ISOPARAMETRIC HYPERSURFACES WITH FOUR PRINCIPAL CURVATURES, II

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Abstract. In this sequel to an earlier article, employing more commutative algebra than previously, we show that an isoparametric hypersurface with four principal curvatures and multiplicities $(3, 4)$ in S^{15} is one constructed by Ozeki and Takeuchi and Ferus, Karcher, and Münzner, referred to collectively as *of OT-FKM type*. In fact, this new approach also gives a considerably simpler proof, both structurally and technically, that an isoparametric hypersurface with four principal curvatures in spheres with the multiplicity constraint $m_2 \geq 2m_1 - 1$ is of OT-FKM type, which left unsettled exactly the four anomalous multiplicity pairs $(4, 5), (3, 4), (7, 8)$, and $(6, 9)$, where the last three are closely tied, respectively, with the quaternion algebra, the octonion algebra, and the complexified octonion algebra, whereas the first stands alone in that it cannot be of OT-FKM type. A by-product of this new approach is that we see that Condition B, introduced by Ozeki and Takeuchi in their construction of inhomogeneous isoparametric hypersurfaces, naturally arises. The cases for the multiplicity pairs $(4, 5), (6, 9)$, and $(7, 8)$ remain open now.

§1. Introduction

An isoparametric hypersurface in a space form is a complete hypersurface whose principal curvatures and their multiplicities are fixed constants. The long history of the study of isoparametric hypersurfaces dates back to 1918, when isoparametric surfaces in Euclidean 3-space arose in the study of geometric optics (see [14], [22], [20]); in contrast, their latest application to integrable systems came as late as 1995 (see [11]), to the author's knowledge. The classification problem of isoparametric hypersurfaces started when Segre [21], for ambient dimension 3, and Levi-Civita [15], for arbitrary ambient dimension n, classified such hypersurfaces in Euclidean space; they are none other than the cylinders $S^k \times \mathbb{R}^{n-k-1}$. Cartan [2] then

took up the task and quickly settled the hyperbolic case; again, the hyperbolic cylinders $S^k \times H^{n-k-1}$ are the only ones. The spherical case amazed Cartan, as such hypersurfaces displayed remarkably deep properties. In fact, he classified the case when the number g of principal curvatures is at most 3. Clifford tori $S^k(r) \times S^{n-k-1}(s) \subset S^n, r^2 + s^2 = 1$, constitute the case when $g = 2$. For $g = 3$, he showed that such hypersurfaces are tubes of constant radii around the Veronese embedding of the projective plane $\mathbb{F}P^2$ in S^{3m+1}, where $m = 1, 2, 4$, or 8 is the dimension of the standard normed algebra $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, or the Cayley algebra $\mathbb{O}$, respectively (see [3], [4]). In particular, this gives a very geometric description of the Cayley plane that appears in the classification of the rank 1 symmetric spaces of compact type that Cartan had classified earlier using Lie group theory he developed. Therefore, all these isoparametric hypersurfaces are homogeneous.

To make a long story short, leaving the beautiful subsequent development of the classification problem to the introductory section in [6], let us simply remark that by [18] it holds that g can only be 1, 2, 3, 4, 6; moreover, the principal values have at most two multiplicities. In fact, if we list these principal values as $\lambda_1 < \cdots < \lambda_g$, and their associated multiplicities as $m_1, \ldots, m_g$, respectively, then $m_i = m_{i+2}$, where the subscripts are modulo g. In particular, $m_1 = m_2$ (actually, $= 1, 2, 4$, or 8 associated with the standard normed algebras) when $g = 3$. We will denote these two multiplicities by m_1 and m_2, with the understanding that $m_1 \leq m_2$. To each isoparametric hypersurface in the sphere, there is indeed a 1-parameter family of isoparametric hypersurfaces in the sphere that degenerates to two submanifolds of the sphere of codimensions $m_1 + 1$ and $m_2 + 1$, called the *focal manifolds of the isoparametric hypersurface* and denoted by M_+ and M_-, respectively. The principal values of the focal manifold of codimension $m_1 + 1$ are $0, 1, -1$ with multiplicities m_1, m_2, m_2, respectively, with m_1 and m_2 interchanged for the other focal manifold.

It is known (see [1]) that $m_1 = m_2 = 1$ or 2 in the case $g = 6$. The case $m_1 = m_2 = 1$ was settled by Dorfmeister and Neher [10] (see also Miyaoka [16]). The other case has recently been settled by Miyaoka [17]. Such isoparametric hypersurfaces are again homogeneous.

Cartan [5] found two homogeneous examples of isoparametric hypersurfaces with $g = 4$ and $(m_1, m_2) = (1, 1)$ or $(2, 2)$. He indicated without proof that his example with multiplicity pair $(2, 2)$ is the only isoparametric hypersurface with the given g and multiplicities; an outline of a proof was given by Ozeki and Takeuchi in [19, Part II, pp. 53–54].

In contrast, inhomogeneous examples do appear in the case $g = 4$. Ozeki and Takeuchi [19, Part I, pp. 541, 549] introduced the notions of Conditions A and B to first construct two families of them [19, Part I], followed by Ferus, Karcher, and Münzner's [12] generalization to infinite families using Clifford representations, where m_1 and m_2 are explicitly given in terms of the dimensions of irreducible Clifford modules. We refer to these examples collectively as of OT-FKM type. These examples, together with the two homogeneous examples with $(m_1, m_2) = (2, 2)$ and $(4, 5)$, not of OT-FKM type, have constituted all known isoparametric hypersurfaces with four principal curvatures in spheres. In fact, using homotopy theory, Stolz [23] proved that the multiplicity pair of an isoparametric hypersurface with $g = 4$ in the sphere is $(2, 2), (4, 5)$, or one of those of isoparametric hypersurfaces of OT-FKM type. In [6], we classified that, except for possibly the four cases with $(m_1, m_2) = (4, 5), (3, 4), (6, 9)$, or $(7, 8)$, all other isoparametric hypersurfaces with $(m_1, m_2) \neq (2, 2)$ are necessarily of OT-FKM type.

It is worth pointing out that isoparametric hypersurfaces of OT-FKM type with multiplicity pairs $(m_1, m_2) = (3, 4), (7, 8)$, and $(6, 9)$ are anomalous in the sense that they are the only ones for which the two symmetric Clifford algebras C'_{m_1+1} and C'_{m_2+1} can act on $\mathbb{R}^{2m_1+2m_2+2}$ to produce incongruent isoparametric hypersurfaces in $S^{2m_1+2m_2+1}$ (see [12]). Their isoparametric structures are tied, respectively, with $\mathbb{H}, \mathbb{O}$, and $\mathbb{O}(\mathbb{C})$ (see [19]). On the other hand, an isoparametric hypersurface with $g = 4$ and multiplicities $(m_1, m_2) = (4, 5)$ stands alone, as it cannot be of OT-FKM type.

The classification theorem in [6] (see also [7]), where it is assumed that $m_2 \geq 2m_1 - 1$, eventually comes down to an estimate on the dimension of certain singular varieties associated with the complexified second fundamental form of the focal manifold of codimension $m_1 + 1$, restricted to the direct sum of the two eigenspaces of the principal values $1, -1$. This restriction to a subspace of a typical tangent space of the focal manifold complicates the dimension estimate, since the restriction does not preserve the constant rank that the second fundamental matrix of a focal manifold enjoys. Moreover, the tracking of the interplay between real and complex varieties in the proof brings in additional technicality.

The primary goal in this sequel to [6] is to prove that an isoparametric hypersurface with $g = 4$ and $(m_1, m_2) = (3, 4)$ is one of OT-FKM type as well, which leaves only three multiplicity pairs $(4, 5), (6, 9)$, and $(7, 8)$ unsettled. In fact, this grew out of an attempt to look at the classification theorem in [6] from a different angle.

Indeed, employing more commutative algebra than that explored in [6], we first present a considerably simpler proof of the classification theorem in [6] by investigating the (complexified) second fundamental form itself without further restriction. An advantage of this approach is that we see that Condition B of Ozeki and Takeuchi, which all isoparametric hypersurfaces of OT-FKM type enjoy as a matter of fact, naturally arises.

The key ingredient on which this new proof of the classification theorem depends is the satisfying result (Proposition 4) that for $m_1 < m_2$, if the components $p_0, p_1, \ldots, p_{m_1}$ of the second fundamental form of M_+ form a regular sequence in the ring of polynomials in $m_1 + 2m_2$ variables (over the complex numbers), then the isoparametric hypersurface is of OT-FKM type. The result sheds light on why the remaining multiplicity pairs $(3, 4), (6, 9)$, and $(7, 8)$ are anomalous. It is because in these cases $p_0, \ldots, p_{m_1}$ no longer form a regular sequence in general, due to the aforementioned fact that incongruent isoparametric hypersurfaces of OT-FKM type do occur in $S^{2m_1+2m_2+1}$; the zero locus cut out by a nonregular sequence $p_0, \ldots, p_{m_1}$ can be wildly untamed, even in the complex category. In fact, Proposition 4 follows from one of the 10 identities defining an isoparametric hypersurface (see [19, Part I, p. 530]). Namely,

$$\sum_{a=0}^{m_1} p_a q_a = 0,$$

where $q_0, q_1, \ldots, q_{m_1}$ are the components of the third fundamental form of M_+. That $p_0, \ldots, p_{m_1}$ form a regular sequence ensures that

$$q_a = \sum_{b=0}^{m_1} r_{ab} p_b,$$

where $0 \leq a \leq m_1$, for some linear homogeneous polynomials r_{ab} satisfying

$$r_{ab} = -r_{ba}$$

for $0 \leq a, b \leq m_1$. This is exactly Condition B of Ozeki and Takeuchi, from which [6, Proposition 19, (8.1)–(8.3)] readily follows. The mild condition $m_1 < m_2$ then warrants that [6, (8.4)] also holds true; the isoparametric hypersurface is then of OT-FKM type (see [6], [7]). That $p_0, \ldots, p_{m_1}$ indeed form a regular sequence, when $m_2 \geq 2m_1 - 1$, is a consequence of a rather straightforward dimension estimate, facilitated by the constant rank of the

shape operators of M_+, over the subset where the Jacobian matrix of these polynomials fails to be of rank $m_1 + 1$ on the variety cut out by them.

What is more important is that in fact the new approach in this paper provides us with a proof that an isoparametric hypersurface with $g = 4$ and $(m_1, m_2) = (3, 4)$ in S^{15} is of OT-FKM type, by showing that there exist on M_+ points of Condition A of Ozeki and Takeuchi. The conclusion then follows from the result of Dorfmeister and Neher [9] (see also [8]) which states that Condition A alone implies that the isoparametric hypersurface is of OT-FKM type.

The salient feature of this approach would appear applicable also to the case $(m_1, m_2) = (7, 8)$. We comment on this in the concluding remarks.

§2. The background commutative algebra

Recall from [13, p. 152] that a *regular sequence* in a commutative ring R with identity is a sequence $a_1, \ldots, a_k$ in R such that the ideal $(a_1, \ldots, a_k)$ is not R and, moreover, that a_1 is not a zero divisor in R and a_{i+1} is not a zero divisor in the quotient ring $R/(a_1, \ldots, a_i)$ for $1 \le i \le k - 1$.

There is a powerful property in commutative algebra that dictates the algebraic independence of a regular sequence (see [13, Proposition 5.10, p. 152]), which is specialized to fit our purpose in the following proposition.

PROPOSITION 1. *Let $a_1, \ldots, a_k$ be a regular sequence in R. Then for any homogeneous polynomial $F(t_1, \ldots, t_k)$ in k variables over R with $F(a_1, \ldots, a_k) = 0$, there always holds that all the coefficients of F belong to $(a_1, \ldots, a_k)$.*

For the convenience of the reader, let us recall a crucial inductive procedure in [6, Proposition 39, p. 57] to generate regular sequences in a polynomial ring.

PROPOSITION 2. *Over the complex numbers, if $p_1, \ldots, p_k, k \ge 2$, are linearly independent homogeneous polynomials of equal degree ≥ 1 in a polynomial ring P such that the ideal $(p_1, \ldots, p_{k-1})$ is prime and that $p_1, \ldots, p_{k-1}$ form a regular sequence in P, then $p_1, \ldots, p_k$ form a regular sequence in P.*

To warrant the primeness of an ideal $(p_1, \ldots, p_s)$ in a polynomial ring, the following (see [6, Proposition 43, p. 59]) is essential.

PROPOSITION 3. *Over the complex numbers, let $p_1, \ldots, p_s$ be a regular sequence of homogeneous polynomials in a polynomial ring, let V be the variety defined by $p_1 = \cdots = p_s = 0$, and let J be the subvariety of V where*

the rank of the Jacobian matrix of $p_1, \ldots, p_s$ is $< s$. If $\dim(J) \le \dim(V) - 2$, then the ideal $(p_1, \ldots, p_s)$ is prime.

Note that the homogeneity of $p_1, \ldots, p_s$ is to guarantee that the variety V is connected.

COROLLARY 1. *Over the complex numbers, let $p_1, \ldots, p_k, k \ge 2$, be linearly independent homogeneous polynomials of equal degree ≥ 1 in a polynomial ring. For $i \le k$, let V_i be the variety defined by $p_1 = \cdots = p_i = 0$, and let J_i be the subvariety of V_i where the rank of the Jacobian matrix of $p_1, \ldots, p_i$ is $< i$. If $\dim(J_i) \le \dim(V_i) - 2$ for $1 \le i \le k-1$, then $p_1, \ldots, p_k$ form a regular sequence in the polynomial ring.*

For a proof, note that p_1 clearly forms a regular sequence, so that Proposition 3 implies that (p_1) is a prime ideal, so the corollary is true when $i = 1$. Repeated applications of Propositions 2 and 3 then deduce that the ideal $(p_1, \ldots, p_i)$ is prime and that $p_1, \ldots, p_i$ form a regular sequence for $1 \le i \le k-1$. From this, Proposition 2 gives that $p_1, \ldots, p_k$ form a regular sequence in the polynomial ring.

§3. The game plan

We will follow closely the notation in [6] for ease of exposition. Let $n_0, n_1, \ldots, n_{m_1}$ be an orthonormal basis of a typical normal space to the focal manifold M_+ of codimension $m_1 + 1$. Let $p_0, p_1, \ldots, p_{m_1}$ be the associated symmetric quadratic forms associated with the second fundamental form S of M_+. That is,

$$p_a(X) := \langle S(X, X), n_a \rangle / 2$$

for $0 \le a \le m_1$. Let $q_0, q_1, \ldots, q_{m_1}$ be the associated symmetric cubic forms of the third fundamental form of M_+. That is,

$$q_a(X) := \langle Q(X, X, X), n_a \rangle / 3,$$

where

$$(1) \qquad \begin{aligned} Q(X, Y, Z) &:= (D_X^\perp S)(Y, Z) \\ &= D_X^\perp \big(S(Y, Z) \big) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z) \end{aligned}$$

for $0 \le a \le m_1$, where $D^\perp$ is the normal connection and ∇ is the Riemannian connection of M_+. These homogeneous polynomials belong to the polynomial ring $\mathcal{P}$ in $m_1 + 2m_2$ variables corresponding to the dimension of M_+.

To apply Proposition 1, suppose that we have established that $p_0, p_1, \ldots,$ p_{m_1} form a regular sequence in $\mathcal{P}$.

Let us recall an identity of Ozeki and Takeuchi [19, Part I, p. 530]. Namely,

$$(2) \qquad p_0 q_0 + p_1 q_1 + \cdots + p_{m_1} q_{m_1} = 0.$$

We can interpret (2) in the spirit of Proposition 1 above. Namely, consider the homogeneous polynomial

$$F(t_0, t_1, \ldots, t_{m_1}) = q_0 t_0 + q_1 t_1 + \cdots + q_{m_1} t_{m_1}$$

over the ring $\mathcal{P}$. Since by assumption $p_0, p_1, \ldots, p_{m_1}$ form a regular sequence with $F(p_0, p_1, \ldots, p_{m_1}) = 0$ by (2), it follows from Proposition 1 that $q_0, q_1,$ $\ldots, q_{m_1}$ belong to the ideal $(p_0, p_1, \ldots, p_{m_1})$. That is, we have

$$(3) \qquad q_a = \sum_{b=0}^{m_1} r_{ab} p_b$$

for $0 \leq a \leq m_1$, where r_{ab} are homogeneous polynomials of degree 1. Substituting (3) into (2), we obtain

$$(4) \qquad \sum_{a \leq b} (r_{ab} + r_{ba}) p_a p_b = 0.$$

Considering the homogeneous polynomial

$$F(t_0, t_1, \ldots, t_{m_1}) = \sum_{a \leq b} (r_{ab} + r_{ba}) t_a t_b$$

over the ring $\mathcal{P}$ and observing that $F(p_0, p_1, \ldots, p_{m_1}) = 0$ by (4), Proposition 1 implies that $r_{ab} + r_{ba}$ belongs to the ideal $(p_0, p_1, \ldots, p_{m_1})$. However, this forces $r_{ab} + r_{ba} = 0$ since it is homogeneous of degree 1, whereas $p_0, p_1, \ldots, p_{m_1}$ are homogeneous of degree 2. We thus conclude that in (3) we have

$$(5) \qquad r_{ab} = -r_{ba}.$$

Now we introduce the Euclidean coordinates of the eigenspaces, with eigenvalues $1, -1, 0$, of the shape operator S_{n_0} to be $u_\alpha, 1 \leq \alpha \leq m_2$, $v_\mu, 1 \leq \mu \leq m_2$, and $w_p, 1 \leq p \leq m_1$, respectively. Set

$$(6) \qquad r_{ab} := \sum_\alpha T_{ab}^\alpha u_\alpha + \sum_\mu T_{ab}^\mu v_\mu + \sum_p T_{ab}^p w_p.$$

By [6, p. 18],

$$2p_0 = \sum_\alpha (u_\alpha)^2 - \sum_\mu (v_\mu)^2,$$

(7)

$$p_a = \sum_{\alpha\mu} S^a_{\alpha\mu} u_\alpha v_\mu + \sum_{\alpha p} S^a_{\alpha p} u_\alpha w_p + \sum_{\mu p} S^a_{\mu p} v_\mu w_p$$

for $1 \le a \le m_1$, where we set

$$S^a_{\alpha\mu} := \langle S(X_\alpha, Y_\mu), n_a \rangle,$$

and so forth, with X_α, Y_μ, and Z_p the orthonormal bases for the coordinates u_α, v_μ, and w_p, respectively. Note that our p_a are different from those in [6, (6.6)], which are truncated versions of ours. We claim that

(8) $$T^\alpha_{a0} = T^\mu_{a0} = 0$$

for $1 \le a \le m_1$. To this end, we calculate q_a in two ways. On the one hand, substituting (6) and (7) into (3), we see that q_a has the term

$$\left(\sum_\alpha T^\alpha_{a0} u_\alpha \right) \left(\sum_\beta (u_\beta)^2 \right) / 2,$$

so that the coefficient of $(u_\alpha)^3$ in q_a, denoted by $q^{\alpha\alpha\alpha}_a$, is

$$q^{\alpha\alpha\alpha}_a = T^\alpha_{a0}/2.$$

On the other hand, when calculating $D^\perp$, we can pick a normal frame so that the normal connection form is zero at any fixed point. Then by (1) and the fact that $S^a_{\alpha\beta} = S^a(X_\alpha, X_\beta) = 0$, we calculate to see that

$$3T^\alpha_{a0}/2 = 3q^{\alpha\alpha\alpha}_a = \langle Q(X_\alpha, X_\alpha, X_\alpha), n_a \rangle$$

$$= dS^a_{\alpha\alpha}(X_\alpha) - \sum_t \theta^t_\alpha(X_\alpha) S^a_{t\alpha} - \sum_t \theta^t_\alpha(X_\alpha) S^a_{\alpha t}$$

(9)

$$= -2 \sum_t \theta^t_\alpha(X_\alpha) S^a_{t\alpha} = -2 \sum_p \theta^p_\alpha(X_\alpha) S^a_{p\alpha} - 2 \sum_\mu \theta^\mu_\alpha(X_\alpha) S^a_{\alpha\mu}$$

$$= 0,$$

where θ^i_j is the Riemannian connection forms of M_+ and the last equality is by [6, (4.18)]. Likewise, $T^\mu_{a0} = 0$. Hence, (8) is proved. The skew-symmetry of r_{ab} in a, b then yields

(10) $$T^\alpha_{0a} = T^\mu_{0a} = 0.$$

Next, let us calculate q_0 in two ways. On the one hand, we expand q_0 by (3), (6), (7), and (10), keeping in mind that q_0 is homogeneous of degree 1 in u_α, v_μ, and w_p, by [19, Part I, p. 537], to obtain that the coefficient of the $u_\alpha v_\mu w_p$-term of q_0, denoted by $q_0^{\alpha\mu p}$, is

$$q_0^{\alpha\mu p} = \sum_{b \geq 1} T_{0b}^p S_{\alpha\mu}^b = 2 \sum_{b \geq 1} T_{0b}^p F_{\alpha b}^\mu,$$

where $S_{\alpha\mu}^b = 2F_{\alpha a}^\mu$ is employed in [6, (6.4)]. On the other hand, by (1) and p_0 in (7), a similar calculation as in (9) yields

$$\begin{aligned}
q_0^{\alpha\mu p} &= 2\langle Q(Y_\mu, Z_p, X_\alpha), n_0 \rangle \\
&= 2\Big(dS_{p\alpha}^0(Y_\mu) - \sum_t \theta_p^t(Y_\mu) S_{t\alpha}^0 - \sum_t \theta_\alpha^t(Y_\mu) S_{pt}^0 \Big) \\
&= -2 \sum_t \theta_p^t(Y_\mu) S_{t\alpha}^0 = -2\theta_p^\alpha(Y_\mu) \\
&= 4F_{\alpha p}^\mu,
\end{aligned}$$

where the last equality follows by [6, (4.18)]. In conclusion, we derive

$$(11) \qquad F_{\alpha p}^\mu = \sum_b f_{pb} F_{\alpha b}^\mu,$$

where

$$(12) \qquad f_{pb} = T_{0b}^p/2,$$

which is exactly [6, (6.13)]. Therefore, we may assume, as in [6, Proposition 11, p. 19], that

$$(13) \qquad F_{\alpha\, a+m_1}^\mu = F_{\alpha a}^\mu$$

once we show that the matrix (f_{pb}) is orthogonal. This is indeed true; the key is another of the 10 identities of Ozeki and Takeuchi [19, Part I, p. 530] defining an isoparametric hypersurface,

$$(14) \qquad \begin{aligned}
16 \sum_{a=0}^{m_1} (q_a)^2 &= 16G\Big(\sum_\alpha (u_\alpha)^2 + \sum_\mu (v_\mu)^2 + \sum_p (w_p)^2 \Big) \\
&\quad - \langle \nabla G, \nabla G \rangle,
\end{aligned}$$

where $G = \sum_{a=0}^{m_1}(p_a)^2$. We can employ the commutative algebra scheme of Proposition 1 again on (14) to rewrite it as a polynomial homogeneous in all $p_a p_b$ with (homogeneous) polynomial coefficients of degree 2, so that these coefficients are linear combinations of all p_a. Specifically, the coefficient of $(p_0)^2$ is

$$16\sum_{a=1}^{m_1}(r_{0a})^2 - 16\Big(\sum_\alpha(u_\alpha)^2 + \sum_\mu(v_\mu)^2 + \sum_p(w_p)^2\Big) + 4\langle\nabla p_0, \nabla p_0\rangle,$$

which is a linear combination of $p_0, p_1, \dots, p_{m_1}$. Knowing that r_{0a} are functions of w_p alone by (6) and (10), we invoke (7) and compare variable types to conclude that

$$(15) \qquad \sum_{a=1}^{m_1}(r_{0a})^2 = \sum_{p=m_1+1}^{2m_1}(w_p)^2,$$

where we agree that p is indexed between m_1+1 and $2m_1$. But then (6) for r_{0a} in terms of (15) says exactly that the matrix (f_{pb}) is orthogonal.

Now with the index choice for p, we see that

$$f_{a+m_1\, b} = \delta_{ab}$$

by (11) and (13). That is, by (6), (10), and (12),

$$r_{0b} = 2\sum_a \delta_b^a w_{a+m_1} = 2w_{b+m_1},$$

and, with the Einstein summation convention,

$$\begin{aligned}
q_0 &= r_{0b}p_b \\
&= 2(\delta_b^a w_{a+m_1})(S_{\alpha\mu}^b u_\alpha v_\mu + S_{\alpha\, c+m_1}^b u_\alpha w_{c+m_1} + S_{\mu\, c+m_1}^b v_\mu w_{c+m_1}).
\end{aligned}$$

Hence, we have

$$\sum_{abc\alpha}(\delta_b^a w_{a+m_1})(S_{\alpha\, c+m_1}^b u_\alpha w_{c+m_1}) = 0$$

or, equivalently, noting that $S_{\alpha\, c+m_1}^a = -F_{c+m_1\, a}^\alpha$ by [6, p. 18],

$$\sum_{ac} F_{c+m_1\, a}^\alpha w_{c+m_1} w_{a+m_1} = 0.$$

In other words, we have

$$(16) \qquad F^{\alpha}_{c+m_1\,a} = -F^{\alpha}_{a+m_1\,c}.$$

Likewise, we have

$$(17) \qquad F^{\mu}_{c+m_1\,a} = -F^{\mu}_{a+m_1\,c}.$$

LEMMA 1. *If $m_1 < m_2$, then the vectors*

$$(F^{\mu}_{\alpha 1}, F^{\mu}_{\alpha 2}, \ldots, F^{\mu}_{\alpha m_1}),$$

for $1 \le \alpha, \mu \le m_2$, span $\mathbb{R}^{m_1}$.

Proof. This is [6, Proposition 7, p. 18]. □

LEMMA 2. *Let $m_1 < m_2$. If (13), (16), and (17) hold, then the hypersurface is of OT-FKM type.*

Proof. It suffices to show that [6, (8.4), p. 28] holds. Then [6, Theorem 24, p. 36] (or see [7] for a conceptual proof of it) will establish the conclusion.

Since all the formulas are given in the proof of [6, Proposition 19], we will be brief and follow all notation there.

Employing equations [6, (8.5)–(8.9)], one establishes, by (13)–(17) with the Einstein summation convention, that

$$F^{\mu}_{\alpha b}(-\theta^b_a + \theta^{b+m_1}_{a+m_1}) = (F^{\mu}_{\alpha\,a\,c} - F^{\mu}_{\alpha\,a+m_1\,c})\theta^c + (F^{\mu}_{\alpha\,a\,c+m_1} - F^{\mu}_{\alpha\,a+m_1\,c+m_1})\theta^{c+m_1}.$$

Then the equations of [6, (8.21)–(8.24)] imply that

$$F^{\mu}_{\alpha\,a\,c} - F^{\mu}_{\alpha\,a+m_1\,c} = F^{\mu}_{\alpha\,a\,c+m_1} - F^{\mu}_{\alpha\,a+m_1\,c+m_1},$$

so that $\theta^b_a - \theta^{b+m_1}_{a+m_1}$ is spanned only by $\theta^c + \theta^{c+m_1}, 1 \le c \le m_1$, by Lemma 1 above. This proves [6, (8.4)]. □

We summarize what we have done so far in the following satisfying proposition.

PROPOSITION 4. *Let $m_1 < m_2$. If $p_0, p_1, \ldots, p_{m_1}$ form a regular sequence, then the isoparametric hypersurface is of OT-FKM type.*

§4. The dimension estimate when $m_2 \geq 2m_1 - 1$

We now show that if $m_2 \geq 2m_1 - 1$, then the assumption in Proposition 4 holds so that the isoparametric hypersurface is of OT-FKM type. Henceforth, all homogeneous polynomials are regarded as being over the complex numbers.

We agree that $\mathbb{C}^{2m_2+m_1}$ consists of points (u, v, w) with coordinates u_α, v_μ, and w_p, where $1 \leq \alpha, \mu \leq m_2$ and $1 \leq p \leq m_1$. For $0 \leq k \leq m_1$, let

$$W_k := \big\{(u, v, w) \in \mathbb{C}^{2m_2+m_1} : p_0(u, v, w) = \cdots = p_k(u, v, w) = 0\big\}.$$

We want to estimate the dimension of the subvariety U_k of $\mathbb{C}^{2m_2+m_1}$, where

$$U_k := \big\{(u, v, w) \in \mathbb{C}^{2m_2+m_1} : \text{rank of the Jacobian of } p_0, \ldots, p_k < k+1\big\}.$$

Similar to [6, p. 68], $p_0, \ldots, p_k$ give rise to a linear system of cones $\mathcal{C}_\lambda$ defined by

$$c_0 p_0 + \cdots + c_k p_k = 0$$

with $\lambda = [c_0 : \cdots : c_k] \in \mathbb{C}P^k$. The singular subvariety of $\mathcal{C}_\lambda$ is

$$\mathscr{S}_\lambda := \big\{(u, v, w) \in \mathbb{C}^{2m_2+m_1} : (c_0 S_{n_0} + \cdots + c_k S_{n_k}) \cdot (u, v, w)^{tr} = 0\big\},$$

where $\langle S_{n_i}(X), Y \rangle = \langle S(X, Y), n_i \rangle$ is the shape operator of the focal manifold M_+ in the normal direction n_i; we have

(18) $$U_k = \bigcup_\lambda \mathscr{S}_\lambda.$$

By Corollary 1, we wish to establish

(19) $$\dim(W_k \cap U_k) \leq \dim(W_k) - 2$$

for $k \leq m_1 - 1$ to verify that $p_0, p_1, \ldots, p_{m_1}$ form a regular sequence.

We first estimate the dimension of $\mathscr{S}_\lambda$. We break it into two cases. If $c_0, \ldots, c_k$ are either all real or all purely imaginary, then

$$\dim(\mathscr{S}_\lambda) = m_1,$$

since $c_0 S_{n_0} + \cdots + c_k S_{n_k} = c S_n$ for some unit normal vector n and some nonzero real or purely imaginary constant c, and we know that the null space of S_n is of dimension m_1. On the other hand, if $c_0, \ldots, c_k$ are not

all real and not all purely imaginary, then similar to [6, p. 69], after a normal basis change, we can assume that $\mathscr{S}_\lambda$ consists of elements (u, v, w) of the form $(S_{n_1^*} - \tau S_{n_0^*}) \cdot (u, v, w)^{tr} = 0$ for some nonzero complex number τ, relative to a new orthonormal normal basis $n_0^*, n_1^*, \ldots, n_k^*$ in the linear span of $n_0, n_1, \ldots, n_k$. That is, in matrix form,

$$(20) \qquad \begin{pmatrix} 0 & A & B \\ A^{tr} & 0 & C \\ B^{tr} & C^{tr} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \tau \begin{pmatrix} I & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where x, y, and z are (complex) eigenvectors of (real) $S_{n_0^*}$ with eigenvalues $1, -1$, and 0, respectively; [6, Lemma 49, p. 64] ensures that we can assume that

$$B = C = \begin{pmatrix} 0 & 0 \\ 0 & \sigma \end{pmatrix},$$

where σ is a nonsingular diagonal matrix of size r by r, with r the rank of B, and that A is of the form

$$(21) \qquad A = \begin{pmatrix} I & 0 \\ 0 & \Delta \end{pmatrix},$$

where $\Delta = \mathrm{diag}(\Delta_1, \Delta_2, \Delta_3, \ldots)$ is of size r by r, in which $\Delta_1 = 0$ and $\Delta_i, i \geq 2$ are nonzero skew-symmetric matrices expressed in the block form $\Delta_i = \mathrm{diag}(\Theta_i, \Theta_i, \Theta_i, \ldots)$ with Θ_i a 2-by-2 matrix of the form

$$\begin{pmatrix} 0 & f_i \\ -f_i & 0 \end{pmatrix}$$

for some $0 < f_i < 1$. We decompose x, y, z into $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2)$ with $x_2, y_2, z_2 \in \mathbb{C}^r$ (by abuse of notation, we do not distinguish column vectors from row vectors). Then (20) comes down to

$$(22) \qquad \begin{aligned} x_1 &= -\tau y_1, & y_1 &= \tau x_1, \\ -\Delta x_2 + \sigma z_2 &= -\tau y_2, & \Delta y_2 + \sigma z_2 &= \tau x_2, \\ \Delta(x_2 + y_2) &= 0. \end{aligned}$$

It follows from the first set of equations of (22) that either $x_1 = y_1 = 0$ or both are nonzero with $\tau = \pm\sqrt{-1}$. In both cases, by the second set of equations of (22), we have

$$(\Delta^2 - \tau^2 I)x_2 = (\Delta - \tau I)\sigma z_2, \qquad (\Delta^2 - \tau^2 I)y_2 = -(\Delta - \tau I)\sigma z_2,$$

which together with the third equation of (22) imply that $x_2 = -y_2$, and so z_2 can be solved in terms of x_2 by the second set of equations of (22). (Note that, conversely, $x_2 = -y_2$ can be solved in terms of z_2 when $\tau \neq \pm f_i \sqrt{-1}$ for all i, so that z can be chosen to be a free variable in this case.) So, either $x_1 = y_1 = 0$, in which case

$$\dim(\mathscr{S}_\lambda) = m_1,$$

or both x_1 and y_1 are nonzero, in which case $y_1 = \pm\sqrt{-1}x_1$ and so

$$(23) \qquad \dim(\mathscr{S}_\lambda) = m_1 + m_2 - r.$$

Since eventually we must estimate the dimension of $W_k \cap U_k$, let us cut $\mathscr{S}_\lambda$ by

$$0 = p_0^* = \sum_\alpha (x_\alpha)^2 - \sum_\mu (y_\mu)^2.$$

Case 1: x_1 and y_1 are both nonzero. This is the case of nongeneric $\lambda \in \mathbb{C}P^k$. We substitute $y_1 = \pm\sqrt{-1}x_1$ and x_2 and y_2 in terms of z_2 into $p_0^* = 0$ to deduce that

$$0 = p_0^* = (x_1)^2 + \cdots + (x_{m_2-r})^2 + z \text{ terms};$$

hence, $p_0^* = 0$ cuts $\mathscr{S}_\lambda$ to reduce the dimension by 1. That is, by (23),

$$(24) \qquad \dim(W_k \cap \mathscr{S}_\lambda) \leq (m_1 + m_2 - r) - 1 \leq m_1 + m_2 - 1,$$

noting that W_k is also cut out by $p_0^*, p_1^*, \ldots, p_k^*$. Meanwhile, similar to [6, p. 71], only a subvariety of λ of dimension $k-1$ in $\mathbb{C}P^k$ assumes that $\tau = \pm\sqrt{-1}$. Therefore, by (24), an irreducible component $\mathcal{W}$ of $W_k \cap \bigcup_\lambda \mathscr{S}_\lambda$ over nongeneric λ will satisfy

$$\dim(\mathcal{W}) \leq \dim(W_k \cap \mathscr{S}_\lambda) + k - 1 \leq m_1 + m_2 + k - 2.$$

Case 2: $x_1 = y_1 = 0$. This is the case of generic λ, where $\dim(\mathscr{S}_\lambda) = m_1$, so that an irreducible component $\mathcal{V}$ of $W_k \cap \bigcup_\lambda \mathscr{S}_\lambda$ over generic λ will satisfy

$$\dim(\mathcal{V}) \leq m_1 + k \leq m_1 + m_2 + k - 2,$$

as we may assume that $m_2 \geq 2$. (The case $m_1 = m_2 = 1$ is straightforward; see [6, p. 61].)

Putting these two cases together, we conclude that

$$(25) \qquad \dim(W_k \cap U_k) \leq m_1 + m_2 + k - 2.$$

On the other hand, since W_k is cut out by $k+1$ equations, we have

$$\text{(26)} \qquad \dim(W_k) \geq m_1 + 2m_2 - k - 1.$$

Therefore,

$$\dim(W_k \cap U_k) \leq \dim(W_k) - 2$$

when $k \leq m_1 - 1$, taking $m_2 \geq 2m_1 - 1$ into account.

In summary, we have established (19) for $k \leq m_1 - 1$, so that the ideal $(p_0, p_1, \ldots, p_k)$ is prime when $k \leq m_1 - 1$. Corollary 1 then implies that $p_0, p_1, \ldots, p_{m_1}$ form a regular sequence. It follows by Proposition 4 that the isoparametric hypersurface is of OT-FKM type.

§5. The case $(m_1, m_2) = (3, 4)$

Recall that a point of M_+ is of Condition A if the shape operators S_n share the same kernel for all unit normal n. Conditions A and B were introduced and explored by Ozeki and Takeuchi [19, Part I, p. 541] in their construction of inhomogeneous isoparametric hypersurfaces in spheres. It was then established by Dorfmeister and Neher [9] (see also [8]) that Condition A alone implies that the isoparametric hypersurface is of OT-FKM type.

THEOREM 1. *Let $(m_1, m_2) = (3, 4)$. Then there exist points of Condition A on M_+; the isoparametric hypersurface is then of OT-FKM type.*

Proof. We follow the notation in Section 4. Suppose that M_+ is free of points of Condition A everywhere. Then one of the three pairs of matrices $(B_1, C_1), (B_2, C_2)$, and (B_3, C_3) of the shape operators S_{n_1}, S_{n_2}, and S_{n_3}, similar to the one given in (20), must be nonzero. By replacing (B_2, C_2) by (B_3, C_3), we may assume that one of (B_1, C_1) and (B_2, C_2) is nonzero in the neighborhood of a given point. For $k = m_1 - 1 = 2$, observe that in (24) if $r > 0$ holds, then in fact

$$\text{(27)} \qquad \dim(W_2 \cap \mathscr{S}_\lambda) \leq m_1 + m_2 - 2.$$

If $r = 0$, then $B_1^* = C_1^* = 0$ and $A^* = I$ in (20) for $S_{n_1^*}$. It follows that $p_0^* = 0$ and $p_1^* = 0$ cut $\mathscr{S}_\lambda$ in the variety

$$\text{(28)} \qquad \left\{ (x, \pm\sqrt{-1}\,x, z) : \sum_\alpha (x_\alpha)^2 = 0 \right\}.$$

(B_2^*, C_2^*) of $S_{n_2^*}$ must be nonzero now. Since z is a free variable in (28), $p_2^* = 0$ will have nontrivial z-terms

$$0 = p_2^* = \sum_{\alpha p} S_{\alpha p} x_\alpha z_p + \sum_{\mu p} T_{\mu p} y_\mu z_p + x_\alpha y_\mu \text{ terms}$$

$$= \sum_{\alpha p} (S_{\alpha p} \pm \sqrt{-1} T_{\alpha p}) x_\alpha z_p + x_\alpha x_\mu \text{ terms},$$

taking $y = \pm\sqrt{-1}x$ into account, where $S_{\alpha p} := \langle S(X_\alpha^*, Z_p^*), n_2^* \rangle$ and $T_{\mu p} := \langle S(Y_\mu^*, Z_p^*), n_2^* \rangle$ are (real) entries of B_2^* and C_2^*, respectively, and where $X_\alpha^*, 1 \le \alpha \le m_2$, $Y_\mu^*, 1 \le \mu \le m_2$, and $Z_p^*, 1 \le p \le m_1$ are orthonormal eigenvectors for the eigenspaces of $S_{n_0^*}$ with eigenvalues $1, -1$, and 0, respectively; hence, the dimension of $\mathscr{S}_\lambda$ will be cut down by 2 by $p_0^*, p_1^*, p_2^* = 0$, so that, again,

$$(29) \qquad \dim(W_2 \cap \mathscr{S}_\lambda) \le m_1 + m_2 - 2,$$

noting that $p_0^*, p_1^*, p_2^* = 0$ also cut out W_2.

In conclusion, for $k = 2$,

$$(30) \qquad \dim(W_k \cap U_k) \le \dim(W_k \cap \mathscr{S}_\lambda) + k - 1 \le m_1 + m_2 + k - 3 = 6,$$

while by (26),

$$\dim(W_k) \ge 8,$$

so that

$$\dim(W_2 \cap U_2) \le \dim(W_2) - 2.$$

(Again, it suffices to consider the nongeneric case for the dimension estimate since the generic case contributes only dimension at most $m_1 + k = 5$, similar to what is detailed in Section 4.)

Meanwhile, (25) and (26) for $k \le 1$ imply that

$$\dim(W_k \cap U_k) \le \dim(W_k) - 2.$$

Therefore, it follows again by (19) that $(p_0), (p_0, p_1)$, and (p_0, p_1, p_2) are all prime ideals, and Corollary 1 asserts that p_0, p_1, p_2, p_3 form a regular sequence, so that Proposition 4 establishes that the isoparametric hypersurface is of the type constructed by Ozeki and Takeuchi [19, Part I], which has points of Condition A on M_+. This contradiction to the assumption made at the outset shows that, indeed, M_+ has points of Condition A.   □

§6. Concluding remarks

Knowing that M_+ has points of Condition A for an isoparametric hypersurface with multiplicities $(7, 8)$ of OT-FKM type (see [12]), one is tempted to apply the upper bound estimate $m_1 + m_2 + k - 3$ in (30) and the lower bound estimate $m_1 + 2m_2 - k - 1$ in (26) to the case, with $k = 6$, to assert (19). This encounters apparent difficulties.

In fact, the upper bound cannot be improved to $m_1 + m_2 + k - 4$ without understanding further properties pertaining to an isoparametric hypersurface with one of the three remaining multiplicity pairs, since otherwise $(m_1, m_2) = (4, 5)$ for $k \leq 3$ would satisfy

$$\dim(W_k \cap U_k) \leq \dim(W_k) - 2,$$

so that the same arguments as above would imply that the isoparametric hypersurface would be of OT-FKM type, which is not the case.

A detailed understanding of the number r in (24) for the multiplicity pair $(7, 8)$ would seem important.

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