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CHARACTERISATION THEOREMS FOR COMPACT HYPERCOMPLEX MANIFOLDS

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Abstract

We have defined and studied some pseudogroups of local diffeomorphisms which generalise the complex analytic pseudogroups. A 4-dimensional (or 8-dimensional) manifold modelled on these 'Fueter pseudogroups' turns out to be a quaternionic (respectively octonionic) manifold.

We characterise compact Fueter manifolds as being products of compact Riemann surfaces with appropriate dimensional spheres. It then transpires that a connected compact quaternionic ($\mathbb{H}$) (respectively $\mathbf{O}$) manifold X, minus a finite number of circles (its 'real set'), is the orientation double covering of the product $Y \times \mathbf{P}^2$, (respectively $Y \times \mathbf{P}^6$), where Y is a connected surface equipped with a canonical conformal structure and $\mathbf{P}^n$ is n-dimensonal real projective space.

A corollary is that the only simply-connected compact manifolds which can allow $\mathbb{H}$ (respectively $\mathbf{O}$) structure are S^4 and $S^2 \times S^2$ (respectively S^8 and $S^2 \times S^6$).

Previous authors, for example Marchiafava and Salamon, have studied very closely-related classes of manifolds by differential geometric methods. Our techniques in this paper are function theoretic and topological.

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1. The Fueter and hypercomplex pseudogroups

Over 50 years ago, R. Fueter [2] had defined a class of mappings whose domain and range are open subsets of $\mathbb{R}^n$, $(n \geqslant 2)$, obtained by a certain transformation from complex analytic functions. Indeed, let ϕ be a holomorphic map whose domain and range are open subsets of the upper half-plane $U = \{z = x_0 + iy$:

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$y > 0\}$. The 'n-dimensional Fueter transform' of ϕ, denoted $F_n(\phi)$, is a C^∞ (in fact C^ω) map from an open domain of $\mathbb{R}^n$ to $\mathbb{R}^n$ obtained by substituting in ϕ the expression $(e_1x_1 + \cdots + e_{n-1}x_{n-1})/\sqrt{x_1^2 + \cdots + x_{n-1}^2}$ for the imaginary unit i. (Here $\mathbb{R}^n$ has coordinates $(x_0, x_1, \ldots, x_{n-1})$ with unit vectors $(e_0, e_1, \ldots, e_{n-1})$. We deliberately identify x_0 as also the real coordinate in the z-plane).

One chief reason for interest in the 'Fueter maps' stems from the fact that $F_4(\phi)$ and $F_8(\phi)$ are expressible as power series in a quaternionic or octonionic variable (respectively) when ϕ has a formally-real expansion around real centres. The precise *definition of* $F_n(\phi)$ is as follows:

Let U denote the upper half-plane, D be a region in U, and $'\mathbb{R}^n = \mathbb{R}^n - \{x_0$ axis$\}$, $n \geqslant 2$. We set

$$(1) \qquad F_n(D) = \left\{(x_0, \ldots, x_{n-1}) \in \mathbb{R}^n \colon \left(x_0, \sqrt{x_1^2 + \cdots + x_{n-1}^2}\right) \in D\right\}.$$

Let $\phi\colon D \to U$ be complex analytic with real and imagnary part decomposition $\phi = \xi + i\eta$. Then $F_n(\phi) = F_n(D) \to \mathbb{R}^n$ is defined by

$$(2) \quad F_n(\phi)(x_0 + e_1x_1 + \cdots + e_{n-1}x_{n-1}) = \xi(x_0, y) + \sum_{j=1}^{n-1} e_j \frac{x_j}{y} \eta(x_0, y)$$

where $y = \sqrt{x_1^2 + \cdots + x_{n-1}^2}$ (the positive square root). Note that $F_2(\phi) \equiv \phi$ on $F_2(D) \cap U$.

We show easily that if ϕ is a diffeomorphism then so is $F_n(\phi)$, and the restrictions of these Fueter diffeomorphisms to arbitrary open subsets of their domains *form a pseudogroup of transformations* on $'\mathbb{R}^n$. Indeed one finds that

$$(3) \qquad F_n(\phi \circ \psi) = F_n(\phi) \circ F_n(\psi).$$

Further, the Fueter transform behaves as a homomorphism of linear spaces of functions, and for $n = 4, 8$ it works as an algebra homomorphism preserving multiplications in $\mathbb{H}$ and $\mathbb{O}$.

The mappings $F_n(\phi)$ are in general not conformal, nor are general Möbius transformations in the Fueter class. Indeed we have calculated that $F_n(\phi)$ is K-quasiconformal in the sense of Ahlfors' 'Möbius transformations in several dimensions' if $|\partial\eta/\partial y - \eta/y| \leqslant 2K$ over the domain D of $\phi = \xi + i\eta$.

If the holomorphic function ϕ has real boundary values where the real axis abuts D then a direct application of *the reflection principle* guarantees that $F_n(\phi)$ can be defined real-analytically on the revolved domain $F_n(D)$ together with the corresponding portions of the x_0-axis.

The fundamental observation of Fueter [2] for our purposes is that *if* ϕ *has a Laurent expansion with real coefficients around real centres*,

$$(4) \qquad \phi(z) = \sum_{n=0}^{\infty} a_n(z-c)^n + \sum_{m=1}^{\infty} b_m \frac{1}{(z-c)^m}$$

(a_n, b_m, c are real, the annulus of convergence is $r < |z - c| < R$) then

$$F_4(\phi)(V) = \sum_{n=0}^{\infty} a_n(V-c)^n + \sum_{m=1}^{\infty} b_m \frac{1}{(V-c)^m} \tag{5}$$

where $V = x_0 + e_1x_1 + e_2x_2 + e_3x_3$ is a quaternionic variable. Similarly, $F_8(\phi)$ will be represented by the 'same' Laurent series with V an octonionic variable. The corresponding domains of convergence are the ring-domains $r < \|V - c\| < R$ in Euclidean spaces $\mathbb{R}^4$ and $\mathbb{R}^8$ respectively. We will allow power series as special cases of the Laurent series, in which case the convergence domains are of course Euclidean balls with centre c.

Note. If the central Laurent series (4) for the complex variable z is a function ϕ mapping U into U then the quaternionic series is precisely $F_4(\phi)$; but if ϕ maps U to the lower half-plane then the corresponding quaternionic series (5) is $-F_4(-\phi)$.

These Laurent (or power) series with central (= real) coefficients in $\mathbb{H}$ or $\mathbb{O}$ variable, together with their restrictions to open subsets of their domains of convergence, will clearly form *pseudogroups of transformations in* $\mathbb{R}^4$ *and* $\mathbb{R}^8$. Manifolds modelled on these as the coordinate transition functions will be christened 1*-dimensional central quaternionic* (*resp. octonionic*) *manifolds*. Briefly we will call them *hypercomplex manifolds* (with $\mathbb{H}$ or $\mathbb{O}$ structure) and their coordinate charts will be identified by the same names.

Our observations show that these pseudogroups are basically sub-pseudogroups of the Fueter pseudogroups in dimensions 4 and 8. Thus hypercomplex manifolds can be studied by investigating *Fueter manifolds*, that is, manifolds with transition functions from the $F_n(\phi)$ class of local diffeomorphisms.

One defines the relevant morphisms and isomorphisms in each of these categories of manifolds in the standard fashion. Notice that a 2-dimensional Fueter manifold is nothing other than a Riemann surface; thus Fueter structures generalise complex structures. It may be interesting to note that $F_n(\phi)$ assigns a consistent meaning to the "power series $\sum \alpha_n(V - \beta)^n$" where V is a $\mathbb{R}^n$-variable, and $\phi = \sum \alpha_n(z - \beta)^n$ (even though $\mathbb{R}^n$ has no algebra structure except for $n = 1, 2, 4, 8$).

We have characterised a Fueter mapping by equations *generalising the Cauchy-Riemann equations*. In particular the Jacobian matrix entries of $F_n(\phi) = (f_0, f_1, \dots, f_{n-1})$ *satisfies the following relations* (here $\partial_k = \partial/\partial x_k$):

$$\begin{cases} \partial_0 f_j = -\partial_j f_0, \ (j > 0) \\ \partial_i f_j = \partial_j f_i, \ (i, j > 0) \\ \langle \nabla f_0, \nabla f_j \rangle = 0, \ (j > 0), \text{ that is, the level hypersurfaces} \\ \text{for } f_0 \text{ always intersect orthogonally the levels for } f_j. \end{cases} \tag{6}$$

REMARK. We can complete this system of partial differential equations for $F_n(\phi)$ to give *six conditions which are necesary and sufficient for* $(f_0, f_1, \dots, f_{n-1})$ *to be a Fueter mapping*. See [1]. In the quaternionic and octonionic cases the equations (6) exhibited above can be established by purely algebraic methods which work uniformly over $\mathbb{H}$, $\mathbb{O}$, and also $\mathbb{C}$, for central-coefficient Laurent series, (see Nag [3]).

2. Characterising Fueter and hypercomplex manifolds

The work and results of this section are mainly the responsibility of the first two authors, Nag and Hillman.

We shall identify $'\mathbb{R}^n = \mathbb{R}^n - \{x_0\text{-axis}\}$ with $U \times S^{n-2}$ in a fixed fashion which will play a crucial role in our theory. Here S^{n-2} is the standard unit sphere in $\mathbb{R}^{n-1}$. Indeed, we map:

(7)

$$(x_0 + e_1x_1 + \cdots + e_{n-1}x_{n-1}) \mapsto \left(x_0 + iy, \left(\frac{x_1}{y}, \frac{x_2}{y}, \dots, \frac{x_{n-1}}{y}\right)\right) \in U \times S^{n-2}$$

where $y = \sqrt{x_1^2 + \cdots + x_{n-1}^2}$ (positive square root).

We will think of $U_\sigma = U \times \{\sigma\}$, for any $\sigma \in S^{n-2}$, as the rotated position of a standard upper half-plane $U \times \{(1, 0, \dots, 0)\}$ in $'\mathbb{R}^n$. The axis of rotation is of course the x_0-axis. We can therefore identify any U_σ with U.

The crux of the matter is that a Fueter mapping $F_n(\phi)$ on $F_n(D)$ is the 'function of revolution' obtained by revolving the function ϕ and its domain D around the x_0-axis. Thus

LEMMA 1 '*Revolution Principle*'. *A Fueter map $F_n(\phi)$ preserves half-planes which have the x_0-axis as boundary, that is, $F_n(\phi)$ maps U_σ into itself and in fact $F_n(\phi)$ restricted to $U_\sigma \cap F_n(D)$ (for any σ) is precisely ϕ on $D \subset U \equiv U_\sigma$.*

PROOF. A simple but important calculation from formulae (1), (2) and (7).

It now follows immediately that *for any n-dimensional Fueter manifold X there is an intrinsically defined C^∞ submersion*

$$(8) \qquad p\colon X \to S^{n-2}.$$

That is, $p(x) = \sigma$ determines the half-plane U_σ in which $x \in X$ lies with respect to any Fueter coordinate chart around x. By Lemma 1 the map p is well-defined and *each fibre of p has the structure of a 1-dimensional complex manifold*. Note that *neighbouring fibres have canonical local (biholomorphic) identifications* determined simply by the Fueter structure of X. Indeed the local identifications are obtained

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by identifying U_σ with $U_{\sigma'}$ (by rotation around the x_0-axis) in the image (in $'\mathbb{R}^n$) of any Fueter chart. Again because of Lemma 1 the identifications do not depend on the Fueter chart used, (charts will always be required to have connected domains).

Clearly if X is compact then the fibre of p is a compact Riemann surface R, and p is a C^∞ fibre bundle (being a proper submersion). By standard compactness arguments we can then show that there are canonical *global* biholomorphic identifications between any two fibres of p. We therefore derive

THEOREM 1. *Any compact n-dimensional Fueter manifold X is Fueter-category isomorphic to the product of a compact Riemann surface R with S^{n-2}. (In fact, $R \times S^{n-2}$ has a canonical Fueter structure for any Riemann surface R in an obvious way. We remark that if $R = U/G$, G an arbitrary Fuchsian group, then $R \times S^{n-2} \equiv {'\mathbb{R}^n}/F_n(G)$ as a Fueter manifold. G can be allowed to possess elliptic elements, and every Riemann surface R then occurs as U/G.)*

Details of the proof of the above theorem are omitted because they are exactly analogous to, (but much simpler than), the proofs for the more subtle hypercomplex manifold results which we explain below.

LEMMA 2. *Any Fueter or hypercomplex manifold is orientable.*

PROOF. A direct calculation or usage of Lemma 1 leads to the relation

$$\det(\operatorname{Jac} F_n(\phi)) = \left(\frac{\eta}{y}\right)^{n-2} \det(\operatorname{Jac}(\phi)) \tag{9}$$

(notations as in equations (1) and (2)).

It follows that the Jacobian determinants of any Fueter or hypercomplex transition function is everywhere positive.

Thus, if we choose a fixed orientation for $\mathbb{R}^4$ any $\mathbb{H}$ manifold then gets a canonical orientation.

DEFINITION. The set of poinits in a hypercomplex manifold X whose image under *any* hypercomplex chart is on the real (x_0)-axis is a closed 1-dimensional submanifold of X called its '*real set*' $\rho_X = \rho$.

Clearly, if X is compact ρ_X is a finite union of circles smoothly embedded in X.

Now, any central $\mathbb{H}$ or $\mathbb{O}$ Laurent series will map any 2-plane containing the x_0 axis into itself; so, using the facts for $F_4(\phi)$, (Lemma 1), we can understand a $\mathbb{H}$ or $\mathbb{O}$ Laurent series function as a 'function of revolution' obtained by revolving a complex analytic function around the real axis. 2-planes in $\mathbb{R}^n$ through x_0 axis are parametrised by the real projective space $\mathbb{P}^{n-2}$, so on the hypercomplex manifold X we can define a natural C^∞ submersion, (in analogy with (8)),

$$p\colon X - \rho_X \to \mathbb{P}^{n-2} \qquad (n = 4 \text{ or } 8). \tag{10}$$

EXAMPLE. Let $X \equiv S^4 = \mathbb{R}^4 \cup \{\infty\}$. We can give this a $\mathbb{H}$-structure analogous to the complex structure of the Riemann sphere, by assigning the identity chart on $\mathbb{R}^4$ and obtaining $V \mapsto 1/V$ as the transition function to the obvious chart covering $(\mathbb{R}^4 - \text{origin}) \cup \{\infty\}$.

Notice ρ is then the 'real circle' $\{x_0\text{-axis}\} \cup \{\infty\}$ and the map $p\colon X - \rho \equiv {}'\mathbb{R}^4 \to \mathbb{P}^2$ is precisely the second component of the identification map (7) followed by the standard double covering $\pi\colon S^2 \to \mathbb{P}^2$. The fibres of p are two disjoint half-planes. (S^8 has similar octonionic description.)

Thus S^4 is quaternionic projective space $\mathbb{P}^1(\mathbb{H})$.

Note that for p to be well defined we must be mapping to $\mathbb{P}^2$, and not to S^2; because *if* (V, ϕ) *is a* $\mathbb{H}$*-chart on* X *then so is* $(V, -\phi)$, and the map ϕ: $V \to {}'\mathbb{R}^4 \equiv U \times S^2$ assigns the S^2 values *antipodal* to those determined by $-\phi$: $V \to {}'\mathbb{R}^4$. Notice further that the upper half-plane element assigned by ϕ to any $x \in V\,(\subset X)$ is the reflection across the y-axis (in U) of the U-element associated to x by $-\phi$.

THEOREM 2. *Let X be a connected compact hypercomplex manfold with real set ρ. Then*

(a) *$p\colon X - \rho \to \mathbb{P}^{n-2}$ ($n = 4$ for $\mathbb{H}$, 8 for $\mathbb{O}$) as defined in* (10) *is a C^∞ fibre bundle which is not globally trivial. Let the fibre $p^{-1}(k)$ for any $k \in \mathbb{P}^{n-2}$ be denoted $X(k)$ $(\subset X - \rho)$.*

(b) *$X(k)$ is an orientable surface with at most two components. It has a canonical conformal structure induced by the hypercomplex structure of X.*

(c) *The closure $\overline{X(k)}$ of $X(k)$ in X is precisely $X(k) \cup \rho$ (for all $k \in \mathbb{P}^{n-2}$). $\overline{X(k)}$ is itself orientable, and if ρ is non-empty then $\overline{X(k)}$ is connected.*

(d) *$\overline{X(k)}$ is a compact surface with a conformal structure, (that is, transition functions are holomorphic or conjugate-holomorphic) and there is a global conformal identification of $\overline{X(k)}$ with $\overline{X(k')}$ for all k' in a neighbourhood of k. These identifications are determined by the hypercomplex structure of X and act as the identity when restricted to ρ.*

PROOF. We deal only with the $\mathbb{H}$-case since no new ideas come in for $\mathbb{O}$.

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First notice that *p is surjective*. Indeed, if ρ is empty then p being submersive and X being compact says p is onto. If ρ has a point ξ in it then any chart ϕ around ξ will map to a 4-dimensional open neighbourhood of $\phi(\xi)$, ($\phi(\xi)$ is on x_0-axis), and already every 2-plane is intersected; so p is onto.

Note that the last argument shows that each point of ρ is a limit point of every fibre $p^{-1}(k)$, and then the first part of (c) follows easily.

Consider now any chart (V, ϕ), $V \subset X$; ϕ assigns to each point of $V - \rho$ a point in $'\mathbb{R}^4 \equiv U \times S^2$. (Recall (7).) *Thus to any $x \in V - \rho$, ϕ assigns an element of U and an element of S^2.* We denote the map ϕ restricted to $(V - \rho) \cap X(k)$ by $\phi(k)$, $(k \in \mathbb{P}^2)$. We can think of $\phi(k)$ as a chart on a small piece of $X(k)$, mapping it to U; (by cutting down the size of V we may assume $(V - \rho) \cap X(k)$ is connected—so $\phi(k)$ maps to exactly *one* half-plane). If (W, ψ) is another chart around $x \in X - \rho$ then ψ assigns to x either the same S^2-value or the opposite S^2-value to that asigned by ϕ. Since the hypercomplex central Laurent series are essentially Fueter mappings we see from the fundamental 'revolution principle' (Lemma 1) that $\phi(k)$ and $\psi(k)$ are holomorphically related near $x \in X(k)$ if the S^2-values coincided, and are anti-holomorphically related if the S^2-values were antipodal.

In any case $X(k)$ has a conformal structure, which, by using charts at points of ρ also, clearly extends to a conformal structure on all of $\overline{X(k)}$.

Now let us explain the local conformal identification of fibres. As for Fueter manifolds, these come by using charts and rotating half-planes to fall on one another. Let (V, ϕ) be a '*small chart*' on X with a 'small' image in $'\mathbb{R}^4$, that is, $\phi(V)$ does not intersect any pair of opposite half-planes. In that case

$$\phi(k')^{-1} \circ \phi(k) \tag{11}$$

for nearby values of k and k' will be a conformal identification of a piece of $X(k)$ with a piece of $X(k')$. Notice that if we use a different chart (W, ψ) the identifications are still the same:

$$\begin{aligned} \psi(k')^{-1} \circ \psi(k) &\equiv \phi(k')^{-1} \circ \phi(k') \circ \psi(k')^{-1} \circ \psi(k) \circ \phi(k)^{-1} \circ \phi(k) \\ &\equiv \phi(k')^{-1} \circ \left[\phi(k') \circ \psi(k')^{-1}\right] \circ \left[\psi(k) \circ \phi(k)^{-1}\right] \circ \phi(k) \\ &= \phi(k')^{-1} \circ \phi(k) \end{aligned} \tag{12}$$

because the square-bracketed mappings cancel each other off by the revolution principle.

Thus, a point $x_1 \in X(k)$ is identified with a point $x_2 \in X(k')$ (k' near k) precisely when the U-values assigned to x_1 and x_2 are the same via any small chart containing both x_1 and x_2 in its domain. It is obvious that $\overline{X(k)}$ can now be conformally identified with $\overline{X(k')}$ (for k' in a small neighbourhood of k in $\mathbb{P}^2$) by extending these canonical identifications to ρ, the extension being the identity on ρ. (Since $\overline{X(k)}$ is compact there is no problem in using a finite number of small hypercomplex charts to cover $X(k)$, and thus get the conformal mappings globally from all of $X(k)$ to each $X(k')$, for k' sufficiently near to k.)

Since we have now got a canonical way to map the fibre $X(k)$ onto $X(k')$, for all k' in a small neighbourhood of k in $\mathbb{P}^2$, it is clear that we have *local triviality*; and so p is a C^∞ fibre bundle.

The bundle cannot be globally trivial since a product $Y \times \mathbb{P}^2$ cannot be orientable for *any* surface Y whatsoever. Since $X - \rho$ is orientable we also see that the fibre $X(k)$ *must be orientable* since the local triviality of the bundle makes $X(k) \times$ (small 2-disc) an open subset of the orientable $X - \rho$. This says nothing, however yet, for orientability of the compact surface $\overline{X(k)} = X(k) \cup \rho$, (for example, a Klein bottle minus a circle can be an annulus). The fibre homotopy exact sequence for p:

$$\cdots \to \pi_1(X - \rho) \to \pi_1(\mathbb{P}^2) \to \pi_0(X(k)) \to \pi_0(X - \rho) \to$$

shows immediately that $X(k)$ *has at most two components.*

To complete the proof of Theorem 2 we need to prove the rather subtle assertions of part (c). We abstract this situation into the following topological proposition.

PROPOSITION 1. *Let X be a connected oriented closed smooth 4-manifold with a non-empty smooth closed 1-submanifold ρ such that there is a bundle projection p: $X - \rho \to \mathbb{P}^2$, with fibre F. Suppose that for each $k \in \mathbb{P}^2$ the closure $\overline{X(k)}$ in X of the fibre $p^{-1}(k) = X(k)$ is $X(k) \cup \rho$, and is a closed 2-submanifold in X. Then*

(i) *ρ is 2-sided in $\overline{X(k)}$,*
(ii) *$\overline{X(k)}$ is orientable,*
(iii) *$\overline{X(k)}$ is connected.*

Note: (ii) $\Rightarrow$ (i) of course.

PROOF. Since X and ρ are orientable, the normal bundle of ρ in X is orientable and therefore trivial; so ρ has a closed product neighbourhood N homeomorphic ($\approx$) to $\rho \times D^3$ in X. We may (either using the geometry of our hypercomplex X or by topology) choose N so that $p \mid X - \operatorname{int} N$ is still a bundle projection. The new fibre G is then a surface with boundary, $\operatorname{int} G \approx F$.

If $\rho_1 (\approx S^1)$ is a component of ρ, and $M_1 = \rho_1 \times S^2$ is the corresponding component of ∂N, the restriction $p \mid M_1$ to $\mathbb{P}^2$ is again a fibre bundle. The fibre homotopy exact sequence says that the fibre of $p \mid M_1$ has either 1 or 2 components—necessarily circles.

But, in fact the fibre must have 2 (circle) components because the total space of any S^1-bundle over $\mathbb{P}^2$ can never be $S^1 \times S^2$ ($\approx M_1$). (The same principle holds for $S^1 \times S^n$ fibering over $\mathbb{P}^n$, any $n \geqslant 2$. The $n = 6$ case is needed for octonionic manifolds. We are grateful to Dr. L. Noakes for supplying us with a proof of this fact.)

Hence the fibre of $p \mid X - \operatorname{int} N$, say $G(k)$ above any $k \in \mathbb{P}^2$, has boundary $\partial G(k) = \rho \times \{-1, 1\}$.

Again by the homotopy exact sequence we see that $\pi_1(M_1)$ maps trivially to $\pi_1(\mathbb{P}^2)$; therefore $p \mid M_1$ factors through the double covering π: $S^2 \to \mathbb{P}^2$ via a map θ: $M_1 \to S^2$ which is itself a bundle map with fibre S^1. Now, the only S^1-bundle over S^2 with total space homeomorphic to $S^1 \times S^2$ is the trivial bundle, so we may choose a homeomorphism h_1: $M_1 \to S^1 \times S^2$ such that $\theta = pr_2 \circ h_1$ and so that $p \mid M_1 = \pi \circ pr_2 \circ h_1$, ($pr_2$ is a projection to the second factor of course).

We make this choice of homeomorphism h_j for each component ρ_j of ρ, and clearly we can 'radially' extend the union of all the h_j to a homeomorphism H: $N \to \rho \times D^3$, so that $H(\rho) = \rho \times \{0\}$ and $p \mid N - \rho$ is $\pi \circ \widehat{pr}_2 \circ H$ where $\widehat{pr}_2$: $\rho \times (D^3 - \{0\}) \to S^2$ projects onto the second factor and then normalizes.

We have therefore proved that *$p \mid N - \rho$ is bundle equivalent to a union of copies of the obvious bundle $S^1 \times (D^3 - \{0\}) \to \mathbb{P}^2$.*

It follows that the closure in N of any fibre of $p \mid N - \rho$ is homeomorphic to $\rho \times [-1, 1]$, and hence that for any $k \in \mathbb{P}^2$ the circles ρ are two-sided in $\overline{X(k)} = G(k) \cup_{\rho \times \{-1,1\}} \rho \times [-1, 1]$.

We do not yet know that the annuli $\rho \times [-1, 1]$ are attached to $G(k)$ so as to produce an orientable $\overline{X(k)}$, but this can now be derived as follows.

Note first that an orientation for a disc neighbourhood of k in $\mathbb{P}^2$ determines a transverse orientation of the normal bundle of $G(k)$ in $X - \operatorname{int} N$, and in particular of $\partial G(k)$ in ∂N. It will suffice to check that the orientations determined in this way by $\pi \circ pr_2$: $S^1 \times S^2 \to \mathbb{P}^2$ on $S^1 \times \{\sigma\}$ and $S^1 \times \{-\sigma\}$ (here $\{\sigma, -\sigma\} = \pi^{-1}(k)$) extend compatibly to $S^1 \times \delta$, where δ is the diameter in $\mathbf{R}^3$ connecting σ and $-\sigma$.

Now, an orientation on the segment δ must point inward at one end and outwards at the other end, so it determines opposing transverse orientations about σ and $-\sigma$. Conversely, since the covering involution of S^2 over $\mathbb{P}^2$ is orientation reversing the transverse orientations we had induced at σ and $-\sigma$ from a local orientation around $k \in \mathbb{P}^2$ must again be opposite,—so they give rise to a consistent orientation of the diameter δ. (It is crucial that we are dealing with even dimensional projective spaces here!)

Thus, $G(k)$ was orientable (being within the orientable $X(k)$), and we now see that the orientation extends over the attached annuli, proving $\overline{X(k)}$ is orientable.

We prove $\overline{X(k)}$ is connected by constructing another bundle over $\mathbb{P}^2$ with closed fibre by a process analogous both to surgery and to blowing-up. Let B be the mapping cylinder of the covering π: $S^2 \to \mathbb{P}^2$. Then B is a 3-manifold with boundary S^2 and it fibres over $\mathbb{P}^2$ with fibre $[-1, 1]$. In fact B is homeomorphic to a closed regular neighbourhood of $\mathbb{P}^2$ in $\mathbb{P}^3$, and therefore to $(\mathbb{P}^3 - \operatorname{int} D^3)$. Let $W = (X - \operatorname{int} N) \cup_{\partial} \rho \times B$. Then W is orientable and fibres over $\mathbb{P}^2$ with fibre $\tilde{F} = G \cup_{\rho \times \{-1,1\}} \rho \times [1, 1]$. Since W is orientable $\tilde{F}$ is too (just as in the proof for orientability of $X(k)$). We cannot straightaway identify $\tilde{F}$ with $\overline{X(k)} = \bar{F}$ but from the construction we see that they have the same number of components (and indeed $\tilde{F} \approx \bar{F}$ if and only if $\bar{F}$ is orientable, which we know it is).

To see that $\tilde{F}$, and hence $\overline{X(k)}$, is connected we note that B contains a loop projecting to the non-trivial element in $\pi_1(\mathbb{P}^2)$. Therefore, since $\rho \neq \phi$, W also has such a loop and $\pi_1(W) \to \pi_1(\mathbb{P}^2)$ is surjective. By the fibre homotopy sequence, therefore, $\tilde{F}$ is connected.

This completes Proposition 1 and Theorem 2.

Our prime example of manifolds with hypercomplex structure we will describe in the next:

PROPOSITION 2. *Let G be torsion-free Fuchsian group operating on U; let Y be the Riemann surface U/G, and $\Omega = U \cup L \cup \tilde{\rho}$ be the full region of discontinuity for G (on $\hat{\mathbb{C}}$). (Here L is the lower half-plane and $\tilde{\rho}$ is the portion (possibly empty) of Ω on $\mathbb{R} \cup \{\infty\}$). Then $X = X_G \equiv {}'\mathbb{R}^n \cup \tilde{\rho}/F_n(G)$ is a manifold with (central) hypercomplex structure, real set ρ being the ideal boundary of Y, and (with previous terminology) $X(k) = U/G \cup L/G = Y \cup (-Y)$, and $\overline{X(k)} = \Omega/G =$ the Schottky double of Y.*

PROOF. This is clear since if $g(z) = (az + b)/(cz + d)$, $g \in G$, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ then $F_n(g)(V) = (aV + b)/(cV + d)$, V being $\mathbb{H}$ or $\mathbb{O}$ variable. Clearly these give transitions in the allowed pseudogroup (expand as Laurent series around $-d/c$).

X_G will be compact precisely when Ω/G is compact.

For a finer description of compact hypercomplex manifolds we need to understand the 'monodromy' of the local identifying maps between the fibers of p; that is, if we take a composition of a finite chain of the identifying maps between nearby fibres, say, $X(k) \to X(k_1) \to X(k_2) \to \cdots \to X(k_n) \to X(k)$, then what will the final conformal automorphism of $X(k)$ look like? (The identifications always extend to $\overline{X(k)}$ by the identity on ρ, and they always preserve angles but not necesarily orientations.) X is assumed to be a compact connected $\mathbb{H}(\mathbb{O})$ manifold in all that follows.

LEMMA 3. *The composition of any finite chain of the canonical maps between nearby fibres of p always produces on any $X(k)$ either the identity or a certain canonical fixed-point free involution $\tau(k)$.*

REMARK. $\tau(k)$ extends to $\overline{X(k)}$ by the identity on ρ, and in fact the proof following shows that near ρ $\tau(k)$ is precisely quaternion (octonion) conjugation. Thus, if ρ is non-empty then $\tau(k)$ is orientation reversing anti-conformal on the connected compact Riemann surface $\overline{X(k)}$.

PROOF. Consider an equivalence relation $\sim$ on $X - \rho$ defined as follows: $x, y \in X - \rho$ are $\sim$ equivalent if there exists a finite sequence of points $x_1 = x, x_2, \dots, x_n = y$ such that each consecutive pair x_i, x_{i+1} are in the (connected) domain of some chart (V_i, ϕ_i) and the U-element assigned by ϕ_i to x_i and x_{i+1} coincide. This is clearly an equivalence relation.

Of course, if V_i is a small chart then $x_i \sim x_{i+1}$ exactly when the canonical identification of fibres maps x_i to x_{i+1}. Thus $\sim$ corresponds precisely to compositions of several local identifying maps of the type (11).

We claim that if $x \sim z$ then there exists a *single* hypercomplex chart (V, ϕ) with $x, z \in V$ and U-values coincident for x and z via ϕ. This will follow if we can fuse together charts and extend them 'in the S^2-factor directions'; we achieve this by applying the revolution principle.

To fix notations suppose $x \sim y$ are in (T, θ) chart and $y \sim z$ are in (W, ψ) chart, both being small charts and without loss of generality assume $W \cap T$ is connected. $y \in W \cap T$ of course. Then let $p(y) = k$, so we may assume $W \cap T \cap X(k)$ is a non-empty connected open subset of the fibre $X(k)$. Define extension of the θ-chart by

$$(13) \qquad \theta^{\text{ext}} = \begin{cases} \theta & \text{on } T, \\ F_4\big(\theta(k) \circ \psi(k)^{-1}\big) \circ \psi & \text{on } \psi^{-1}\big(\psi(W) \cap F_4(D)\big), \end{cases}$$

where $\psi(k)(W \cap T \cap X(k)) = D \subset U$ (upper half-plane).

Note, we have arranged $\theta(k) \circ \psi(k)^{-1}$ to be holomorphic on $D \subset U$ (into U) by replacing θ by its negative if necesary (see paragraph preceding Theorem 2). Then clearly θ^{ext} will be in the hypercomplex atlas of X and its domain contains x and z with same U-value being assigned by θ^{ext} to both points. We can thus establish our claim by induction.

We can show that on any fibre $X(k)$ the relation $\sim$ identifies points in pairs, —and this is the involutory automorphism $\tau(k)$ of $X(k)$ which we have in the statement of Lemma 3.

In fact, let $x_1 \in X(k)$ be within a small chart (V, ϕ) around it. Let $\phi(x_1) = (\zeta, \sigma) \in U \times S^2 \equiv {}'\mathbb{R}^4$. Thus $k = \mathbb{P}^2$-class of σ. Connect σ to $-\sigma$ by a half-circle γ on S^2. We claim that there exists a chart $(W, \tilde{\phi})$ which extends ϕ and

$\tilde{\phi}(W) \subset \mathbb{R}^4$ intersects all the half-planes corresponding to points of γ. If this were not true there would be a first point σ_1 on γ (starting from σ) which is not included in any such chart. But by taking limits and using compactness we see there is some point in $X - \rho$ corresponding to (ζ, σ_1). We take any small chart around this point and use the previous fusing of charts argument to extend ϕ a little further in the S^2-direction; that is, σ_1 cannot exist. It is not hard to see that σ_1 cannot be $-\sigma$ either.

Thus, we will have a ('big') chart (V, ϕ) containing x_1 and also containing a point x_2 such that $\phi(x_2) = (\zeta, -\sigma)$. Thus $x_1 \sim x_2$ (both on $X(k)$), and $\tau(k)$ interchanges x_1 and x_2 on $X(k)$. Because of the revolution principle a different choice of charts makes no effect on the definition of $\tau(k)$; the proof of this is similar to the equations (12) in the proof of Theorem 2.

Lemma 3 is proved. Indeed, note that our γ on S^2 represents the non-trivial element of $\pi_1(\mathbb{P}^2)$, and continuation of charts along γ has led to the involution $\tau(k)$ on $X(k)$. Continuation of charts along a γ_1 which represents the trivial element of $\pi_1(\mathbb{P}^2)$ would produce the identity identification on $X(k)$. This closely resembles a 'monodromy' map $\pi_1(\mathbb{P}^2) \to \mathrm{Aut}(X(k))$.

LEMMA 4. *If ρ is non-empty then the fibres $X(k)$ must have two components, Y and $-Y$, (mirror images of each other), each with ideal boundary ρ. $\overline{X(k)}$ must be the Schottky double of Y (and hence connected), and $\tau(k)$ is the canonical reflection on the double.*

PROOF. Consider any component A of $X(k)$. $\tau(k)$ ($= \tau$ say) maps components to components, so, if $\tau(A) \cap A$ is non-empty then $\tau(A) = A$.

But A must have pieces of ρ as its boundary since we proved $\overline{X(k)} = X(k) \cup \rho$ was connected. Now, τ acts as reflection near points of ρ, (remark following Lemma 3), and this is impossible if $\tau(A) = A$. So there must be a component distinct from A and all the assertions follow.

THEOREM 3. *Let X be a connected closed hypercomplex manifold with real set ρ. Then there is a natural C^∞ mapping*

$$\beta \colon X - \rho \to Y \times \mathbb{P}^{n-2}, \qquad (n = 4, 8)$$

which is the orientation double covering map; here $Y = X(k)/\tau(k)$ is a connected surface with a conformal structure (fix any $k \in \mathbb{P}^{n-2}$).

If ρ is not empty then Y must be simply a component of the fibre $X(k)$ and the compact manifold X is isomorphic to the manifold X_G of Proposition 2 with $Y = U/G$ and $X - \rho$ is isomorphic to $Y \times S^{n-2}$.

ADDENDUM. If ρ is empty we may separate the cases (i) $X(k)$ has 2-components, (ii) $X(k)$ is connected. In case (i) each component is a compact Riemann surface Y, $\tau(k)$ maps one component to the other and X is diffeomorphic to $Y \times S^{n-2}$. (If genus$(Y) > 1$ then X is isomorphic to $X_G = {}'\mathbb{R}^n/F_n(G)$, as before.) In case (ii) $X(k)$ is a compact Riemann surface. If $\tau(k)$ is orientation preserving then $Y = X(k)/\tau(k)$ is itself a compact Riemann surface and X is diffeomorphic to $Y \times S^{n-2}$. If $\tau(k)$ is orientation reversing then Y is a compact non-orientable surface with conformal structure, and X is the orientation double covering of $Y \times \mathbb{P}^{n-2}$.

PROOF. The second-factor of the map β is our original fibre bundle p. Fix any $k_0 \in \mathbb{P}^2$, and define $Y = X(k_0)/\tau(k_0)$. We now take any $x \in X - \rho$, say $x \in X(k)$; identify $X(k)$ with $X(k_0)$ by any chain of the canonical local fibre-identifications. Then the image of x in $X(k_0)$ is well-defined when we go modulo the 'monodromy' $\tau(k_0)$. This defines β and shows it to be two-to-one, and therefore a covering space. Since $X - \rho$ is oriented, but $Y \times \mathbb{P}^2$ is never orientable for any surface Y, we see β must be the orientation double covering. (Recall that any oriented covering space of a non-orientable manifold factors through the orientation double covering.) Y is connected because X is.

The last statement of the theorem follows by pulling back the bundle p: $X - \rho \to \mathbb{P}^2$ over S^2 and noting that the new total space $\widetilde{X - \rho}$ has two components, since by Lemma 4, $X(k_0) =$ fibre of p has two components; (use the exact homotopy sequence of the pullback bundle). So each component of $\widetilde{X - \rho}$ must be a copy of $X - \rho$ itself, and since $\widetilde{X - \rho}$ is a double covering of $Y \times S^2$ (by pulling back β) we see that $X - \rho \cong Y \times S^2$.

In view of our previous results all the claims are now established without difficulty.

COROLLARY 1. *The only simply-connected compact manifolds which can allow hypercomplex structure are S^4 and $S^2 \times S^2$ (for quaternionic); (S^8 and $S^2 \times S^6$ for octonionic).*

PROOF. $\pi_1(X) = 0$ implies $\pi_1(X - \rho) = 0$. Then comparing β with the double covering $Y \times S^2 \to Y \times \mathbb{P}^2$ shows $X - \rho \equiv Y \times S^2$ for some surface Y. But simple connectivity says $\pi_1(Y) = 0$, and the only simply connected surfaces are U or S^2. Since ρ is a finite union of circles compactifying $Y \times S^2$ it must have exactly one component for $Y = U$ case and no components for $Y = S^2$. The results follow.

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REMARKS. We do not know whether $S^2 \times S^{n-2}$ carries $\mathbb{H}$ (or $\mathbb{O}$) structure.

Our central quaternionic manifolds are extremely akin to, (but nevertheless distinct from), the integrable almost quaternionic manifolds studied by Marchiafava [5] and Salamon [6] *et al.*. Actually the derivatives of our transition mappings (5) do *not* in general lie in the group $GL(1, \mathbb{H}) = \mathbb{H}^*$. Indeed, the Jacobian of one of our coordinate transitions falls in $\mathbb{H}^*$ only for $\phi(z) = az + b$, (ϕ as in (4)). See Datta-Nag [1]. Thus the hypercomplex manifolds which are the concern of the present paper need not be integrable almost quaternionic manifolds—despite the marked similarity.

It has been proved, (see [5], [6]), that amongst the class of integrable almost quaternionic manifolds the only compact simply-connected one is S^4. The reader may compare with this our corollary above.

As general references for the work of previous authors we quote [4], [5], [6], [7].

3. Higher dimensional Fueter analysis and associated Lie groups

Functions of several complex variables may also be subjected to a 'Fueter transform' as below. Let $\phi = (\phi_1, \phi_2)$: $D \to \mathbb{C}^2$ be holomorphic from a domain $D \subseteq {}'\mathbb{C}^2 = \{(z_1, z_2) \in \mathbb{C}^2\colon \operatorname{Im} z_1 > 0,\ \operatorname{Im} z_2 > 0\}$. We define its Fueter transform $F_n^{(2)}(\phi)$: $F_n^{(2)}(D) \to \mathbb{R}^n \times \mathbb{R}^n$ by

$$F_n^{(2)}(\phi)(x_{0,1} + e_1 x_{1,1} + \cdots + e_{n-1} x_{n-1,1}, x_{0,2} + e_1 x_{1,2} + \cdots + e_{n-1} x_{n-1,2})$$
$$= \left(\xi_1 + \frac{e_1 x_{1,1} + \cdots + e_{n-1} x_{n-1,1}}{y_1}\eta_1, \xi_2 + \frac{e_1 x_{1,2} + \cdots + e_{n-1} x_{n-1,2}}{y_2}\eta_2\right)$$

where $y_j = \sqrt{x_{1,j}^2 + \cdots + x_{n-1,j}^2}$, $\phi_j = \xi_j + i\eta_j$, $j = 1, 2$. The domain $F_n^{(2)}(D) \subset {}'\mathbb{R}^{2n} = \{\text{points in } \mathbb{R}^n \times \mathbb{R}^n \text{ with } y_1 > 0 \text{ and } y_2 > 0\}$ is self evident.

As before, the $F_n^{(2)}(\phi)$ mappings can be described geometrically from ϕ by revolving the ${}'\mathbb{C}^2$ on which ϕ was defined around the pair of real directions. Thus we *do get new pseudogroups and a corresponding theorem that a compact "2nd-order" Fueter $2n$-manifold will be $F_n^{(2)}$-isomorphic to the product of a compact $2_{\mathbb{C}}$-dimensional complex manifold with $S^{n-2} \times S^{n-2}$.* Unfortunately however the $F_n^{(2)}(\phi)$, $n = 4, 8$, maps do *not* occur as power series in two $\mathbb{H}$ or $\mathbb{O}$ variables even when (ϕ_1, ϕ_2) has formally-real power series expansion in (z_1, z_2). (With $k > 2$ complex variables the situation is of course exactly the same.)

We have been able to characterise the matrices which are Jacobians of $F_n^{(k)}(\phi)$ mappings. We can therefore define an almost-Fueter$_n^{(k)}$ structure on a nk-dimensional C^∞ manifold as reduction of the structure group of the tangent bundle to these matrix groups of Jacobians. The invertible Jacobians form *explicit families of $(2k^2 + k)$-dimensional Lie subgroups of $GL(nk, \mathbb{R})$*. All groups in the family are mutually isomorphic and the family is parametrised by a k-fold product of spheres $S^{n-2} \times \cdots \times S^{n-2}$.

The description of non-compact Fueter or hypercomplex manifolds, and also several-variables hypercomplex manifolds, remain unknown to us.

More details of these matters will appear later.

Added in Proof

An application of the Fueter theory developed here will appear in a forthcoming paper by B. Datta and S. Nag in the Bulletin of the London Mathematical Society entitled 'Zero sets of quaternionic and octonionic analytic functions with central coefficients'.

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