_{\varphi,u}(z)\\
&= I_{1,n} + I_{2,n}.
\end{aligned}$$

In view of

$$|R_n f(z)|^2 = |\langle f, R_n K_z^\omega\rangle|^2 \le \|f\|^2_{\mathcal{A}^2_\omega}\|R_n K_z^\omega\|^2_{\mathcal{A}^2_\omega} = \|f\|^2_{\mathcal{A}^2_\omega}\sum_{k=n+1}^\infty \frac{|z|^{2k}}{\omega_k}, \tag{4.4}$$

we get

$$I_{1,n} \le \|f\|^2_{\mathcal{A}^2_\omega}\mu^\omega_{\varphi,u}(D_r)\sum_{k=n+1}^\infty \frac{|r|^{2k}}{\omega_k}.$$

Therefore,

$$\limsup_{n\to\infty}\sup_{\|f\|_{\mathcal{A}^2_\omega}\le 1} I_{1,n} = 0.$$

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By Lemma 2.4, $\mu^{\omega}_{\varphi,u,r}$ is an $\mathcal{A}^2_\omega$-Carleson measure and

$$I_{2,n} = \int_{\mathbb{D}\setminus D_r} |R_n f(z)|^2 d\mu^{\omega}_{\varphi,u,r}(z) \lesssim \|\mu^{\omega}_{\varphi,u,r}\|_\omega \|R_n f\|^2_{\mathcal{A}^2_\omega}$$

$$\lesssim \|f\|^2_{\mathcal{A}^2_\omega} \sup_{|a|\geq r} \frac{(1-|a|^2)^\delta}{\omega(a)} \int_{\mathbb{D}} |\phi'_a(z)|^{2+\delta} d\mu^{\omega}_{\varphi,u}(z).$$

Thus,

$$\limsup_{n\to\infty} \|uC_\varphi \circ R_n\|^2_{\mathcal{A}^2_\omega \to \mathcal{A}^2_\omega} = \limsup_{n\to\infty} \sup_{\|f\|_{\mathcal{A}^2_\omega}\leq 1} \|uC_\varphi \circ R_n f\|^2_{\mathcal{A}^2_\omega}$$

$$\lesssim \sup_{|a|\geq r} \frac{(1-|a|^2)^\delta}{\omega(a)} \int_{\mathbb{D}} |\phi'_a(z)|^{2+\delta} d\mu^{\omega}_{\varphi,u}(z).$$

Letting $r \to 1^-$, by (4.3), we conclude that

$$\|uC_\varphi\|_{e;\mathcal{A}^2_\omega \to \mathcal{A}^2_\omega} \lesssim \limsup_{|a|\to 1^-} \frac{(1-|a|^2)^\delta}{\omega(a)} \int_{\mathbb{D}} |\phi'_a(z)|^{2+\delta} d\mu^{\omega}_{\varphi,u}(z). \tag{4.5}$$

By the similar arguments given in the proof of Theorem 2.1, we have

$$\begin{aligned}\|uC_\varphi \circ R_n f\|^2_{\mathcal{A}^2_\omega} &= \int_{\mathbb{D}} |R_n f(z)|^2 d\mu^{\omega}_{\varphi,u}(z) \\ &\lesssim \int_{D_r} \frac{|R_n f(\xi)|^2 \mu^{\omega}_{\varphi,u}(E(\xi,1/2))}{(1-|\xi|^2)^2\omega(\xi)} \omega(\xi) dA(\xi) \\ &\quad + \int_{\mathbb{D}\setminus D_r} \frac{|R_n f(\xi)|^2 \mu^{\omega}_{\varphi,u}(E(\xi,1/2))}{(1-|\xi|^2)^2\omega(\xi)} \omega(\xi) dA(\xi) \\ &= J_{1,n} + J_{2,n}.\end{aligned}$$

Clearly, $\limsup_{n\to\infty} J_{1,n} = 0$. Since

$$\limsup_{n\to\infty} \sup_{\|f\|_{\mathcal{A}^2_\omega}\leq 1} J_{2,n} \lesssim \sup_{|a|\geq r} \frac{\mu^{\omega}_{\varphi,u}(E(a,1/2))}{(1-|a|^2)^2\omega(a)},$$

we have

$$\|uC_\varphi\|_{e;\mathcal{A}^2_\omega \to \mathcal{A}^2_\omega} \lesssim \limsup_{|a|\to 1^-} \frac{\mu^{\omega}_{\varphi,u}(E(a,1/2))}{(1-|a|^2)^2\omega(a)}. \tag{4.6}$$

Now, let $\{a_n\}$ be a sequence in $\mathbb{D}$ with $|a_n| \geq 1/2$ and $|a_n| \to 1^-$ such that

$$\lim_{n\to\infty} \frac{\mu^{\omega}_{\varphi,u}(E(a_n,1/2))}{(1-|a_n|^2)^2\omega(a_n)} = \limsup_{|a|\to 1^-} \frac{\mu^{\omega}_{\varphi,u}(E(a,1/2))}{(1-|a|^2)^2\omega(a)}.$$

Defining $f_n(z) = \frac{1}{\sqrt{\omega(a_n)}} \frac{(1-|a_n|^2)^{1+\delta}}{(1-\overline{a_n}z)^{2+\delta}}$, we have that $\{f_n\}$ is a bounded sequence in $\mathcal{A}^2_\omega$ converging to zero uniformly on compact subsets of $\mathbb{D}$. Fix a compact operator

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$T : \mathcal{A}^2_\omega \to \mathcal{A}^2_\omega$. It follows from [3, Lemma 3.3] that $Tf_n \to 0$ in $\mathcal{A}^2_\omega$ whenever $n \to \infty$. Therefore,

$$\begin{aligned}\|uC_\varphi - T\|^2_{\mathcal{A}^2_\omega \to \mathcal{A}^2_\omega} &\geq \limsup_{n\to\infty} \|uC_\varphi f_n\|^2_{\mathcal{A}^2_\omega} \\ &= \limsup_{n\to\infty} \int_{\mathbb{D}} |u(z)|^2 |f_n(\varphi(z))|^2 \omega(z) dA(z) \\ &= \limsup_{n\to\infty} \int_{\mathbb{D}} \frac{|u(z)|^2}{\omega(a_n)} \frac{(1-|a_n|^2)^{2+2\delta}}{|1-\overline{a_n}\varphi(z)|^{4+2\delta}} \omega(z) dA(z) \\ &= \limsup_{n\to\infty} \frac{(1-|a_n|^2)^\delta}{\omega(a_n)} \int_{\mathbb{D}} \frac{(1-|a_n|^2)^{2+\delta}}{|1-\overline{a_n}z|^{4+2\delta}} d\mu^\omega_{\varphi,u}(z) \\ &\gtrsim \limsup_{n\to\infty} \frac{(1-|a_n|^2)^\delta}{\omega(a_n)} \int_{E(a_n,1/2)} \frac{(1-|a_n|^2)^{2+\delta}}{|1-\overline{a_n}z|^{4+2\delta}} d\mu^\omega_{\varphi,u}(z) \\ &\gtrsim \limsup_{n\to\infty} \frac{(1-|a_n|^2)^{2+2\delta}}{\omega(a_n)} \frac{\mu^\omega_{\varphi,u}(E(a_n,1/2))}{(1-|a_n|^2)^{4+2\delta}}.\end{aligned}$$

Thus,

$$\|uC_\varphi\|^2_{e;\mathcal{A}^2_\omega \to \mathcal{A}^2_\omega} = \inf_T \|uC_\varphi - T\|^2_{\mathcal{A}^2_\omega \to \mathcal{A}^2_\omega} \gtrsim \limsup_{|a|\to 1^-} \frac{\mu^\omega_{\varphi,u}(E(a,1/2))}{(1-|a|^2)^2 \omega(a)}. \tag{4.7}$$

By the similar arguments given in the proof of Theorem 2.1, one can state that

$$\limsup_{|a|\to 1^-} \frac{\mu^\omega_{\varphi,u}(E(a,1/2))}{(1-|a|^2)^2 \omega(a)} \gtrsim \limsup_{|a|\to 1^-} \frac{(1-|a|^2)^\delta}{\omega(a)} \int_{\mathbb{D}} |\phi'_a(z)|^{2+\delta} d\mu^\omega_{\varphi,u}(z). \tag{4.8}$$

Applying relations (4.5)–(4.8), we obtain the desired result. ■

As an immediate consequence of Theorem 4.2, we have the following result.

Corollary 4.3. *Let ω be an almost standard weight. Let $uC_\varphi : \mathcal{A}^2_\omega \to \mathcal{A}^2_\omega$ be bounded. Then, the following statements are equivalent:*

(i) *uC_φ is compact.*

(ii) $\displaystyle\limsup_{|a|\to 1^-} \frac{\mu(E(a,1/2))}{(1-|a|^2)^2 \omega(a)} = 0.$

(iii) $\displaystyle\limsup_{|a|\to 1^-} \frac{(1-|a|^2)^\delta}{\omega(a)} \int_{\mathbb{D}} |\phi'_a(z)|^{2+\delta} d\mu^\omega_{\varphi,u} = 0.$

Corollary 4.4. *Let ω be an almost standard weight. Let $uC_\varphi : \mathcal{A}^2_\omega \to \mathcal{A}^2_\omega$ be bounded. Then,*

$$\limsup_{n\to\infty} \frac{\|u\varphi^n\|^2_{\mathcal{A}^2_\omega}}{\|z^n\|^2_{\mathcal{A}^2_\omega}} \lesssim \|uC_\varphi\|^2_{e;\mathcal{A}^2_\omega \to \mathcal{A}^2_\omega} \lesssim \limsup_{n\to\infty} \frac{n\|u\varphi^n\|^2_{\mathcal{A}^2_\omega}}{\|z^n\|^2_{\mathcal{A}^2_\omega}}.$$

Proof Since $\left\{\frac{z^n}{\|z^n\|_{\mathcal{A}^2_\omega}}\right\}$ is a bounded sequence in $\mathcal{A}^2_\omega$, for every compact operator $T:\mathcal{A}^2_\omega\to\mathcal{A}^2_\omega$,

$$\lim_{n\to\infty}\left\|T\frac{z^n}{\|z^n\|_{\mathcal{A}^2_\omega}}\right\|^2_{\mathcal{A}^2_\omega}=0.$$

Therefore,

$$\|uC_\varphi-T\|^2_{\mathcal{A}^2_\omega\to\mathcal{A}^2_\omega}\geq\limsup_{n\to\infty}\left\|uC_\varphi\left(\frac{z^n}{\|z^n\|_{\mathcal{A}^2_\omega}}\right)\right\|^2_{\mathcal{A}^2_\omega},$$

which implies that

$$\|uC_\varphi\|^2_{e;\mathcal{A}^2_\omega\to\mathcal{A}^2_\omega}\gtrsim\limsup_{n\to\infty}\frac{\|u\varphi^n\|^2_{\mathcal{A}^2_\omega}}{\|z^n\|^2_{\mathcal{A}^2_\omega}}. \tag{4.9}$$

Let $M=\limsup_{n\to\infty}n\|u\varphi^n\|^2_{\mathcal{A}^2_\omega}/\|z^n\|^2_{\mathcal{A}^2_\omega}$ and $\varepsilon>0$ be arbitrary. Then, for some $n_0\in\mathbb{N}$ and every $n\geq n_0$,

$$\frac{n\|u\varphi^n\|^2_{\mathcal{A}^2_\omega}}{\|z^n\|^2_{\mathcal{A}^2_\omega}}<M+\varepsilon.$$

Arguing as in the proof of Theorem 3.3 and applying Theorem 4.2, we have

$$\begin{aligned}\|uC_\varphi\|^2_{e;\mathcal{A}^2_\omega\to\mathcal{A}^2_\omega}\lesssim\lim_{r\to1}\sup_{|a|>r}\frac{(1-|a|^2)^{2\delta+2}}{\omega(a)}&\left[\sum_{n=0}^{n_0}n^{2\delta+3}|a|^{2n}\|u\varphi^n\|^2_{\mathcal{A}^2_\omega}\right.\\ &\left.+\sum_{n=n_0+1}^{\infty}n^{2\delta+3}\omega_n|a|^{2n}\frac{\|u\varphi^n\|^2_{\mathcal{A}^2_\omega}}{\omega_n}\right].\end{aligned} \tag{4.10}$$

Fix $r\in(0,1)$. Since

$$\sup_{|a|>r}\frac{(1-|a|^2)^{2\delta+2}}{\omega(a)}\leq\frac{(1-r^2)^{2\delta+2}}{\omega(r)}\leq\frac{(1-r^2)^{\delta+1}}{\omega(0)}=(1-r^2)^{\delta+1}.$$

Therefore,

$$\sup_{|a|>r}\frac{(1-|a|^2)^{2\delta+2}}{\omega(a)}\sum_{n=0}^{n_0}n^{2\delta+3}|a|^{2n}\|u\varphi^n\|^2_{\mathcal{A}^2_\omega}\leq C(\varepsilon)(1-r^2)^{\delta+1}. \tag{4.11}$$

In view of Lemma 3.2, we get, on the other hand,

$$\begin{aligned}&\sup_{|a|>r}\frac{(1-|a|^2)^{2\delta+2}}{\omega(a)}\sum_{n=n_0+1}^{\infty}n^{2\delta+3}\omega_n|a|^{2n}\frac{\|u\varphi^n\|^2_{\mathcal{A}^2_\omega}}{\omega_n}\\ &\lesssim\sup_{|a|>r}\left(\sum_{n=0}^{\infty}n^{2\delta+2}\omega_n|a|^{2n}\right)^{-1}\left(\sum_{n=n_0+1}^{\infty}n^{2\delta+2}\omega_n|a|^{2n}\right)(M+\varepsilon)\leq M+\varepsilon.\end{aligned} \tag{4.12}$$

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Using relations (4.10)–(4.12) and letting $r \to 1$, we obtain

$$\|uC_\varphi\|^2_{e;\mathcal{A}^2_\omega \to \mathcal{A}^2_\omega} \lesssim M + \varepsilon,$$

and since $\varepsilon > 0$ is arbitrary, we have

$$\|uC_\varphi\|^2_{e;\mathcal{A}^2_\omega \to \mathcal{A}^2_\omega} \lesssim \limsup_{n\to\infty} \frac{n\|u\varphi^n\|^2_{\mathcal{A}^2_\omega}}{\omega_n}. \qquad \blacksquare$$

In the proceeding theorem, we estimate the essential norm of weighted composition operators on $\mathcal{D}^2_\omega$.

Theorem 4.5 *Let ω be an almost standard weight, let $u \in \mathcal{D}^2_\omega$, and let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic. If $\mu^\omega_{\varphi,u\varphi'}$ and $\mu^\omega_{\varphi,u'}$ are $\mathcal{A}^2_\omega$-Carleson measures, then*

$$\|uC_\varphi\|^2_{e;\mathcal{D}^2_\omega \to \mathcal{D}^2_\omega} \lesssim \limsup_{|a|\to 1^-} \frac{(1-|a|^2)^\delta}{\omega(a)} \int_{\mathbb{D}} |\varphi_a'(z)|^{2+\delta} d\mu^\omega_{\varphi,u'}(z)$$
$$+ \limsup_{|a|\to 1^-} \frac{(1-|a|^2)^\delta}{\omega(a)} \int_{\mathbb{D}} |\varphi_a'(z)|^{2+\delta} d\mu^\omega_{\varphi,u\varphi'}(z).$$

Proof Clearly, $uC_\varphi : \mathcal{D}^2_\omega \to \mathcal{D}^2_\omega$ is bounded and $u \in \mathcal{D}^2_\omega$. As in the proof of Theorem 4.2,

$$\|uC_\varphi\|_{e;\mathcal{D}^2_\omega \to \mathcal{D}^2_\omega} \le \liminf_{n\to\infty} \|uC_\varphi \circ R_n\|_{\mathcal{D}^2_\omega \to \mathcal{D}^2_\omega}.$$

For $f \in \mathcal{D}^2_\omega$ with $\|f\|_{\mathcal{D}^2_\omega} \le 1$, we have

$$\|uC_\varphi \circ R_n f\|^2_{\mathcal{D}^2_\omega} \lesssim |u(0)R_n f(\varphi(0))|^2 + \int_{\mathbb{D}} |u'(z)|^2 |R_n f(\varphi(z))|^2 \omega(z) dA(z)$$
$$+ \int_{\mathbb{D}} |u(z)|^2 |(R_n f)'(\varphi(z))|^2 |\varphi'(z)|^2 \omega(z) dA(z)$$
$$= |u(0)R_n f(\varphi(0))|^2 + \|u'C_\varphi \circ R_n f\|^2_{\mathcal{A}^2_\omega} + \|u\varphi' C_\varphi \circ R_n f'\|^2_{\mathcal{A}^2_\omega}.$$

By (4.4), we have

$$\limsup_{n\to\infty} \sup_{\|f\|_{\mathcal{D}^2_\omega} \le 1} |u(0)R_n f(\varphi(0))|^2 \le \|u\|^2_{\mathcal{D}^2_\omega} \limsup_{n\to\infty} \sum_{k=n+1}^{\infty} \frac{|f(\varphi(0))|^{2k}}{\|z^n\|^2_{\mathcal{D}^2_\omega}} = 0.$$

Fix $r \in (0,1)$. Arguing as in the proof of Theorem 4.2, since $\mu^\omega_{\varphi,u\varphi'}$ is an $\mathcal{A}^2_\omega$-Carleson measure, we have

$$\limsup_{n\to\infty} \sup_{\|f\|_{\mathcal{D}^2_\omega} \le 1} \|u\varphi' C_\varphi \circ R_n f'\|^2_{\mathcal{A}^2_\omega} \lesssim \sup_{|a|>r} \frac{(1-|a|^2)^\delta}{\omega(a)} \int_{\mathbb{D}} |\phi_a'(z)|^{2+\delta} d\mu^\omega_{\varphi,u\varphi'}(z).$$

Since $\mu^\omega_{\varphi,u'}$ is an $\mathcal{A}^2_\omega$-Carleson measure, by (3.9), we have

$$\limsup_{n\to\infty} \sup_{\|f\|_{\mathcal{D}^2_\omega} \le 1} \|u'C_\varphi \circ R_n f\|^2_{\mathcal{A}^2_\omega} \lesssim \sup_{|a|>r} \frac{(1-|a|^2)^\delta}{\omega(a)} \int_{\mathbb{D}} |\phi_a'(z)|^{2+\delta} d\mu^\omega_{\varphi,u'}(z).$$

Thus,

$$\limsup_{n\to\infty} \|uC_\varphi \circ R_n\|^2_{\mathcal{D}^2_\omega \to \mathcal{D}^2_\omega} = \limsup_{n\to\infty} \sup_{\|f\|_{\mathcal{D}^2_\omega} \le 1} \|uC_\varphi \circ R_n f\|^2_{\mathcal{D}^2_\omega}$$

$$\lesssim \sup_{|a|>r} \frac{(1-|a|^2)^\delta}{\omega(a)} \int_{\mathbb{D}} |\phi_a'(z)|^{2+\delta} d\mu^\omega_{\varphi,u'}(z) + \sup_{|a|>r} \frac{(1-|a|^2)^\delta}{\omega(a)} \int_{\mathbb{D}} |\phi_a'(z)|^{2+\delta} d\mu^\omega_{\varphi,u\varphi'}(z).$$

Letting $r \to 1^-$, we obtain the result. ■

Corollary 4.6. *Let ω be an almost standard weight, let $u \in \mathcal{D}^2_\omega$, and let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic. If $\mu^\omega_{\varphi,u\varphi'}$ and $\mu^\omega_{\varphi,u'}$ are $\mathcal{A}^2_\omega$-Carleson measures, then*

$$\|uC_\varphi\|_{e;\mathcal{D}^2_\omega \to \mathcal{D}^2_\omega} \lesssim \limsup_{n\to\infty} \frac{n\|u'\varphi^n\|^2_{\mathcal{A}^2_\omega}}{\|z^n\|^2_{\mathcal{A}^2_\omega}} + \limsup_{n\to\infty} \frac{n\|u\varphi'\varphi^n\|^2_{\mathcal{A}^2_\omega}}{\|z^n\|^2_{\mathcal{A}^2_\omega}}. \tag{4.13}$$

Proof By Theorems 4.2 and 4.4,

$$\limsup_{|z|\to 1^-} \frac{\mu^\omega_{\varphi,u\varphi'}(E(z,1/2))}{(1-|z|^2)^2\omega(z)} \approx \limsup_{|a|\to 1^-} \frac{(1-|a|^2)^\delta}{\omega(a)} \int_{\mathbb{D}} |\varphi_a'(z)|^{2+\delta} d\mu^\omega_{\varphi,u\varphi'}(z)$$

$$\lesssim \limsup_{n\to\infty} \frac{n\|u\varphi'\varphi^n\|^2_{\mathcal{A}^2_\omega}}{\|z^n\|^2_{\mathcal{A}^2_\omega}}$$

and

$$\limsup_{|z|\to 1^-} \frac{\mu^\omega_{\varphi,u'}(E(z,1/2))}{(1-|z|^2)^2\omega(z)} \approx \limsup_{|a|\to 1^-} \frac{(1-|a|^2)^\delta}{\omega(a)} \int_{\mathbb{D}} |\varphi_a'(z)|^{2+\delta} d\mu^\omega_{\varphi,u'}(z)$$

$$\lesssim \limsup_{n\to\infty} \frac{n\|u'\varphi^n\|^2_{\mathcal{A}^2_\omega}}{\|z^n\|^2_{\mathcal{A}^2_\omega}}.$$

Therefore, by Theorem 4.5, we get (4.13), and the proof is done. ■

Corollary 4.7. *Let ω be an almost standard weight. Let $C_\varphi : \mathcal{D}^2_\omega \to \mathcal{D}^2_\omega$ be bounded. Then,*

$$\limsup_{n\to\infty} \frac{\|\varphi'\varphi^n\|^2_{\mathcal{A}^2_\omega}}{\|z^n\|^2_{\mathcal{A}^2_\omega}} \lesssim \|C_\varphi\|^2_{e;\mathcal{D}^2_\omega \to \mathcal{D}^2_\omega} \lesssim \limsup_{n\to\infty} \frac{n\|\varphi'\varphi^n\|^2_{\mathcal{A}^2_\omega}}{\|z^n\|^2_{\mathcal{A}^2_\omega}}.$$

Proof Clearly, $\|C_\varphi\|_{e;\tilde{\mathcal{D}}_\omega \to \mathcal{D}^2_\omega} \approx \|C_\varphi\|_{e;\mathcal{D}^2_\omega \to \mathcal{D}^2_\omega}$. By Theorems 4.2 and 4.5, we have $\|C_\varphi\|_{e;\mathcal{D}^2_\omega \to \mathcal{D}^2_\omega} \lesssim \|\varphi' C_\varphi\|_{e;\mathcal{A}^2_\omega \to \mathcal{A}^2_\omega}$. We define $B_\varphi = DC_\varphi D^{-1}$, where the differentiation operator D is defined by $Df(z) = f'(z)$ and its inverse by $(D^{-1}f)(z) = \int_0^z f(\xi)d\xi$. Both D and D^{-1} establish an isomorphism between $\tilde{\mathcal{D}}_\omega$ and $\mathcal{A}^2_\omega$. Using standard arguments gives that

$$\|C_\varphi\|_{e;\tilde{\mathcal{D}}_\omega \to \mathcal{D}^2_\omega} \ge \|B_\varphi\|_{e;\mathcal{A}^2_\omega \to \mathcal{A}^2_\omega} = \|\varphi' C_\varphi\|_{e;\mathcal{A}^2_\omega \to \mathcal{A}^2_\omega}.$$

Therefore, $\|C_\varphi\|_{e;\mathcal{D}^2_\omega\to\mathcal{D}^2_\omega} \approx \|\varphi' C_\varphi\|_{e;\mathcal{A}^2_\omega\to\mathcal{A}^2_\omega}$. The proof is done by Theorem 4.4. ∎

As an immediate consequence of Corollary 4.7, we have the following result.

Corollary 4.8. *Let ω be an almost standard weight, and let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic such that $C_\varphi : \mathcal{D}^2_\omega \to \mathcal{D}^2_\omega$ is bounded. Then, C_φ is compact if*

$$\limsup_{n\to\infty} \frac{n\|\varphi'\varphi^n\|^2_{\mathcal{A}^2_\omega}}{\|z^n\|^2_{\mathcal{A}^2_\omega}} = 0.$$

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References

[1] G. Bao and H. Wulan, *Hankel matrices acting on Dirichlet spaces*. J. Math. Anal. Appl. **409**(2014), no. 1, 228–235.
[2] O. Constantin, *Carleson embeddings and some classes of operators on weighted Bergman spaces*. J. Math. Anal. Appl. **365**(2010), no. 1, 668–682.
[3] M. D. Contreras, J. A. Peláez, C. Pommerenke, and J. Rättyä, *Integral operators mapping into the space of bounded analytic functions*. J. Funct. Anal. **271**(2016), no. 10, 2899–2943.
[4] C. Cowen and B. MacCluer, *Composition operators on spaces of analytic functions*, CRC Press, Boca Raton, 1995.
[5] Z. Čučković and R. Zhao, *Weighted composition operators on the Bergman space*. J. Lond. Math. Soc. **70**(2004), no. 2, 499–511.
[6] Z. Čučković and R. Zhao, *Weighted composition operators between weighted Bergman spaces and different hardy spaces*. Ill. J. Math. **51**(2007), no. 2, 479–498.
[7] P. Duren and A. Schuster, *Bergman spaces*, Mathematical Surveys and Monographs, 100, American Mathematical Society, Providence, RI, 2004.
[8] O. El-Fallah, H. Mahzouli, I. Marrhich, and H. Naqos, *Toeplitz operators on harmonically weighted Bergman spaces and applications to composition operators on Dirichlet spaces*. J. Math. Anal. Appl. **466**(2018), no. 1, 471–489.
[9] W. W. Hastings, *A Carleson measure theorem for Bergman spaces*. Proc. Amer. Math. Soc. **52**(1975), 237–241.
[10] K. Kellay and P. Lefèvre, *Compact composition operators on weighted Hilbert spaces of analytic functions*. J. Math. Anal. Appl. **386**(2012), 718–727.
[11] T. L. Kriete III and B. MacCluer, *Composition operators on large weighted Bergman spaces*. Indiana Univ. Math. J. **41**(1992), no. 3, 755–788.
[12] M. Lindström and E. Saukko, *Norm of weighted composition operators and difference of composition operators between standard weighted Bergman spaces*. Complex Anal. Oper. Theory **9**(2015), no. 6, 1411–1432.
[13] D. H. Luecking, *Forward and reverse Carleson inequalities for functions in Bergman spaces and their derivatives*. Amer. J. Math. **107**(1985), 85–111.
[14] J. Pau and P. A. Pérez, *Composition operators acting on weighted Dirichlet spaces*. J. Math. Anal. Appl. **401**(2013), no. 2, 682–694.
[15] S. Stevic and A. Sharma, *Generalized composition operators on weighted hardy spaces*. Appl. Math. Comput. **218**(2012), 8347–8352.
[16] K. Zhu, *Positive Toeplitz operators on weighted Bergman spaces of bounded symmetric domains*. J. Operator Theory **20**(1988), no. 2, 329–357.
[17] K. Zhu, *Operator theory in function spaces*. 2nd ed., Mathematical Surveys and Monographs, 138, American Mathematical Society, Providence, RI, 2007.

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