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Cone-Monotone Functions: Differentiability and Continuity

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Abstract. We provide a porosity-based approach to the differentiability and continuity of real-valued functions on separable Banach spaces, when the function is monotone with respect to an ordering induced by a convex cone K with non-empty interior. We also show that the set of nowhere K-monotone functions has a σ-porous complement in the space of continuous functions endowed with the uniform metric.

1 Introduction

The fact that σ-directionally porous sets and porous sets arise naturally in the study of differentiability of Lipschitz functions has been well illustrated by Preiss and Zajiceck [6, 7]. It is our goal in this note to provide a σ-directional porosity-based approach to the differentiability and continuity of cone-monotone functions on a Banach space X.

Cone-monotone functions have been considered by Ward, Chabrillac–Crouzeix, and Saks on $\mathbb{R}^n$ [4, 10], Borwein, Burke, and Lewis [2] on separable spaces — for K having non-empty interior. The key positive result is: Suppose X is separable and $K \subset X$ is a convex cone with non-empty interior. If $f : X \to R \cup \{+\infty\}$ is K-monotone, then f is Gâteaux differentiable a.e. [2]. As shown in Borwein and Goebel [3], if K has empty interior, almost anything can happen for K-monotone functions.

The paper is organized as follows. In Section 2, we illustrate that the class of cone-monotone functions is significantly broad; it includes Lipschitz functions, quasiconvex functions, and marginal value functions. In Section 3, we give an alternative proof to the differentiability theorem of cone-monotone functions on separable Banach spaces (due to Borwein, Burke and Lewis [2]) using the notion of σ-directionally porous sets. Section 4 deals with continuity, measurability, and extendibility of cone-monotone functions. In Section 5, we discuss the relationships among upper hull, lower hull, and the original monotone functions with regards to continuity and to differentiability. Section 6 details an application to quasiconvex functions. In Section 7, we show that the family of functions which are K-monotone functions on some open subset is σ-porous in the space of continuous functions endowed with the uniform metric. We conclude the paper with some open questions.

In the remainder of this introduction we give the basic notions and definitions used in the sequel.

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Let X be a Banach space, let $A \subset X$ be a non-empty open set, and let $K \subset X$ be a closed convex cone with $\operatorname{int}(K) \neq \varnothing$. Here $\operatorname{int}(K)$ denotes the interior of K. We say that $f\colon A \to \mathbb{R} \cup \{+\infty\}$ is *K-increasing* on A if $f(x+k) \geq f(x)$ whenever $x \in A$, $x+k \in A$ and $k \in K$. We say that f is *strictly K-increasing* on A if $f(x+k) > f(x)$ whenever $x+k, x \in A$ and $k \in K \setminus \{0\}$. For $x \in A$, we define the one-sided derivatives

$$f^{+}(x;v) := \limsup_{t \downarrow 0} \frac{f(x+tv)-f(x)}{t}, \quad \text{and} \quad f_{+}(x;v) := \liminf_{t \downarrow 0} \frac{f(x+tv)-f(x)}{t}.$$

We note that both $f^{+}(x;\,\cdot\,)$ and $f_{+}(x;\,\cdot\,)$ are K-increasing whenever f is K-increasing. When $f^{+}(x;v) = f_{+}(x;v)$ is finite, we write

$$f_{+}'(x;v) = \lim_{t \downarrow 0} \frac{f(x+tv)-f(x)}{t}.$$

The two-sided directional derivative $f'(x;v)$ is defined by

$$f'(x;v) := \lim_{t \to 0} \frac{f(x+tv)-f(x)}{t}.$$

We use $\underline{f}$ and $\overline{f}$ to denote the *lower (semi-continuous) envelope* and *upper envelope* of f respectively. For $a, b \in X$, we let $\mathbb{B}(a,r)$ denote the open ball with center a and radius r, and write $a \leq_K b$ if $b - a \in K$,

$$(a,b) := (a + \operatorname{int}(K)) \cap (b - \operatorname{int}(K)), \quad \text{and} \quad [a,b] := (a+K) \cap (b-K).$$

Definition 1 Let X be a Banach space and $M \subset X$.

(i) The set M is *porous* at a if there exists $1 > c > 0$ such that for every $\epsilon > 0$ there is some point $b \in X$ such that $\|b-a\| < \epsilon$, $\mathbb{B}(b,r) \cap M = \varnothing$, and $r > c\|b-a\|$.

(ii) M is *directionally porous* at a if one can always use $b = a + tv$ for some $t \geq 0$ and a fixed direction $v \in X$.

(iii) M is *porous (resp. directionally porous)* if it is porous (resp. directionally porous) at all points of M.

(iv) The set M is *σ-porous (resp. directionally σ-porous)* if it is a countable union of porous (resp. directionally porous) subsets of X.

We note that in $\mathbb{R}^n$, porous sets and directionally porous sets are the same. We also need the definition of Aronszajn null sets.

Definition 2 Let X be a separable Banach space and let $0 \neq v \in X$ be given. We define

(i) $\mathcal{A}(v)$ as the system of all Borel sets $B \subset X$ such that $B \cap (a + \mathbb{R}v)$ is Lebesgue null on each line $a + \mathbb{R}v$, $a \in X$.

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(ii) If $\{x_n\}$ is a finite or infinite sequence of nonzero elements in X, we denote by $\mathcal{A}(\{x_n\})$ the collection of all Borel sets A which can be decomposed as $A = \bigcup A_n$, where $A_n \in \mathcal{A}(x_n)$ for every n.

(iii) A set $A \subset X$ is called *Aronszajn null* if for every given complete (*i.e.,* densely spanning) sequence (x_n) in X, *i.e.,*

$$\overline{\operatorname{span}\{x_1, x_2, x_3, \dots\}} = X,$$

the set A belongs to $\mathcal{A}(\{x_n\})$.

Note that when X is separable, directionally porous sets are Aronszajn null [6].

2 Why K-Monotone Functions?

An easy but key observation is that Lipschitz functions decompose as a sum of linear and monotone functions (this may be viewed as a strong analogue of being of bounded variation).

Proposition 1 *Let A be a non-empty open subset of a Banach space X, and let $f\colon A \to \mathbb{R}$ be Lipschitz on A. Then there exists an element $x^* \in X^*$ such that $f + x^*$ is K-monotone on A with respect to some convex cone K with* $\operatorname{int}(K) \neq \varnothing$.

Proof We follow the idea from [2]. Fix $v_0 \in S_X$ and $\phi \in X^*$ such that $\phi(v_0) = 1$. For $\epsilon > 0$ small, when $\|v - v_0\| \leq \epsilon$ we have $\phi(v) \geq 1/2$. Then

$$\phi(v) \geq \frac{1}{2} \geq \frac{1}{2}\frac{1}{1+\epsilon}(1+\epsilon) \geq \frac{1}{2(1+\epsilon)}\|v\|,$$

for $\|v - v_0\| \leq \epsilon$. Let $K := \bigcup_{l \geq 0} l\mathbb{B}(v_0, \epsilon)$. By the homogeneity of ϕ, $\phi(v) \geq C\|v\|$ for $v \in K$ and $C = 1/(2(1+\epsilon))$. Since f is Lipschitz, for $x \in A, k \in K$, for $x + k \in A$ we have

$$f(x+k) - f(x) \leq L\|k\| \leq \frac{L}{C}C\|k\| \leq \frac{L}{C}\phi(k).$$

That is,

$$\left(f - \frac{L}{C}\phi\right)(x+k) \leq \left(f - \frac{L}{C}\phi\right)(x).$$

whenever $x, x+k \in A$ and $k \in K$. Hence $(f - \frac{L}{C}\phi)$ is $-K$-increasing. ■

Recall that a function $f\colon X \to \mathbb{R} \cup \{+\infty\}$ is *quasiconvex* if the lower level set $S_\lambda(f) = \{x \in A \mid f(x) \leq \lambda\}$ is convex for every $\lambda \in \mathbb{R}$.

Proposition 2 *Assume f is quasiconvex and lower semicontinuous (l.s.c.) on a Banach space X. Suppose that S_λ has non-empty interior. Then for every $a \in X$ with $f(a) > \lambda$, there exist an open neighborhood V of a and a convex cone K with* $\operatorname{int}(K) \neq \varnothing$, *such that f is K-monotone on V.*

Proof Consider $c = a + \alpha(a - b)$ with $\alpha > 0$ and $b \in \operatorname{int}(S_\lambda)$. Choose $\epsilon > 0$ such that $\mathbb{B}(b, \epsilon) \in S_\lambda$, and define

$$K = \bigcup_{l \geq 0} l[\mathbb{B}(b, \epsilon) - c].$$

Since f is l.s.c. at a, there exists an open neighborhood V of a such that $f(x) > \lambda$ if $x \in V$ and $V \subset c + K$. For $x \in V, x + k \in V$, there exists $y \in \mathbb{B}(b, \epsilon)$ such that $x + k = \xi x + (1 - \xi)y$ for some $0 < \xi < 1$. We have

$$f(x + k) \leq \max\{f(y), f(x)\} = f(x),$$

because $f(y) \leq \lambda$ and $f(x) > \lambda$. Hence f is $-K$-increasing on V. ∎

As a final example, let $f\colon X \to \mathbb{R}$ be bounded below and $g\colon X \to Y$, where Y is a Banach space partially ordered by a closed convex cone K. The *optimal value function* $V(p)$ for the inequality constraints minimization problem

$$\min\{f(x) : g(x) \leq_K p\}$$

is $-K$-increasing on Y. When K has non-empty interior, and the Slater condition is verified, *i.e.*, there exists $\hat{x} \in X$ such that $-g(\hat{x}) \in \operatorname{int}(K)$, $V(p)$ is moreover finite-valued around 0.

3 Main Result

Let $\mathbb{Q}$ denote the rational numbers, and $\mathbb{Q}^+$ denote the nonnegative rationals. We continue with a few preparatory results.

Lemma 3 *Let f be a real valued function defined on a Banach space X and fix $v_1, v_2 \in X$. For $k, l, m \in \mathbb{N}$ and $y, z \in \mathbb{R}$, the set $A(k, l, m, y, z)$ of all $x \in X$ verifying*

(i) $$\frac{f(x + tu) - f(x)}{t} - y < \frac{1}{l} \quad \text{for } \|u - v_1\| < 1/m \text{ and } 0 < t < 1/k,$$

(ii) $$\frac{f(x + tu) - f(x)}{t} - z < \frac{1}{l} \quad \text{for } \|u - v_2\| < 1/m \text{ and } 0 < t < 1/k,$$

(iii) $$\frac{f(x + s(v_1 + v_2)) - f(x)}{s} - (y + z) > \frac{3}{l} \quad \text{occurs for arbitrarily small } s > 0,$$

is directionally porous in X.

Proof Let $x \in A(k, l, m, y, z)$. Choose $0 < s < 1/k$ such that

$$\frac{f(x + s(v_1 + v_2)) - f(x)}{s} - (y + z) > \frac{3}{l}.$$

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We claim that
$$\mathbb{B}(x+sv_1,\frac{s}{m})\cap A(k,l,m,y,z)=\varnothing.$$
Indeed, for $\|h\|<\frac{1}{m}$, if $x+sv_1+sh$ satisfies (ii), we have

$$\frac{f(x+s(v_1+h)+su)-f(x+s(v_1+h))}{s}<z+\frac{1}{l},\ \text{for } \|u-v_2\|<\frac{1}{m}. \tag{1}$$

By (i),

$$\frac{f(x+s(v_1+h))-f(x)}{s}<y+\frac{1}{l}. \tag{2}$$

Adding inequalities (1) and (2), we get

$$\frac{f(x+s(v_1+h)+su)-f(x)}{s}<y+z+\frac{2}{l},\ \text{for } \|u-v_2\|<\frac{1}{m}.$$

Taking $u=v_2-h$, we have

$$\frac{f(x+sv_1+sv_2)-f(x)}{s}<y+z+\frac{2}{l}.$$

This contradicts the choice of s. ∎

Define

$$A_{(v_1,v_2)}:=\bigcup\{A_{(k,l,m,y,z)}\mid k,l,m\in\mathbb{N},\,y,z\in\mathbb{Q}\}. \tag{3}$$

Then by definition $A_{(v_1,v_2)}$ is σ-directionally porous in X.

Lemma 4 *Assume that X is a Banach space and $f\colon X\to\mathbb{R}$ is K-increasing, with $\operatorname{int}(K)\neq\varnothing$. For $u,v\in\operatorname{int}(K)$, define the sets*

$$E:=\{x\in X\mid f'(x;u)\text{ and } f'(x;v)\text{ exist and are finite}\},$$

$$S:=\{x\in E\mid f^+(x;d_1u+d_2v)\le f'(x;u)d_1+f'(x;v)d_2\text{ holds for all }(d_1,d_2)\in\mathbb{R}^2\}.$$

Then the set $E\setminus S$ is σ-directionally porous in X.

Proof (a) Let D be a countable dense subset in $\mathbb{R}^2$. We claim that

$$S:=\bigcap_{(d_1,d_2)\in D}E_{(d_1,d_2)},$$

where $E_{(d_1,d_2)}:=\{x\in E\mid f^+(x;d_1u+d_2v)\le d_1f'(x;u)+d_2f'(x;v)\}$. Clearly, S is a subset of the latter. We show the reverse inclusion. Given $(d_1,d_2)\in\mathbb{R}^2$, we may find arbitrarily close $(\hat d_1,\hat d_2)\in D$ such that $d_1\le\hat d_1$, $d_2\le\hat d_2$. Then

$$f^+(x;d_1u+d_2v)\le f^+(x;\hat d_1u+\hat d_2v)\le\hat d_1f'(x;u)+\hat d_2f'(x;v).$$

Let $(\hat{d}_1, \hat{d}_2) \to (d_1, d_2)$ to obtain

$$f^+(x; d_1 u + d_2 v) \le d_1 f'(x; u) + d_2 f'(x; v).$$

(b) We show that for each $(d_1, d_2) \in D$, the set $E \setminus E_{(d_1,d_2)}$ is σ-directionally porous. First, by (3), $A_{(d_1 u, d_2 v)}$ is σ-directionally porous. We claim

$$E \setminus A_{(d_1 u, d_2 v)} \subset E_{(d_1,d_2)}.$$

Indeed, for $x \in E \setminus A_{(d_1 u, d_2 v)}$, both $f'(x; u)$ and $f'(x; v)$ exist. For $1/l > 0$, we have

$$f'(x; d_1 u) = d_1 f'(x; u) < d_1 f'(x; u) + \frac{1}{2l},$$
$$f'(x; d_2 v) = d_2 f'(x; v) < d_2 f'(x; v) + \frac{1}{2l}.$$

Because $f^+(x; \cdot)$ is continuous at $d_1 u, d_2 v \in \operatorname{int}(K) \cup \operatorname{int}(-K)$, for some $\delta > 0$,

$$f^+(x; d_1 u + \delta u) < d_1 f'(x; u) + \frac{1}{2l},$$
$$f^+(x; d_2 v + \delta v) < d_2 f'(x; v) + \frac{1}{2l}.$$

For some $k \in \mathbb{N}$, when $0 < t < 1/k$ we have

$$\frac{f(x + t(d_1 u + \delta u)) - f(x)}{t} < d_1 f'(x; u) + \frac{1}{2l},$$
$$\frac{f(x + t(d_2 v + \delta v)) - f(x)}{t} < d_2 f'(x; v) + \frac{1}{2l}.$$

Since $d_1 u + \delta u - K$, $d_2 v + \delta v - K$ are neighborhoods of $d_1 u$ and $d_2 v$ respectively, there exist $m \in \mathbb{N}$ such that

$$\mathbb{B}(d_1 u, 1/m) \subset d_1 u + \delta u - K \quad \text{and} \quad \mathbb{B}(d_2 v, 1/m) \subset d_2 v + \delta v - K.$$

By the K-monotonicity of f we have

$$\frac{f(x + th) - f(x)}{t} < d_1 f'(x; u) + \frac{1}{2l} \quad \text{for } \|h - d_1 u\| < \frac{1}{m},$$
$$\frac{f(x + th) - f(x)}{t} < d_2 f'(x; v) + \frac{1}{2l} \quad \text{for } \|h - d_2 v\| < \frac{1}{m}.$$

Choose $y, z \in \mathbb{Q}$ such that

$$|y - d_1 f'(x; u)| < \frac{1}{2l}, \quad \text{and} \quad |z - d_2 f'(x; v)| < \frac{1}{2l}.$$

We have

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(i)
$$\frac{f(x+th)-f(x)}{t} < y + \frac{1}{l},$$
if $\|h - d_1u\| < 1/m$ and $0 < t < 1/k$.

(ii)
$$\frac{f(x+th)-f(x)}{t} < z + \frac{1}{l},$$
if $\|h - d_2v\| < 1/m$ and $0 < t < 1/k$.

Because $x \in E \setminus A_{(d_1u, d_2v)}$, we have
$$\frac{f(x+t(d_1u+d_2v))-f(x)}{t} < y + z + \frac{3}{l} \quad \text{for small } t > 0.$$

Therefore, for small $t > 0$,
$$\begin{aligned}
&\frac{f(x+t(d_1u+d_2v))-f(x)}{t} - (d_1 f'(x;u) + d_2 f'(x;v)) \\
&= \Big[\frac{f(x+t(d_1u+d_2v))-f(x)}{t} - (y+z)\Big] \\
&\quad + (y - d_1 f'(x;u)) + (z - d_2 f'(x;v)) \\
&< \frac{4}{l}.
\end{aligned}$$

Hence $f^+(x; d_1u + d_2v) \le d_1 f'(x;u) + d_2 f'(x;v)$. ■

Lemma 5 *Assume that X is a Banach space and $f\colon X \to \mathbb{R}$ is K-monotone, with* $\operatorname{int}(K) \neq \varnothing$*. Fix $u, v \in \operatorname{int}(K)$. Let*
$$E := \{x \in X \mid \text{ both } f'(x;u) \text{ and } f'(x;v) \text{ exist and are finite}\},$$
$$S := \{x \in E \mid f'(x; d_1u + d_2v) = d_1 f'(x;u) + d_2 f'(x;v) \text{ for all } (d_1, d_2) \in \mathbb{R}^2\}.$$

Then the set $E \setminus S$ is σ-directionally porous in X.

Proof By Lemma 4, for
$$S_1 := \{x \in E \mid f^+(x; d_1u + d_2v) \le d_1 f'(x;u) + d_2 f'(x;v) \text{ for all } (d_1, d_2) \in \mathbb{R}^2\}$$
the set $E \setminus S_1$ is σ-directionally porous in X. Applied to $-f$, for
$$S_2 := \{x \in E \mid f_+(x; d_1u + d_2v) \ge d_1 f'(x;u) + d_2 f'(x;v) \text{ for all } (d_1, d_2) \in \mathbb{R}^2\},$$
the set $E \setminus S_2$ is σ-directionally porous in X. When $x \in S := S_1 \cap S_2$,
$$f'(x; d_1u + d_2v) = d_1 f'(x;u) + d_2 f'(x;v),$$
for all $(d_1, d_2) \in \mathbb{R}^2$. ■

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Proposition 6 *Assume that X is a Banach space, and $K \subset X$ is a closed convex cone with $\operatorname{int}(K) \neq \varnothing$. Let $f : X \to \mathbb{R}$ be K-increasing. For $k_i \in \operatorname{int}(K)$, $1 \leq i \leq n$, define*

$$D_n := \{r_1 k_1 + \cdots + r_n k_n \mid r_i \in \mathbb{Q}^+ \text{ for } 1 \leq i \leq n\} \setminus \{0\},$$
$$E_n := \{x \in X \mid f'(x; d) \text{ exists and is finite for all } d \in D_n\}.$$

Then the set $E_n \setminus S_n$ is σ-directionally porous in X, where

$$S_n := \{x \in E_n \mid f^+(x; \cdot) = f_+(x; \cdot) \text{ is finite and linear on } \operatorname{span}\{k_1, \ldots, k_n\}\}.$$

Proof By Lemma 5, for $d_1, d_2 \in D_n$, the set

$$S(d_1, d_2) := \{x \in E_n \mid f'(x; rd_1 + sd_2) = rf'(x; d_1) + sf'(x; d_2) \text{ for } (r, s) \in \mathbb{R}^2\},$$

has $E_n \setminus S(d_1, d_2)$ being σ-directional porous in X. Thus

$$S_n := \bigcap \{S(d_1, d_2) \mid d_1, d_2 \in D_n\},$$

has $E_n \setminus S_n$ being σ-directional porous in X. For $x \in S_n$, we will show that $f_+(x; \cdot) = f^+(x; \cdot)$ and is linear on $\operatorname{span}\{k_1, k_2, \ldots, k_n\}$.

To see this, for $l_1, l_2, \ldots, l_n \in \mathbb{R}$, choose nonzero rational numbers

$$\hat{l}_1 \geq l_1, \ldots, \hat{l}_n \geq l_n.$$

As $f^+(x; \cdot)$ is K-increasing,

$$f^+(x; l_1 k_1 + \cdots + l_n k_n) \leq f^+(x; \hat{l}_1 k_1 + \cdots + \hat{l}_n k_n).$$

Without loss of any generality, write

$$\hat{l}_1 k_1 + \cdots + \hat{l}_n k_n = \hat{l}_1 k_1 + \cdots + \hat{l}_m k_m - (-\hat{l}_{m+1} k_{m+1} - \cdots - \hat{l}_n k_n),$$

where $\hat{l}_1, \ldots, \hat{l}_m \geq 0$, $-\hat{l}_{m+1}, \ldots, -\hat{l}_n \geq 0$. As $x \in S_n$, we have

$$\begin{aligned} f^+(x; \hat{l}_1 k_1 + \cdots + \hat{l}_n k_n) &= f'(x; \hat{l}_1 k_1 + \cdots + \hat{l}_m k_m) - f'(x; -\hat{l}_{m+1} k_{m+1} - \cdots - \hat{l}_n k_n) \\ &= \hat{l}_1 f'(x; k_1) + \cdots + \hat{l}_n f'(x; k_n). \end{aligned}$$

Then $f^+(x; l_1 k_1 + \cdots + l_n k_n) \leq \hat{l}_1 f'(x; k_1) + \cdots + \hat{l}_n f'(x; k_n)$. Letting $\hat{l}_1 \to l_1, \ldots, \hat{l}_n \to l_n$, we obtain

$$f^+(x, l_1 k_1 + \cdots + l_n k_n) \leq l_1 f'(x; k_1) + \cdots + l_n f'(x; k_n).$$

Similarly, one may show

$$f_+(x; l_1 k_1 + \cdots + l_n k_n) \geq l_1 f'(x; k_1) + \cdots + l_n f'(x; k_n).$$

Since $f_+(x; \cdot) \leq f^+(x; \cdot)$, we conclude that $f^+(x; \cdot) = f_+(x; \cdot)$ and is linear on $\operatorname{span}\{k_1, \ldots, k_n\}$. ■

Lemma 7 *Let X be a Banach space and $K \subset X$ be a closed convex cone with* $\operatorname{int}(K) \neq \varnothing$*. Suppose that $D \subset X$ is dense. Then for every $u \in X$ there exist $u_n, v_n \in D$ such that*

$$u_n \leq_K u \leq_K v_n, \quad \text{and} \quad u_n \to u,\ v_n \to u \quad \text{in norm as } n \to \infty.$$

Proof As $u \pm K$ has non-empty interior, and D is dense in X, we easily find u_n and v_n. ∎

The following result is Proposition 6.29 [1, p. 144]. We include it for completeness.

Lemma 8 *Let F be an n-dimensional subspace of X, and let $\{y_k\}_{k=1}^n$ be a basis for F. Let λ_n be the Lebesgue measure on F, and let A be a Borel subset of X such that $\lambda_n(F \cap (A + x)) = 0$ for every $x \in X$. Then $A \in \mathcal{A}\big(\{y_k\}_{k=1}^n\big)$.*

We are now ready to prove our main result:

Theorem 9 *Let X be a separable Banach space, $K \subset X$ be a closed convex cone with* $\operatorname{int}(K) \neq \varnothing$*. Suppose that $f\colon X \to \mathbb{R}$ is lower semicontinuous and K-monotone. Then f is Gâteaux differentiable on X except for a Aronszajn null set.*

Proof Without loss of generality, we assume that f is K-increasing (otherwise consider $-K$). Let (x_n) be a complete sequence in X. Because $\operatorname{int}(K) \neq \varnothing$ and $\overline{\operatorname{span}\{x_1, x_2, x_3, \dots\}} = X$, we may take nonzero

$$\{k_i\}_{i=1}^{\infty} \subset \operatorname{span}\{x_1, x_2, x_3, \dots\},$$

such that

$$\overline{\{k_i \mid i \in \mathbb{N}\}} = K, \text{ and } k_i \in \operatorname{int}(K) \text{ for } i \in \mathbb{N}.$$

Define

$$D := \bigcup_{n=1}^{\infty} \{r_1 k_1 + \cdots + r_n k_n \mid r_i \in \mathbb{Q}^+ \text{ for } 1 \leq i \leq n\} \setminus \{0\}.$$

(a) Let $d \in D$. Because f is l.s.c., both $f^+(\,\cdot\,, d)$ and $f_+(\,\cdot\,, d)$ are Borel measurable. Therefore, the set

$$E_d := \Big\{ x \in X \mid f^+(x; d) = f_+(x; d),\ f^+(x; -d) = f_+(x; -d) \text{ exist} \\ \text{and } f_+'(x; -d) + f_+'(x; d) = 0 \Big\},$$

is Borel measurable. For n large, we have

$$d \in \operatorname{span}\{x_1, x_2, \dots, x_n\}.$$

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We claim that $X \setminus E_d$ belongs to $\mathcal{A}\big(\{x_i\}_{i=1}^n\big)$. To see this, we observe that for every $a \in X$, the set $X \setminus E_d$ intersect each line $a + \mathbb{R}d$ in a set of null one-dimensional Lebesgue measure. Write

$$F := \operatorname{span}\{x_1, x_2, \dots, x_n\}, \quad \chi_S(x) := \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Let λ_n denote Lebesgue measure on F. For $a \in X$, we have

$$\begin{aligned} \lambda_n\big(F \cap ((X \setminus E_d) + a)\big) &= \int \chi_{F\cap((X\setminus E_d)+a)}\, d\lambda_n \\ &= \int \lambda_1\big([F \cap ((X \setminus E_d) + a)] \cap (u + \mathbb{R}d)\big)\, d\lambda_{n-1}(u) = 0. \end{aligned}$$

By Lemma 8, we conclude that $X \setminus E_d \in \mathcal{A}\big(\{x_i\}_{i=1}^n\big)$.

Now, the set defined by

$$E := \bigcap_{d\in D} E_d = \{x \in X \mid f'(x; d) \text{ is finite for all } d \in D\},$$

is Borel measurable and $X \setminus E$ belongs to $\mathcal{A}\big(\{x_i\}_{i=1}^\infty\big)$.

(b) Write $Y_n := \operatorname{span}\{k_1, \dots, k_n\}$. By Proposition 6, for

$$S_n := \{x \in E \mid f^+(x; \cdot) = f_+(x; \cdot) \text{ is finite and linear on } Y_n\},$$

the set $E \setminus S_n$ is σ-directionally porous in X. Let $S := \bigcap_{n=1}^\infty S_n$. Then $E \setminus S$ is σ-directionally porous in X, in particular, $E \setminus S \in \mathcal{A}\big(\{x_i\}_{i=1}^\infty\big)$. For $x \in S$, $f^+(x; \cdot) = f_+(x; \cdot)$ is finite and linear on $Y := \bigcup_{n=1}^\infty Y_n$. Since

$$Y \supset \{k_i \mid i \in \mathbb{N}\} - \{k_i \mid i \in \mathbb{N}\},$$

we have $\overline{Y} \supset K - K = X$, *i.e.,* Y is dense in X. Let $x \in S$. We will show that f is Gâteaux differentiable at x. Take $e \in Y \cap \operatorname{int}(K)$. Then $f^+(x; e)$ is finite and

$$f^+(x; y) \le f^+(x; e) \quad \text{for } y \le_K e.$$

Since $\{y \in X \mid y \le_K e\}$ contains 0 as an interior point, by the Hahn–Banach extension theorem, $f^+(x; \cdot)$ can be extended linearly from Y to X, denoted by λ. That is, $\lambda \in X^*$ and $f^+(x; y) = f_+(x; y) = \lambda(y)$ for $y \in Y$. For every $u \in X$, by Lemma 7 there exist $u_n, v_n \in Y$ such that $u_n \le_K u \le_K v_n$ and $u_n \to u, v_n \to u$ in norm. We have

$$f^+(x; u) \le f^+(x; v_n) = \lambda(v_n),$$
$$f_+(x; u) \ge f_+(x; u_n) = f^+(x; u_n) = \lambda(u_n).$$

Let $n \to \infty$ to obtain $f_+(x; u) = f^+(x; u) = \lambda(u)$. Therefore, f is Gâteaux differentiable at $x \in S$. ■

We remark that in separable Banach spaces, Aronszajn null sets, Gaussian null sets, and cubic null sets coincide [1, pp. 142–145] or [6]. Theorem 9 is an extension to separable Banach spaces of the differentiability theorem concerning monotone functions on $\mathbb{R}^n$ given by Chabrillac, and Crouzeix [4]. The following example shows that Theorem 9 fails if $\operatorname{int}(K) = \varnothing$.

Example 10 Let c_0 be the space consisting of the sequences which converge to 0, endowed with the uniform norm given by $\|x\| := \sup_{n\geq 1} |x_n|$. Then c_0 is a separable Banach space (in fact an Asplund space). The closed convex cone c_0^+, *i.e.*, the non-negative sequences, has no interior, and c_0^+ is not Aronszajn's null. Define $f\colon c_0 \to \mathbb{R}$ by $f(x) = \sqrt{\|x^+\|}$. Then f is c_0^+-increasing. However, f is not Gâteaux differentiable on $-c_0^+$. Indeed, for $x \in -c_0^+$, $f(x) = 0$. If x has $x_n = 0$ for some n, then for $t > 0$,

$$\frac{f(x + te_n) - f(x)}{t} \geq \frac{\sqrt{t}}{t} \to \infty \quad \text{as } t \downarrow 0.$$

If x has $x_n < 0$ for all n, take $t_n = 2\sqrt{-x_n}$, and $h = (\sqrt{-x_n})$, we have $t_n \downarrow 0$ and

$$\frac{f(x + t_n h) - f(x)}{t_n} \geq \frac{\sqrt{x_n + t_n h_n}}{t_n} = \frac{1}{2} \quad \text{for all } n.$$

Therefore f is not Gâteaux differentiable at x. However, f is generically Fréchet differentiable on $c_0 \setminus (-c_0^+)$ because $\|x^+\|$ is convex.

More pathological examples concerning K-monotone functions when K has empty interior can be found in [2, 3].

Example 11 (Singular functions on separable spaces) Assume that X is a separable Banach space and $K \subset X$ is a closed convex cone with $K \cap -K = \{0\}$ and $\operatorname{int}(K) \neq \varnothing$. Then there exists a continuous $g\colon X \to \mathbb{R}$ such that g is strictly K-increasing and has Gâteaux derivative $\nabla g = 0$ throughout X except at points of a Aronszajn null set.

To see this, we take $f\colon \mathbb{R} \to \mathbb{R}$, strictly increasing and continuous, such that $f'(x) = 0$ on $\mathbb{R}$ a.e. When X is separable, there exists $x^* \in K^+$ such that $\langle x^*, k\rangle > 0$ for every $k \in K\setminus\{0\}$. Indeed, because the dual ball $\mathbb{B}_{X^*}(0)$ is weak* separable, we may choose a countable weak* dense set $\{x_n^*\}_{n=1}^{\infty}$ in $K^+ \cap \mathbb{B}_{X^*}(0)$, and let $x^* := \sum_{n=1}^{\infty} \frac{x_n^*}{2^n}$. If $\langle x^*, x\rangle = 0$ for some $x \in K$, then $\langle x_n^*, x\rangle = 0$ for each $n \in \mathbb{N}$, and so $\langle y^*, x\rangle = 0$ for every $y^* \in K^+$. Thus $x \in K \cap (-K)$, and so $x = 0$.

Define $g\colon X \to \mathbb{R}$ by $g(x) := f(\langle x^*, x\rangle)$. Because f is strictly increasing, we have g strictly K-increasing on X. For each $k \in K$ and $x \in X$, the function $h\colon \mathbb{R} \to \mathbb{R}$ given by

$$h(t) := g(x + tk) = f\big(\langle x^*, x\rangle + t\langle x^*, k\rangle\big),$$

is strictly increasing and $h'(t) = 0$ a.e. on $\mathbb{R}$. By Theorem 9, g is Gâteaux differentiable on X with $\nabla g(x) = 0$ except for an Aronszajn null set.

4 Continuity, Measurability and Extendibility

The following result improves Theorem 6 [4] in which the authors showed that a cone-monotone function $f\colon \mathbb{R}^n \to \mathbb{R}$ is continuous almost everywhere.

Proposition 12 *Let X be a Banach space. Assume that the closed convex cone $K \subset X$ has $\operatorname{int}(K) \neq \varnothing$ and $f\colon X \to \mathbb{R}$ is K-monotone. Then*

$$D := \{x \in X \mid f \text{ is discontinuous at } x\},$$

is σ-directionally porous in X. When X is separable, D is Aronszajn null.

Proof Without loss of any generality, we assume that f is K-increasing. We have $D = \{x \in X \mid \underline{f}(x) < \overline{f}(x)\}$. Write

$$S_1 := \{x \in X \mid \underline{f}(x) < f(x)\}, \text{ and } S_2 := \{x \in X \mid f(x) < \overline{f}(x)\}.$$

We claim S_2 is σ-directionally porous. The proof of the σ-directional porosity of S_1 is similar. Write $S_2 = \bigcup_{p\in\mathbb{Q}} D_p$ where

$$D_p := \{x \in X \mid f(x) < p < \overline{f}(x)\}.$$

For $x \in D_p$, $f(x) < p$. For $y \in x - \operatorname{int}(K)$, $f(y) \leq f(x) < p$. For every $y \in x - \operatorname{int}(K)$, $\overline{f}(y) \leq f(x) < p$, so $y \notin D_p$. That is,

$$[x - \operatorname{int}(K)] \cap D_p = \varnothing.$$

Since this holds for each $x \in D_p$, D_p is directionally porous, and so S_2 is σ-directionally porous. ■

On the other hand, Proposition 12 fails if $\operatorname{int}(K) = \varnothing$:

Example 13 For the Hilbert space l_2 with norm $\|x\| := \sqrt{\sum_{n=1}^{\infty} x_n^2}$, the closed convex cone l_2^+, *i.e.*, the set of nonnegative sequences, has no interior. We define

$$f(x) := \begin{cases} 1 & \text{if } x \in l_2 \text{ has infinitely many positive terms,} \\ 0 & \text{otherwise.} \end{cases}$$

Then f is l_2^+-increasing. For $x = (x_1, x_2, \dots) \in l_2$, choose N large such that

$$\sqrt{\sum_{i=N}^{\infty} x_n^2} < \epsilon/2.$$

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Consider

$$y := (x_1, \dots, x_N, \frac{\epsilon}{2^2}, \frac{\epsilon}{2^3}, \dots) \in l_2,$$
$$z := (x_1, \dots, x_N, -\frac{\epsilon}{2^2}, -\frac{\epsilon}{2^3}, \dots) \in l_2.$$

Then $\|y - x\| < \epsilon$ and $\|z - x\| < \epsilon$. It follows that $f(y) = 1$ and $f(z) = 0$. Since $\epsilon > 0$ is arbitrary, we conclude that f is not continuous at x. Thus, f is *nowhere* continuous on X.

Another preparatory decomposition result is in order.

Proposition 14 *Let X be a Banach space and $K \subset X$ be a closed convex cone with $\operatorname{int}(K) \neq \varnothing$. Assume that $f\colon X \to \mathbb{R}$ is K-monotone. Then for every $r \in \mathbb{R}$, the level set $S_r := \{x \mid f(x) \leq r\}$, can be written as $O \cup T$ where O is open and T is directionally porous. Hence f is Gaussian measurable when X is separable .*

Proof Without loss of generality, we assume that f is K-increasing. Write

$$\partial S_r = S_r \setminus \operatorname{int}(S_r).$$

We show that ∂S_r is directionally porous. For $x \in \partial S_r$, we have $x - \operatorname{int}(K)$ open. Since f is K-increasing, we know $f(y) \leq f(x) \leq r$ for $y \in [x - \operatorname{int}(K)]$, so $x - \operatorname{int}(K) \subset \operatorname{int}(S_r)$. This shows

$$[x - \operatorname{int}(K)] \cap \partial S_r = \varnothing,$$

so ∂S_r is directionally porous. When X is separable, a directionally porous set is Gaussian null, so S_r is Gaussian measurable. Since this holds for every r, f is Gaussian measurable on X. ■

We now discuss the extendibility of K-monotone functions. As usual, for a closed convex cone $K \subset X$, its *indicator function* is defined by:

$$I_K(x) := \begin{cases} 0 & \text{if } x \in K, \\ +\infty & \text{otherwise.} \end{cases}$$

Note that I_K is K-decreasing.

Proposition 15 *Let X be a Banach space and $K \subset X$ be a closed convex cone. Assume that $f\colon A \subset X \to \mathbb{R} \cup \{\pm\infty\}$ is K-increasing. We define*

$$g(x) := \inf\{f(y) + I_K(y - x) : y \in \operatorname{dom}(f)\} = \inf\{f(y) : y \geq_K x,\ y \in \operatorname{dom} f\},$$

$$h(x) := \sup\{f(y) - I_K(x - y) : y \in \operatorname{dom}(f)\} = \sup\{f(y) : y \leq_K x,\ y \in \operatorname{dom} f\}.$$

Then g and h satisfy

(i) *g and h are K-increasing on X and $g|_{\operatorname{dom}(f)} = f = h|_{\operatorname{dom}(f)}$;*

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(ii) *g is the largest, and h is the smallest, K-monotone extension of f;*
(iii) *if f is quasiconvex (resp. convex), then g is quasiconvex (resp. convex).*

Proof (i) and (ii): Let $k \in K$. Since I_K is K-decreasing, we have

$$\begin{aligned} g(x+k) &= \inf\{f(y) + I_K(y-(x+k)) \mid y \in \operatorname{dom} f\} \\ &\geq \inf\{f(y) + I_K(y-x) \mid y \in \operatorname{dom} f\} = g(x). \end{aligned}$$

Now for $x \in \operatorname{dom} f$, we have $g(x) = f(x)$. By definition, for $x \in \operatorname{dom} f$, we have $g(x) \leq f(x)$. But for $y - x \in K$, $f(y) \geq f(x)$ so $f(y) + I_K(y-x) \geq f(x)$. This gives $g(x) \geq f(x)$. Hence $g|_{\operatorname{dom} f} = f$. Assume l is an extension of f and K-increasing. We show that $g \geq l$. Since l is K-increasing, we have

$$l(x) \leq l(y) + I_K(y-x), \quad \text{so,}$$
$$l(x) \leq f(y) + I_K(y-x) \quad \text{for } y \in \operatorname{dom} f.$$

By definition, we have $l(x) \leq g(x)$. Hence g is the largest K-increasing extension of f. The claims for h are verified similarly.

(iii): Let f be quasiconvex. We show that g is quasiconvex. Assume $g(x), g(z) \leq \alpha$. For $\epsilon > 0$, there exist $\hat{x}$ and $\hat{z}$ such that

$$f(\hat{x}) + I_K(\hat{x} - x) \leq g(x) + \epsilon, \quad \text{and} \quad f(\hat{z}) + I_K(\hat{z} - z) \leq g(z) + \epsilon.$$

This gives $\hat{x} \geq_K x$ and $\hat{z} \geq_K z$. For $0 \leq \lambda \leq 1$ we have $\lambda\hat{x} + (1-\lambda)\hat{z} \geq_K \lambda x + (1-\lambda)z$, and $f(\lambda\hat{x} + (1-\lambda)\hat{z})) \leq \max\{f(\hat{x}), f(\hat{z})\}$. Then

$$g(\lambda x + (1-\lambda)z) \leq f(\lambda\hat{x} + (1-\lambda)\hat{z}) \leq \max\{g(z), g(x)\} + \epsilon,$$

so $g(\lambda x + (1-\lambda)z) \leq \alpha + \epsilon$. Since ϵ is arbitrary, we have $g(\lambda x + (1-\lambda)z) \leq \alpha$. Hence g is quasiconvex. Similarly, one can prove that g is convex when f is convex. ■

5 Upper Hull, Lower Hull and Robust Continuity

When $f \colon X \to \mathbb{R}$ is K-monotone with $\operatorname{int}(K) \neq \varnothing$, the continuity and differentiability of f is closely related to the continuity and differentiability of its upper or lower hull. The following is a generalization of Chabrillac and Crouzeix [4] from $\mathbb{R}^n$ to general Banach spaces.

Proposition 16 *Let X be a Banach space and $K \subset X$ be a closed convex cone with $\operatorname{int}(K) \neq \varnothing$. Suppose that $f \colon X \to \mathbb{R} \cup \{\pm\infty\}$ is K-monotone and $f(a)$ is finite. Then*

(i) *f is continuous at a if and only if $\underline{f}$ (resp. $\overline{f}$) is continuous at a. In particular, $\overline{f}(a) = f(a)$ (resp. $\underline{f}(a) = f(a)$) whenever $\overline{f}$ (resp. $\underline{f}$) is continuous at a.*
(ii) *f is continuous at a if and only if for some $e \in \operatorname{int}(K)$ the function $\phi \colon \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ given by $\phi(t) := f(a + te)$ is continuous at $t = 0$.*

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(iii) *f is Gâteaux differentiable at a if and only if $\underline{f}$ (resp. $\overline{f}$) is Gâteaux differentiable at a.*

(iv) *If f is Gâteaux differentiable at a, then it is also Hadamard differentiable (*i.e., *uniformly on norm-compact sets) at a.*

Proof Without loss of any generality, we assume that f is K-increasing.

(i): Fix $a \in X$. Let $e \in \operatorname{int}(K)$. For $\epsilon > 0$, the set $(a - \epsilon e, a + \epsilon e)$ is a neighborhood of a. We have

$$\overline{f}(a - \epsilon e) \leq f(a) \leq \overline{f}(a + \epsilon e), \quad \text{and} \quad \overline{f}(a - \epsilon e) \leq f(y) \leq \overline{f}(a + \epsilon e),$$

for $y \in (a - \epsilon e, a + \epsilon e)$. Hence, if $\overline{f}$ is continuous at a, then f is continuous at a, so $\overline{f}(a) = f(a)$. Conversely, assume f is continuous at a. For $\epsilon > 0$ and $e \in \operatorname{int}(K)$, we have

$$f(a - \epsilon e) \leq \overline{f}(a) \leq f(a + \epsilon e), \quad \text{and} \quad f(a - \epsilon e) \leq \overline{f}(y) \leq f(a + \epsilon e),$$

for $y \in (a - \epsilon e, a + \epsilon e)$. Hence $\overline{f}$ is continuous at a. The arguments for $\underline{f}$ are similar.

(ii): Assume ϕ is continuous at $t = 0$. We have

$$f(a - \epsilon e) - f(a) \leq f(y) - f(a) \leq f(a + \epsilon e) - f(a),$$

for $y \in (a - \epsilon e, a + \epsilon e)$. Since the latter is a neighborhood of a and $\epsilon > 0$ is arbitrary, we conclude that f is continuous at a. The other direction is obvious.

(iii): Assume $\overline{f}$ is Gâteaux differentiable at a. By (ii), $\overline{f}$ is continuous at a, so $\overline{f}(a) = f(a)$ by (i). Fix $u \in X$. For $\epsilon, t > 0, e \in \operatorname{int}(K)$, since $a + tu - t\epsilon e \in \operatorname{int}(a + tu - K)$ we have

$$\frac{\overline{f}(a + tu - t\epsilon e) - \overline{f}(a)}{t} \leq \frac{f(a + tu) - f(a)}{t} \leq \frac{\overline{f}(a + tu) - \overline{f}(a)}{t}.$$

Let $t \to 0$. We obtain

$$\langle \nabla \overline{f}(a), u - \epsilon e \rangle \leq f_{+}(a; u) \leq f^{+}(a, u) \leq \langle \nabla \overline{f}(a), u \rangle.$$

Let $\epsilon \downarrow 0$. We have $f'_{+}(a; u) = \langle \nabla \overline{f}(a), u \rangle$.

Now assume that f is Gâteaux differentiable at a. By (ii), f is continuous at a, so $\overline{f}(a) = f(a)$. Fix $u \in X$. Take $\epsilon, t > 0$ and $e \in \operatorname{int}(K)$. We have

$$\frac{f(a + tu) - f(a)}{t} \leq \frac{\overline{f}(a + tu) - \overline{f}(a)}{t} \leq \frac{f(a + tu + t\epsilon e) - f(a)}{t}.$$

Let $t \downarrow 0$. We have

$$\left\langle \nabla f(a), u \right\rangle \leq \overline{f}_{+}(a; u) \leq \overline{f}^{+}(a; u) \leq \left\langle \nabla f(a), u + \epsilon e \right\rangle.$$

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Let $\epsilon \to 0$. We have

$$\overline{f}'_{+}(a; u) = \langle \nabla f(a), u \rangle.$$

Thus $\overline{f}$ is Gâteaux differentiable at a. The arguments for $\underline{f}$ is similar.

(iv): Recall that f is Hadamard differentiable at a if, for each $v \in X$, whenever $t_n \downarrow 0$ and $v_n \to v$ in norm, we have

$$\lim_{t_n \downarrow 0, v_n \to v} \frac{f(a + t_n v_n) - f(a) - t_n \langle f'(a), v \rangle}{t_n} = 0.$$

Assume that f is Gâteaux differentiable at a. Choose $\epsilon > 0$ and $e \in \operatorname{int}(K)$. For n sufficiently large, we have $\pm(v_n - v) + \epsilon e \in \operatorname{int}(K)$, and so

$$\frac{f(a + t_n(v - \epsilon e)) - f(a)}{t_n} \leq \frac{f(a + t_n v_n) - f(a)}{t_n} \leq \frac{f(a + t_n(v + \epsilon e)) - f(a)}{t_n}.$$

When $n \to \infty$, we obtain

$$\limsup_{t_n \downarrow 0, v_n \to v} \frac{f(a + t_n v_n) - f(a)}{t_n} \leq \langle \nabla f(a), v + \epsilon e \rangle,$$
$$\liminf_{t_n \downarrow 0, v_n \to v} \frac{f(a + t_n v_n) - f(a)}{t_n} \geq \langle \nabla f(a), v - \epsilon e \rangle.$$

Letting $\epsilon \to 0$ to obtain

$$\lim_{t_n \downarrow 0, v_n \to v} \frac{f(a + t_n v_n) - f(a)}{t_n} = \langle \nabla f(a), v \rangle. \qquad \blacksquare$$

An upper semicontinuous function $k \colon X \to \mathbb{R}$ is called *topologically robust upper semicontinuous* on X if $k(x) = \limsup_{y \in D, y \to x} k(y)$ for every $x \in X$, where D is the set of points at which k is continuous.

Proposition 17 *Let X be a Banach space and $K \subset X$ be a closed convex cone with $\operatorname{int}(K) \neq \varnothing$. Suppose that $f \colon X \to \mathbb{R}$ is K-monotone. Then $\overline{f}$ and $\underline{f}$ are K-monotone and $\overline{f}$ is topologically robust upper semicontinuous.*

Proof Without loss of generality, we assume f is K-increasing. Let $k \in \operatorname{int}(K)$ and $x \in X$. The set $x + k - \operatorname{int}(K)$ is a neighborhood of x, and $f(x + k) \geq f(y)$ for every $y \in (x + k - \operatorname{int}(K))$. It follows that

$$\overline{f}(x + k) \geq f(x + k) \geq \overline{f}(x),$$

so $\overline{f}(x + k) \geq \overline{f}(x)$. For arbitrary $k \in K$, we take $k_n \in \operatorname{int}(K)$ such that $k_n \to k$. Then

$$\overline{f}(x + k) \geq \limsup_{n \to \infty} \overline{f}(x + k_n) \geq \overline{f}(x).$$

Hence $\overline{f}$ is K-increasing. The proof for $\underline{f}$ being K-increasing is similar.

For $x \in X$, $x+K$ has non-empty interior. Since $\overline{f}$ is u.s.c., there exists $y \in \operatorname{int}(x+K)$ arbitrarily near by x such that $\overline{f}(y) \geq \overline{f}(x)$ and $\overline{f}$ is continuous at y. Then

$$\overline{f}(x) \leq \limsup_{y \in D, y \to x} \overline{f}(y) \leq \limsup_{y \to x} \overline{f}(y) = \overline{f}(x).$$

Hence $\overline{f}$ is topologically robust u.s.c. ■

Proposition 16(iii), (iv), Proposition 17, and Theorem 9 conspire to show that:

Theorem 18 *Let X be a separable Banach space, $K \subset X$ be a closed convex cone with $\operatorname{int}(K) \neq \varnothing$. Suppose that $f\colon X \to \mathbb{R}$ is K-monotone. Then f is Hadamard differentiable on X except perhaps at points of an Aronszajn null set.*

6 Continuity and Differentiability of Quasiconvex Functions

In this section, we apply earlier results to quasiconvex functions. For a convex set C, we denote by $\dim(C)$ the dimension of the affine hull of C. Let $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be a quasiconvex function. Following Crouzeix [5], we define $\overline{\lambda}$ as the value such that

$$\dim(S_\mu(f)) < n \leq \dim(S_\lambda(f)), \text{ whenever } \mu < \overline{\lambda} < \lambda.$$

Theorem 19 *Assume $f\colon \mathbb{R}^n \to \mathbb{R}$ is l.s.c. and quasiconvex. Then*

(i) *f is continuous except for a σ-porous set;*

(ii) *f is Fréchet differentiable except for a Lebesgue null set.*

Proof Consider the sets

$$A := \{x \in \mathbb{R}^n \mid f(x) < \overline{\lambda}\},$$
$$B := \{x \in \mathbb{R}^n \mid f(x) = \overline{\lambda}\},$$
$$C := \{x \in \mathbb{R}^n \mid f(x) \leq \overline{\lambda}\}.$$

Now $A = \bigcup_{n=1}^{\infty} A_n$ where

$$A_n := \left\{x \in \mathbb{R}^n \mid f(x) \leq \overline{\lambda} - \frac{1}{n}\right\}.$$

Because f is l.s.c. and quasiconvex, A_n is closed convex set with empty interior. By [7, Theorem 2], A_n is porous, so A is σ-porous. For the boundary of B, denoted by ∂B, we note that $\partial B \subset (\partial A \cup \partial C)$. Because the distance function associated with a convex set is not differentiable at any boundary point, by [7, Theorem 1], ∂A and ∂C are σ-porous. On C, the possible discontinuity points and the possible non-Fréchet

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differentiability points of f are a subset of $A \cup \partial A \cup \partial C$, which is σ-porous. For $x \in \mathbb{R}^n \setminus C$, $f(x) > \overline{\lambda}$, when $\overline{\lambda} < \lambda < f(x)$, the set $S_\lambda(f)$ has non-empty interior. By Proposition 2, there exists a neighborhood V containing x such that f is monotone with respect to a convex cone with non-empty interior.

For (i), on $\mathbb{R}^n \setminus C$, we apply Proposition 12. For (ii), on $\mathbb{R}^n \setminus C$, we apply Theorem 18. ■

While (ii) is given in [5], (i) appears to be new.

7 Porosity Results for the Class of K-Monotone Functions

Our first result concerns nowhere K-monotone functions in $C(A)$, the continuous functions defined on A. Here A is a nonempty open subset of separable Banach space X. On $C(A)$ we define $\|f - g\|_\infty := \sup_{x \in A} |f(x) - g(x)|$,

$$\rho(f, g) := \min\{1, \|f - g\|_\infty\} \quad \text{for } f, g \in C(A).$$

As usual, $(C(A), \rho)$ is a complete metric space.

Theorem 20 *Let X be a separable space. Assume that $K \subset X$ is a convex cone with $\operatorname{int}(K) \neq \varnothing$ and $K \cap (-K) = \{0\}$. In $C(A)$, the set*

$$\{f : f \in C(A) \text{ is not } K\text{-monotone on any open subset of } A\},$$

has a σ-porous complement in $C(A)$.

Proof Choose $l^+ \in X^*$ such that $l^+(k) > 0$ for every $k \in K \setminus \{0\}$ (see Example 11). Fix $k \in K$ such that $0 < l^+(k) < 1/4$. Define

$$I_O := \{f \in C(A) : f \text{ is } K\text{-increasing on open set } O\}.$$

We show that I_O is porous in $C(A)$. For this, we need $\alpha > 0$ such that for every $1 > r > 0$, $f \in C(A)$, there exists $h_2 \in C(A)$ such that

$$\mathbb{B}(h_2, \alpha r) \subset \mathbb{B}(f, r) \setminus I_O.$$

For given $f \in I_O$, choose $\delta > 0$ and $x_0 \in O$ such that $x_0 + \delta k \in O$ and $f(x_0 + \delta k) - f(x_0) < r/8$. Define

$$h_1(x) := \min\{f(x_0) - \frac{r}{4} - \frac{r}{2\delta} l^+(x - x_0), f(x)\},$$
$$h_2(x) := \max\left\{h_1(x), f(x) - \frac{r}{2}\right\}.$$

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We have $\|h_2 - f\|_\infty \le r/2 < 1$, so $\rho(h_2, f) \le r/2$. Since f is K-increasing on O, for $x \in (x_0 + K) \cap O$, we have

$$f(x_0) - \frac{r}{4} - \frac{r}{2\delta} l^+(x - x_0) \le f(x_0) \le f(x), \quad \text{so}$$
$$h_1(x) = f(x_0) - \frac{r}{4} - \frac{r}{2\delta} l^+(x - x_0) \quad \text{for } x \in (x_0 + K) \cap O.$$

It follows that

$$h_1(x_0 + \delta k) = f(x_0) - \frac{r}{4} - \frac{r}{2} l^+(k) > f(x_0) - \frac{3r}{8},$$
$$f(x_0 + \delta k) - \frac{r}{2} = f(x_0 + \delta k) - f(x_0) + f(x_0) - \frac{r}{2} < \frac{r}{8} + f(x_0) - \frac{r}{2} = f(x_0) - \frac{3r}{8}.$$

This shows that

$$h_2(x_0 + \delta k) = f(x_0) - \frac{r}{4} - \frac{r}{2} l^+(k), \quad h_2(x_0) = f(x_0) - \frac{r}{4}.$$

Now for $\|g - h_2\|_\infty < \alpha r$, we have

$$\begin{aligned} g(x_0 + \delta k) - g(x_0) &= (g - h_2)(x_0 + \delta k) - (g - h_2)(x_0) + h_2(x_0 + \delta k) - h_2(x_0) \\ &\le 2\alpha r - \frac{r}{2} l^+(k) = r\Big(2\alpha - \frac{l^+(k)}{2}\Big). \end{aligned}$$

When $0 < \alpha < l^+(k)/4$, we have $g(x_0 + \delta k) - g(x_0) < 0$, so $g \notin I_O$. Moreover, when $\rho(g, h_2) < \alpha r$ we have $\|g - h_2\|_\infty < \alpha r$. Then

$$\|g - f\|_\infty \le \|g - h_2\|_\infty + \|h_2 - f\|_\infty \le \alpha r + \frac{r}{2} < r.$$

This shows $\mathbb{B}(h_2, \alpha r) \subset \mathbb{B}(f, r) \setminus I_O$. Hence, I_O is porous in $C(A)$. When X is separable, take a countable dense set $\{x_i\} \subset A$ and rational numbers $\{r_i\}$ dense in $(0, \infty)$. Define

$$S^+_{mn} := \{f \in C(A) : f \text{ is } K\text{-increasing on } \mathbb{B}(x_n, r_m)\},$$
$$S^-_{mn} := \{f \in C(A) : f \text{ is } K\text{-decreasing on } \mathbb{B}(x_n, r_m)\}.$$

Then $S^+ = \bigcup S^+_{mn}$ collects all functions which are K-increasing on some open subset of A, and S^+ is σ-porous. Similarly, $S^- = \bigcup S^-_{mn}$ collects all functions which are K-decreasing on some open subsets of A, S^- is σ-porous. Each $f \in [C(A) \setminus (S^+ \cup S^-)]$ is nowhere K-monotone througout A. ■

Next, we consider strictly K-increasing functions in $I_K(A)$ where

$$I_K(A) := \{f \in C(A) \mid f \text{ is } K\text{-increasing on } A\},$$

is happily a complete subspace of $(C(A), \rho)$.

The following result is essentially due to Rubinov and Zaslavski [9]. Here we take the opportunity to improve their proof by using the metric ρ on $I_K(A)$.

Theorem 21 *Let X be a separable space, $K \subset X$ be a closed convex cone with $K \cap -K = \{0\}$ and $\operatorname{int}(K) \neq \varnothing$. Then the set*

$$\{f \in I_K(A) : f \text{ is strictly } K\text{-increasing on } A\},$$

has a σ-directionally porous complement in $(I_K(A), \rho)$.

Proof By assumption, we may choose $l \in X^*$ such that $l(k) > 0$ for every $k \in K \setminus \{0\}$. Therefore, l is strictly K-increasing on A. The function $f_0 \colon X \to \mathbb{R}$ defined by

$$f_0(x) := \frac{2}{\pi} \arctan(l(x)) \text{ is strictly } K\text{-increasing on } A,$$

and $\|f_0\|_\infty \leq 1$. Define

$$A_n := \left\{ (x, y) \mid x, y \in A, y - x \in K, f_0(y) - f_0(x) \geq \frac{1}{n} \right\}, \text{ and}$$

$$\mathcal{F}_n := \Big\{ f \in I_K(A) : \text{ there exists } \delta > 0 \text{ such that } f(y) > f(x) + \delta \\ \text{for every } (x, y) \in A_n. \Big\}$$

We claim $I_K(A) \setminus \mathcal{F}_n$ is directionally porous in $I_K(A)$. Let $f \in I_K(A)$. For $0 < r < 1$, we set $h := f + \frac{r}{8} f_0$. Then

$$\rho(h, f) = \min\left\{ \left\| \frac{r f_0}{8} \right\|_\infty, 1 \right\} \leq r/8.$$

For $g \in I_K(A)$ and $\rho(g, h) \leq \alpha r$, we have

$$\rho(g, f) \leq \alpha r + \frac{r}{8} \leq r,$$

by requiring $\alpha < 7/8$. Whenever $(x, y) \in A_n$, we have

$$\begin{aligned} g(y) - g(x) &= (g - h)(y) - (g - h)(x) + h(y) - h(x) \\ &\geq -2\alpha r + f(y) - f(x) + \frac{r}{8}(f_0(y) - f_0(x)) \\ &\geq -2\alpha r + \frac{r}{8n} = r(\frac{1}{8n} - 2\alpha). \end{aligned}$$

On setting $\alpha = \frac{1}{32n}$, we have

$$g(y) - g(x) > \frac{r}{16n} \quad \text{whenever } (x, y) \in A_n.$$

Therefore

$$\mathbb{B}(h, r/(32n)) \subset \mathbb{B}(f, r) \cap \mathcal{F}_n.$$

Since this holds for every $f \in I_K(A)$ and $0 < r < 1$, we conclude $I_K(A) \setminus \mathcal{F}_n$ is directionally porous.

Now define $\mathcal{F} := \bigcap_{n=1}^{\infty} \mathcal{F}_n$. Then $\mathcal{F}$ has a σ-directional porous complement. Let $f \in \mathcal{F}$. Whenever $y \geq_K x$ and $y \neq x$, we have $f_0(y) - f_0(x) > 1/n$ for some n. Since $f \in \mathcal{F}_n$, we have

$$f(y) - f(x) > \delta \quad \text{for some } \delta > 0,$$

in particular $f(y) > f(x)$. Hence f is strictly K-increasing on A. ■

8 Open Questions

To stimulate further study on K-monotone functions, we finish with two open questions.

The proof of Theorem 9 uses the separability of Banach space X and K having non-empty interior. Preiss [8] has shown that Lipschitz functions $f\colon X \to \mathbb{R}$ on β-smooth Banach spaces X are densely β-differentiable. Lipschitz functions on Banach spaces are K-monotone with K having non-empty interior by Proposition 1. This leaves us:

Conjecture 1 *Let X be a Banach space with an equivalent β-smooth renorm and $K \subset X$ a closed convex cone with* $\operatorname{int}(K) \neq \varnothing$. *Suppose that $f\colon X \to \mathbb{R}$ is continuous and K-monotone. Then f is β-differentiable on X densely.*

And we should greatly appreciate an answer to:

Conjecture 2 *There is a continuous $f\colon l_2$ (or merely on c_0) $\to \mathbb{R}$ such that f is l_2^+ (resp. c_0^+)-increasing but f is* nowhere *Gâteaux differentiable.*

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