THE $L(r, t)$ SUMMABILITY TRANSFORM

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1. Introduction. In a recent article Cheney and Sharma **(1)** studied the linear operator P_n defined by

$$P_n(f, x) = \sum_{k=n}^{\infty} b_{n,k} f\left(\frac{k-n}{k}\right)$$

where

$$b_{n,k} = \begin{cases} 0 & \text{if } k < n, \\ (1-r)^{n+1} \exp\left(\frac{tr}{1-r}\right) L_{k-n}^{(n)}(t) r^{k-n} & \text{if } k \geqslant n; \end{cases}$$

here $L_j^{(n)}(t)$ denotes the Laguerre polynomial of degree j. Cheney and Sharma proved that if f is continuous on $[0, 1]$, then $P_n(f, x)$ converges uniformly to $f(x)$ on $[0, a]$ where $0 < a < 1$.

In this paper we consider the matrix $L(r, t) = (b_{n,k})$ as a summability matrix and determine some of its properties. The special case $L(r, 0)$ is the well-known Taylor matrix $T(r)$ **(2)**. Thus, $L(r, t)$ is a generalization of $T(r)$.

In §2 we examine the regularity of $L(r, t)$. In §§3 and 4 we examine the summability of the geometric series and a series of Legendre polynomials (respectively) by means of the $L(r, t)$ transform. In §5 we determine sufficient conditions on r_1 and r_2 which ensure that each sequence that is summable $T(r_1)$ is summable $L(r_2, t)$ to the same value.

2. Regularity. A matrix $C = (c_{n,k})$ is regular if and only if the well-known Silverman–Toeplitz conditions:

(2.1) $$\lim_{n\to\infty} c_{n,k} = 0, \qquad k = 0, 1, \ldots,$$

(2.2) $$\lim_{n\to\infty} \sum_{k=0}^{\infty} c_{n,k} = 1,$$

and

(2.3) $$\sup_n \left\{ \sum_{k=0}^{\infty} |c_{n,k}| \right\} < \infty$$

are satisfied.

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THEOREM 2.1. (i) *If* $L(z, t)$ *is regular for some real or complex* t, *then* $|z| \leqslant 1$.

(ii) *If* $L(z, t)$ *is regular for some* $t \leqslant 0$, *then* $\mathrm{Im}(z) = 0$ *and* $0 \leqslant \mathrm{Re}(z) < 1$.

(iii) *For a given value of* z, $L(z, t)$ *is regular for each* t *if and only if* $\mathrm{Im}(z) = 0$ *and* $0 \leqslant \mathrm{Re}(z) < 1$.

(iv) *If* $t \leqslant 0$, $L(z, t)$ *is regular if and only if* $\mathrm{Im}(z) = 0$ *and* $0 \leqslant \mathrm{Re}(z) < 1$.

Proof. (i) By (**7**, (5.1.9)),

$$\sum_{k=0}^{\infty} L_k^{(n)}(t) z^k$$

is a power series in z with radius of convergence equal to one. Hence,

$$\sum_{k=0}^{\infty} b_{n,k} \quad \text{and} \quad \sum_{k=0}^{\infty} |b_{n,k}|$$

can converge for $|z| \leqslant 1$ only. Thus, we must have $|z| \leqslant 1$.

(ii) By (i) we have $|z| \leqslant 1$. For $t \leqslant 0$, $L_{k-n}^{(n)}(t) \geqslant 0$ for $k \geqslant n = 0, 1, \ldots$. Hence,

$$\sum_{k=0}^{\infty} |b_{n,k}| = |1 - z|^{n+1} \left| \exp\left(\frac{tz}{1-z}\right) \right| \sum_{k=n}^{\infty} L_{k-n}^{(n)}(t) |z|^{k-n}.$$

Suppose $|z| < 1$. Then, by (**7**, (5.1.9)),

$$\begin{aligned}\sum_{k=0}^{\infty} |b_{n,k}| &= |1 - z|^{n+1} \left| \exp\left(\frac{tz}{1-z}\right) \right| (1 - |z|)^{-n-1} \exp\left(\frac{-t|z|}{1-|z|}\right) \\ &= \left(\frac{|1-z|}{1-|z|}\right)^{n+1} \left| \exp\left(\frac{tz}{1-z}\right) \right| \exp\left(\frac{-t|z|}{1-|z|}\right)\end{aligned}$$

which is uniformly bounded for $n \geqslant 0$ if and only if

$$\frac{|1-z|}{1-|z|} \leqslant 1.$$

However, $|1 - z| \geqslant 1 - |z|$; thus we must have $\mathrm{Im}(z) = 0$ and $0 \leqslant \mathrm{Re}(z) < 1$. Now, suppose $|z| = 1$. Then

$$\sum_{k=0}^{\infty} |b_{n,k}| = |1 - z|^{n+1} \left| \exp\left(\frac{tz}{1-z}\right) \right| \sum_{k=n}^{\infty} L_{k-n}^{(n)}(t).$$

But, by Abel's theorem,

$$\sum_{k=n}^{\infty} L_{k-n}^{(n)}(t)$$

diverges for $t \leqslant 0$ since $L_{k-n}^{(n)}(t) \geqslant 0$ and

$$\lim_{x \uparrow 1} \sum_{k=n}^{\infty} L_{k-n}^{(n)}(t) x^k = \lim_{x \uparrow 1} (1 - x)^{-n-1} e^{-xt/(1-x)} = +\infty.$$

So, we cannot have $|z| = 1$.

(iii) Let z be given. If $L(z, t)$ is regular for each t, it is regular for some $t \leqslant 0$. Hence, by (ii), $\operatorname{Im}(z) = 0$ and $0 \leqslant \operatorname{Re}(z) < 1$.

Now, let $\operatorname{Im}(z) = 0$ and $0 \leqslant \operatorname{Re}(z) < 1$. Condition (2.1) holds for $L(z, t)$ without restriction on z. Condition (2.2) is satisfied if $|z| < 1$; cf. (**7**, (5.1.9)). Furthermore,

$$\sum_{k=0}^{\infty} |b_{n,k}| = |1 - z|^{n+1} \left|\exp\left(\frac{tz}{1 - z}\right)\right| \sum_{k=n}^{\infty} |L_{k-n}^{(n)}(t)|\,|z|^{k-n}$$
$$\leqslant |1 - z|^{n+1} \left|\exp\left(\frac{tz}{1 - z}\right)\right| \sum_{k=n}^{\infty} \sum_{j=0}^{k-n} \binom{k}{k-n-j} \frac{|t|^j}{j!} |z|^{k-n}$$
$$= |1 - z|^{n+1} \left|\exp\left(\frac{tz}{1 - z}\right)\right| \sum_{j=0}^{\infty} \frac{|tz|^j}{j!} \left(\frac{1}{1 - |z|}\right)^{n+j+1}$$
$$= \left(\frac{|1 - z|}{1 - |z|}\right)^{n+1} \left|\exp\left(\frac{tz}{1 - z}\right)\right| \exp\left(\frac{|tz|}{1 - |z|}\right),$$

which is uniformly bounded for $n \geqslant 0$. Thus, Condition (2.3) holds. So $L(z, t)$ is regular for each t.

(iv) Let $t \leqslant 0$. If $\operatorname{Im}(z) = 0$ and $0 \leqslant \operatorname{Re}(z) < 1$, then, by (iii), $L(z, t)$ is regular. If $L(z, t)$ is regular, then, by (ii), $\operatorname{Im}(z) = 0$ and $0 \leqslant \operatorname{Re}(z) < 1$.

3. Summability of the geometric series.

THEOREM 3.1. *Let $|r| < 1$. For each t, the $L(r, t)$ transform continues the geometric series analytically into the region*

$$\left\{z : \left|\frac{(1 - r)z}{1 - rz}\right| < 1\right\} \cap \{z : |rz| < 1\}.$$

Proof. Let $|r| < 1$ and define

$$\sigma_n(z) = \sum_{k=n}^{\infty} b_{n,k}\, s_k(z),$$

where $s_k(z)$ is the kth partial sum of the geometric series. It is clear that

$$\sigma_n(z) = \frac{1}{1 - z} - \frac{1}{1 - z} \sum_{k=n}^{\infty} b_{n,k}\, z^{k+1}$$

since, as in Theorem 2.1 (iii), if $|r| < 1$, we have Condition (2.2) satisfied. So

$$\sum_{k=n}^{\infty} b_{n,k}\, z^{k+1} = \sum_{k=n}^{\infty} (1 - r)^{n+1} \exp\left(\frac{tr}{1 - r}\right) L_{k-n}^{(n)}(t) r^{k-n} z^{k+1}$$
$$= [(1 - r)z]^{n+1} \exp\left(\frac{tr}{1 - r}\right) \sum_{k=n}^{\infty} L_{k-n}^{(n)}(t)\,(rz)^{k-n}$$
$$= \left[\frac{(1 - r)z}{1 - rz}\right]^{n+1} \exp\left(\frac{tr}{1 - r}\right) \exp\left(-\frac{trz}{1 - rz}\right)$$

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if $|rz| < 1$. Hence

$$\lim_{n\to\infty} \sum_{k=n}^{\infty} b_{n,k}\, z^{k+1} = 0$$

if

$$\left|\frac{(1-r)z}{1-rz}\right| < 1 \quad \text{and} \quad |rz| < 1.$$

These regions are identical with those of the $T(r)$ transform for like values of r **(2)**.

The region in which the $L(r, t)$ transform provides the analytic continuation of an arbitrary Taylor series may be determined by the Okada theorem **(6)**.

4. Summability of a series of Legendre polynomials. Let $P_n(z)$ and $Q_n(w)$ denote the Legendre polynomials of the first and second kind (respectively) of degree n. Then it is known **(8)** that

$$\frac{1}{w-z} = \sum_{n=0}^{\infty} (2n+1)P_n(z)Q_n(w),$$

for fixed w, in the interior of the ellipse E with foci ± 1 and passing through w. Let

$$s_k = \sum_{n=0}^{k} (2n+1)P_n(z)Q_n(w) \tag{4.1}$$

and

$$d_n = P_{n+1}(z)Q_n(w) - P_n(z)Q_{n+1}(w). \tag{4.2}$$

Then, by the Christoffel formula,

$$\frac{1}{w-z} = s_n + (n+1)\frac{1}{w-z}d_n.$$

Choose the branch of $(\beta^2-1)^{\frac{1}{2}}$ such that $\beta + (\beta^2-1)^{\frac{1}{2}}$ lies in the exterior of the unit circle and let

$$\mu = \mu(\phi) = z + (z^2-1)^{\frac{1}{2}}\cos\phi$$

and

$$\nu = \nu(\alpha) = w + (w^2-1)^{\frac{1}{2}}\cosh\alpha.$$

Then, the Laplace integral representations of $P_n(z)$ and $Q_n(w)$ are

$$P_n(z) = \frac{1}{\pi}\int_0^{\pi} \mu^n\, d\phi \tag{4.3}$$

and

$$Q_n(w) = \int_0^{\infty} \nu^{-n-1}\, d\alpha. \tag{4.4}$$

From (4.2), (4.3), and (4.4) we obtain

$$d_n = \frac{1}{\pi}\int_0^{\infty}\int_0^{\pi}\left(\frac{\mu}{\nu}\right)^n\left[\frac{\mu}{\nu} - \frac{1}{\nu^2}\right] d\phi\, d\alpha. \tag{4.5}$$

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LEMMA 4.1. *If* $|r\theta| < 1$, *then*

$$\sum_{k=n}^{\infty} (k+1)b_{n,k}\theta^k = \left[n + 1 - \frac{tr\theta}{1 - r\theta}\right] \frac{(1-r)}{(1-r\theta)^2} \exp\left(\frac{tr}{1-r}\right) \times \exp\left(\frac{tr\theta}{r\theta - 1}\right)\left[\frac{(1-r)\theta}{1-r\theta}\right]^n.$$

Proof. We have

$$\sum_{k=n}^{\infty} b_{n,k}\theta^{k+1} = \sum_{k=n}^{\infty} (1-r)^{n+1} \exp\left(\frac{tr}{1-r}\right) L_{k-n}^{(n)}(t) r^{k-n}\theta^{k+1} = \left[\frac{(1-r)\theta}{1-r\theta}\right]^{n+1} \exp\left(\frac{tr}{1-r}\right) \exp\left(-\frac{tr\theta}{1-r\theta}\right)$$

for $|r\theta| < 1$. The desired result is obtained by differentiation.

THEOREM 4.2. *The sequence* $\{s_k\}$ *of partial sums* (4.1) *is* $L(r, t)$*-summable to* $(w-z)^{-1}$ *for each* t *and* $0 \leqslant r < 1$ *whenever*

$$\left|\frac{\mu(\phi)}{\nu(\alpha)}\right| \leqslant \lambda < \frac{1}{r}$$

for all $0 \leqslant \alpha < \infty, 0 \leqslant \phi \leqslant \pi$, *and*

$$\sup_{\phi,\alpha} \left|\frac{\mu - r\mu}{\nu - r\mu}\right| < 1.$$

Proof. Let

$$\tau_n = \sum_{k=n}^{\infty} b_{n,k}\, s_k = \frac{1}{w-z} - \frac{1}{w-z} \sum_{k=n}^{\infty} (k+1) b_{n,k}\, d_k.$$

Then

$$\lim_{n\to\infty} \tau_n = \frac{1}{w-z}$$

if and only if

$$\lim_{n\to\infty} \sum_{k=n}^{\infty} (k+1) b_{n,k}\, d_k = 0.$$

From (4.5) and since $|\mu/\nu| \leqslant \lambda < 1/r$ for all $0 \leqslant \alpha < \infty, 0 \leqslant \phi \leqslant \pi$, we have

$$\sum_{k=n}^{\infty} (k+1)b_{n,k}\, d_k = \sum_{k=n}^{\infty} (k+1)b_{n,k} \frac{1}{\pi}\int_0^{\infty}\int_0^{\pi} \left(\frac{\mu}{\nu}\right)^k \left[\frac{\mu}{\nu} - \frac{1}{\nu^2}\right] d\phi\, d\alpha = \frac{1}{\pi}\int_0^{\infty}\int_0^{\pi} \left[\sum_{k=n}^{\infty} (k+1)b_{n,k}\left(\frac{\mu}{\nu}\right)^k\right]\left[\frac{\mu}{\nu} - \frac{1}{\nu^2}\right] d\phi\, d\alpha.$$

From Lemma 4.1 we obtain

$$\left|\sum_{k=n}^{\infty} (k+1)b_{n,k}\, d_k\right| \leqslant \frac{1}{\pi}\left|\exp\left(\frac{tr}{1-r}\right)\right| \left(n + 1 + \frac{|t|}{1-r\lambda}\right)\exp\left(\frac{|t|}{1-r\lambda}\right) \times \sup_{\phi,\alpha}\left|\frac{\mu - r\mu}{\nu - r\mu}\right|^n \int_0^{\infty}\int_0^{\pi} \left|\frac{\mu\nu - 1}{(\nu - r\mu)^2}\right| d\phi\, d\alpha.$$

Hence

$$\lim_{n\to\infty} \sum_{k=n}^{\infty} (k+1) b_{n,k}\, d_k = 0$$

whenever

$$\left|\frac{\mu(\phi)}{\nu(\alpha)}\right| \leqslant \lambda < \frac{1}{r}$$

for all $0 \leqslant \alpha < \infty, 0 \leqslant \phi \leqslant \pi$, and

$$\sup_{\phi,\alpha} \left|\frac{\mu - r\mu}{\nu - r\mu}\right| < 1.$$

The last inequality yields three cases: (i) $r = \frac{1}{2}$, (ii) $r < \frac{1}{2}$, and (iii) $r > \frac{1}{2}$. Cases (i) and (ii) are identical with those studied by Cowling and King **(3)**; and the regions of summability of the sequence (4.1) by means of the transformation $L(r, t)$ are given in Theorem 2.1 for $r = \frac{1}{2}$ and Theorem 2.3 for $r < \frac{1}{2}$.

In case (iii) we have

$$\left|\frac{\mu - r\mu}{\nu - r\mu}\right| < 1$$

if and only if

$$|\mu|^2 - \frac{2r}{2r-1} \operatorname{Re}(\mu\bar{\nu}) > \frac{1}{1-2r} |\nu|^2,$$

which is equivalent to μ being in the exterior of the circle

$$K_\nu^r = \left\{ \mu : \left| \mu - \frac{r}{2r-1} \nu \right| = \left(\frac{1-r}{2r-1} |\nu| \right)^2 \right\}$$

for fixed ν. Let $\operatorname{ext}(K_\nu^r)$ denote the exterior of K_ν^r. It follows readily that $\{z: |z| \leqslant 1\} \subseteq \operatorname{ext}(K_\nu^r)$. Let h_ν^r and l_ν^r be the internal common tangents to the unit circle and K_ν^r, and let H_ν^r and L_ν^r be the open half-planes having h_ν^r and l_ν^r as boundaries (respectively) and containing the unit circle. Let J_ν^r be the finite area exterior to K_ν^r and bounded by K_ν^r and the lines h_ν^r and l_ν^r. Let $C_\nu^r = H_\nu^r \cup L_\nu^r \cup J_\nu^r$. Now, $\{z: |z| \leqslant 1\} \subseteq \bigcap_\nu C_\nu^r$ and, if $\mu \in \bigcap_\nu C_\nu^r$, then

$$\left|\frac{\mu - r\mu}{\nu - r\mu}\right| < 1.$$

Since $\mu(\phi)$ describes the line segment with end points $z - (z^2-1)^{\frac{1}{2}}$ and $z + (z^2-1)^{\frac{1}{2}}$ for $0 \leqslant \phi \leqslant \pi$, and since

$$z - (z^2-1)^{\frac{1}{2}} \in \bigcap_\nu C_\nu^r,$$

we need only require

$$z + (z^2-1)^{\frac{1}{2}} \in \bigcap_\nu C_\nu^r$$

(by construction of C_ν^r). This is equivalent to requiring $z \in B_w^r$ where B_w^r is the image of $\bigcap_\nu C_\nu^r$ under the mapping $w = \frac{1}{2}(s + 1/s)$.

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The requirement that

$$\left|\frac{\mu(\phi)}{\nu(\alpha)}\right| \leqslant \lambda < \frac{1}{r}$$

holds for all $0 \leqslant \phi \leqslant \pi$, $0 \leqslant \alpha < \infty$, provided

$$|z + (z^2 - 1)^{\frac{1}{2}}| \leqslant \lambda |w + (w^2 - 1)^{\frac{1}{2}}|,$$

i.e., $z + (z^2 - 1)^{\frac{1}{2}}$ must lie strictly inside the circle with centre at the origin and radius $r^{-1}|w + (w^2 - 1)^{\frac{1}{2}}|$. This is equivalent to requiring that z lie strictly inside the ellipse E_w^r with foci ± 1, passing through

$$\frac{1}{r}\left[w - \frac{1 - r^2}{2(w + (w^2 - 1)^{\frac{1}{2}})}\right].$$

Notice that E_w^r and B_w^r contain the ellipse E with foci ± 1, passing through w.

We have proved

THEOREM 4.3. *The sequence of partial sums of the series*

$$\sum_{n=0}^{\infty} (2n + 1) P_n(z) Q_n(w)$$

is $L(r, t)$-summable to $(w - z)^{-1}$ for fixed w and $\frac{1}{2} < r < 1$ whenever z lies in any closed subdomain contained in the region $B_w^r \cap E_w^r$ for each t.

The domains in which the $L(r, t)$ transform provide the analytic continuation of a general series of Legendre polynomials

$$f(z) = \sum_{n=0}^{\infty} a_n P_n(z), \qquad a_n = \frac{2n + 1}{2\pi i} \int_{\gamma} f(w) Q_n(w)\, dw, \tag{4.6}$$

for the cases $r = \frac{1}{2}$ and $r < \frac{1}{2}$ are the same as those determined by Cowling and King **(3)** in Theorems 2.2 and 2.4, respectively. In a recent paper **(4)**, Jakimovski proved a general result which gives the domain in which the series (4.6) is A-summable to $f(z)$, provided the matrix A and the domain in which the sequence (4.1) is A-summable to $(w - z)^{-1}$ have certain properties. However, Jakimovski's result does not apply to the $L(r, t)$ matrix for $\frac{1}{2} < r < 1$ since the domain D in which $L(r, t)$ is efficient is not a generating domain (Condition (iv) of Definition 1.1 is not satisfied).

Because of the computational difficulties involved, the author has not yet determined this domain.

5. The relation $T(r_1) \subset L(r_2, t)$. In the following we assume that $\{x_n\}$ is $T(r_1)$-summable to x. Let $\{\sigma_n\}$ be the $T(r_1)$ transform of $\{x_n\}$, i.e.,

$$\sigma_n = \sum_{k=n}^{\infty} c_{n,k}\, x_k$$

where $(c_{n,k})$ is the $T(r_1)$ matrix. It is known **(2)** that if $r_1 \neq 1$, then the $T(r_1)$ matrix has as its inverse the $T(-r_1/(1-r_1))$ matrix. Let $(d_{n,k})$ be this matrix. Let $(b_{n,k})$ be the $L(r_2, t)$ matrix.

LEMMA 5.1. *If* $r_1 \neq 1, r_2 \neq 1$, *and* $r_1 \neq r_2$ *and*

$$l_{n,j} = \begin{cases} 0 & \text{if } n > j, \\ \sum_{k=n}^{j} b_{n,k}\, d_{k,j} & \text{if } n \leqslant j, \end{cases}$$

then $(l_{n,j})$ *is the* $L((r_2 - r_1)/(1-r_1), tr_2/(r_2 - r_1))$ *matrix.*

Proof. If either $r_1 = 0$ or $r_2 = 0$, the result follows immediately. Suppose $r_1 \neq 0$ and $r_2 \neq 0$. Then

$$\begin{aligned}
&\sum_{k=n}^{j} b_{n,k}\, d_{k,j} \\
&= (1-r_2)^{n+1} \exp\left(\frac{tr_2}{1-r_2}\right)\left(\frac{1}{1-r_1}\right)^{j+1}\left(\frac{1}{r_2}\right)^{n}(-r_1)^j \sum_{k=n}^{j} L_{k-n}^{n}(t)\left(-\frac{r_2}{r_1}\right)^{k}\binom{j}{k} \\
&= (1-r_2)^{n+1} \exp\left(\frac{tr_2}{1-r_2}\right)\left(\frac{1}{1-r_1}\right)^{j+1}\left(\frac{1}{r_2}\right)^{n}(-r_1)^j \\
&\qquad\qquad \times\left[\left(-\frac{r_2}{r_1}\right)^{n}\left(\frac{r_1-r_2}{r_1}\right)^{j-n} L_{j-n}^{(n)}\left(\frac{tr_2}{r_2-r_1}\right)\right] \\
&= \left(\frac{1-r_2}{1-r_1}\right)^{n+1} \exp\left(\frac{tr_2}{1-r_2}\right)\left(\frac{r_2-r_1}{1-r_1}\right)^{j-n} L_{j-n}^{(n)}\left(\frac{tr_2}{r_2-r_1}\right).
\end{aligned}$$

LEMMA 5.2. *If* $|r_1| + |r_2| < |1 - r_1|$, *then*

$$\sum_{k=n}^{\infty} |b_{n,k}| \sum_{j=k}^{\infty} |d_{k,j}||\sigma_j|$$

converges.

Proof. Since $\{\sigma_j\}$ converges, there exists $M > 0$ such that $|\sigma_j| \leqslant M$ for all $j = 1, 2, \ldots$. So

$$\begin{aligned}
\sum_{k=n}^{\infty} |b_{n,k}| \sum_{j=k}^{\infty} |d_{k,j}||\sigma_j| &\leqslant M \sum_{k=n}^{\infty} |b_{n,k}| \sum_{j=k}^{\infty} |d_{k,j}| \\
&= M|1-r_2|^{n+1}\left|\exp\left(\frac{tr_2}{1-r_2}\right)\right| \sum_{k=n}^{\infty} |L_{k-n}^{(n)}(t)||r_2|^{k-n}\left[\frac{1}{|1-r_1|-|r_1|}\right]^{k+1} \\
&\leqslant M\left[\frac{|1-r_2|}{|1-r_1|-|r_1|-|r_2|}\right]^{n+1}\left|\exp\left(\frac{tr_2}{1-r_2}\right)\right| \\
&\qquad\qquad \times \exp\left(-\frac{|tr_2|}{|1-r_1|-|r_1|-|r_2|}\right).
\end{aligned}$$

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THEOREM 5.3. *If*
(i) $|r_1| < 1$, $r_2 \neq 1$,
(ii) $|r_1| < |r_2|$,
(iii) $|r_1| + |r_2| < |1 - r_1|$,
and
(iv) $|1 - r_2| + |r_1 - r_2| = |1 - r_1|$,
then $\{x_n\}$ *is* $L(r_2, t)$*-summable to* x.

Proof. We have

$$\sigma_n = \sum_{k=n}^{\infty} c_{n,k}\, x_k.$$

Let

$$s_n = \sum_{k=n}^{\infty} d_{n,k}\, \sigma_k.$$

By Conditions (i), (ii), and (iii) and a result of Laush **(5)**, we have $s_n = x_n$. Let

$$\tau_n = \sum_{k=n}^{\infty} b_{n,k}\, x_k.$$

So

$$\tau_n = \sum_{j=n}^{\infty} \left(\sum_{k=n}^{j} b_{n,k}\, d_{k,j} \right) \sigma_j$$

by Lemma 5.2 and Condition (iii). By Lemma 5.1 and Conditions (i) and (ii),

$$\tau_n = \sum_{j=n}^{\infty} l_{n,j}\, \sigma_j$$

where $(l_{n,j})$ is the $L((r_2 - r_1)/(1 - r_1),\ tr_2/(r_2 - r_1))$ matrix. By Theorem 2.1, this matrix is regular if and only if

$$0 \leqslant \frac{r_2 - r_1}{1 - r_1} < 1,$$

which follows by Condition (iv). Therefore

$$\lim_{n \to \infty} \tau_n = x.$$

If $t = 0$ in Theorem 5.3, we have the case $T(r_1) \subset T(r_2)$ studied by Laush **(5)**.

COROLLARY 5.4. *Let* r_1 *and* r_2 *be real. If* $0 \leqslant r_1 < r_2 < 1$ *and* $r_1 + r_2 < 1 - r_1$, *then* $\{x_n\}$ *is* $L(r_2, t)$*-summable to* x.

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