WEIGHTED QUADRATIC PARTITIONS OVER A FINITE FIELD

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Introduction. Using some known results on Gauss sums in a finite field, it is shown that the sum (1.3) defined below can either be evaluated explicitly or expressed in terms of a Kloosterman sum. The same result applies to the more general sum $S(\alpha, \lambda, Q)$ defined in (5.1). The latter sum also satisfies the reciprocity formula (5.5). Some related sums are discussed in §§6, 7.

1. The weighted sum S. Let $q = p^n$, where p is an odd prime. Assuming $\alpha \in GF(q)$, we put

$$e(\alpha) = e^{2\pi i t(\alpha)/p}, \tag{1.1}$$

where

$$t(\alpha) = \alpha + \alpha^p + \ldots + \alpha^{p^{n-1}}.$$

Then as is well known

$$\sum_\beta e(\alpha\beta) = \begin{cases} q & (\alpha = 0), \\ 0 & (\alpha \neq 0). \end{cases} \tag{1.2}$$

By $\sum_\beta$, $\sum_\xi$, etc. will be understood summations over the numbers of $GF(q)$.

Let $\alpha_1, \ldots, \alpha_s$ be non-zero numbers of $GF(q)$ and consider the sum

$$S = \sum_{\alpha_1\xi_1^2+\ldots+\alpha_s\xi_s^2=\alpha} e(2\lambda_1\xi_1 + \ldots + 2\lambda_s\xi_s), \tag{1.3}$$

where α, λ_1 are arbitrary and the summation is extended over all sets $\xi_1, \ldots, \xi_s$ satisfying $\alpha_1\xi_1^2 + \ldots + \alpha_s\xi_s^2 = \alpha$. Using (1.2) we may write

$$\begin{aligned} qS &= \sum_{\xi_1,\ldots,\xi_s} \sum_\beta e\{\beta(\alpha_1\xi_1^2 + \ldots + \alpha_s\xi_s^2 - \alpha) + 2\lambda_1\xi_1 + \ldots + 2\lambda_s\xi_s\} \\ &= \sum_\beta e(-\alpha\beta) \prod_{i=1}^{s} \sum_\xi e(\alpha_1\beta\xi^2 + 2\lambda_i\xi). \end{aligned} \tag{1.4}$$

Now for $\beta \neq 0$ we have

$$\sum_\xi e(\beta\xi^2 + 2\lambda\xi) = \sum_\xi e\left(\beta\left(\xi + \frac{\lambda}{\beta}\right)^2\right) e\left(-\frac{\lambda^2}{\beta}\right) = e\left(-\frac{\lambda^2}{\beta}\right) G(\beta), \tag{1.5}$$

where

$$G(\beta) = \sum_\xi e(\beta\xi^2) \qquad (\beta \neq 0). \tag{1.6}$$

It is known that [**2**, §3]

$$G(\beta) = \psi(\beta)G(1), \quad G^2(1) = q\psi(-1), \tag{1.7}$$

Received March 17, 1952.

where $\psi(\beta) = +1$ or -1 according as β is a square or a non-square in $GF(q)$. We have also

$$\sum_{\beta} \psi(\beta)e(\alpha\beta) = \psi(\alpha)G(1) \tag{1.8}$$

for all α, provided we define $\psi(0) = 0$.

Making use of (1.5) and (1.6), we see that (1.4) becomes

$$qS = \sum_{\xi_1,\ldots,\xi_s} e(2\lambda_1\xi_1 + \ldots + 2\lambda_s\xi_s) + \sum_{\beta\neq 0} e\left(-\alpha\beta - \frac{\omega}{\beta}\right)\prod_{i=1}^{s} G(\alpha_i\beta), \tag{1.9}$$

where for brevity we put

$$\omega = \frac{\lambda_1^2}{\alpha_1} + \ldots + \frac{\lambda_s^2}{\alpha_s}. \tag{1.10}$$

The first sum in the right member of (1.9) vanishes unless all $\lambda_i = 0$. Next using (1.7), the second sum in (1.9) becomes

$$\sum_{\beta\neq 0} e\left(-\alpha\beta - \frac{\omega}{\beta}\right)\psi(\alpha_1\ldots\alpha_s\beta^s)G^s(1). \tag{1.11}$$

2. Kloosterman sums. To evaluate (1.11) we consider separately $s = 2t$, $s = 2t + 1$. For $s = 2t$, (1.11) becomes

$$q^t\psi((-1)^t\alpha_1\ldots\alpha_{2t})\sum_{\beta\neq 0} e\left(-\alpha\beta - \frac{\omega}{\beta}\right); \tag{2.1}$$

for $s = 2t + 1$, we get

$$q^tG(1)\psi((-1)^t\alpha_1\ldots\alpha_{2t+1})\sum_{\beta\neq 0}\psi(\beta)e\left(-\alpha\beta - \frac{\omega}{\beta}\right). \tag{2.2}$$

For $\alpha = 0$, (2.1) and (2.2) are easily evaluated. For $\alpha \neq 0$, we define the Kloosterman sums

$$K(\alpha, \omega) = \sum_{\beta\neq 0} e\left(\alpha\beta + \frac{\omega}{\beta}\right), \quad K(\alpha) = K(\alpha, 1), \tag{2.3}$$

$$L(\alpha, \omega) = \sum_{\beta\neq 0} \psi(\beta)e\left(\alpha\beta + \frac{\omega}{\beta}\right), \quad L(\alpha) = L(\alpha, 1). \tag{2.4}$$

If $\omega = 0$ we have at once

$$K(\alpha, 0) = -1, \quad L(\alpha, 0) = G(\alpha). \tag{2.5}$$

We note also that for $\gamma \neq 0$,

$$K(\alpha\gamma, \omega\gamma^{-1}) = K(\alpha, \omega), \quad L(\alpha\gamma, \omega\gamma^{-1}) = \psi(\gamma)L(\alpha, \omega). \tag{2.6}$$

In particular for $\gamma \neq 0$,

$$K(\alpha, \omega) = K(\alpha\omega), \quad L(\alpha, \omega) = \psi(\omega)L(\alpha\omega). \tag{2.7}$$

We can easily evaluate $L(\alpha)$ (compare [**4**, p. 102]). (For $\alpha = 0$ it is evident from (1.8) that $L(0) = G(1)$.) For $\alpha \neq 0$ we have, using (1.5) and (1.7),

$$\sum_{\xi} e(-\beta\xi^2 + 2\xi) = G(-\beta)e\left(\frac{1}{\beta}\right) = G(-1)\psi(\beta)e\left(\frac{1}{\beta}\right),$$

$$G(-1)\sum_{\beta\neq 0}\psi(\beta)e\left(\alpha\beta + \frac{1}{\beta}\right) = \sum_{\xi}\sum_{\beta} e(\beta(\alpha - \xi^2) + 2\xi).$$

Summing on the right side first with respect to β, we get

$$\text{(2.8)} \qquad \begin{cases} L(\alpha) = 0 & (\psi(\alpha) = -1), \\ L(\alpha^2) = G(1)(e(2\alpha) + e(-2\alpha)). \end{cases}$$

3. Evaluation of S. We now collect these results. There are several cases. Using (1.9), (2.1), (2.2) we have first, for $\alpha = \omega = 0$,

$$\text{(3.1)} \qquad S = \begin{cases} q^{s-1}l + q^{t-1}(q-1)\psi((-1^t)\,\delta) & (s = 2t), \\ q^{s-1}l & (s = 2t+1), \end{cases}$$

where $\delta = \alpha_1 \ldots \alpha_s$ and $l = 1$ if all $\lambda_i = 0$, $l = 0$ otherwise. Next if $\alpha \neq 0$, $\omega = 0$, the sum in (2.1) reduces to -1, while the sum in (2.2) $= G(-\alpha)$. Hence

$$\text{(3.2)} \qquad S = \begin{cases} q^{s-1}l - q^{t-1}\psi((-1)^t\delta) & (s = 2t), \\ q^{s-1}l + q^t\psi((-1)^t\alpha\delta) & (s = 2t+1). \end{cases}$$

If $\alpha = 0$, $\omega \neq 0$, we get

$$\text{(3.3)} \qquad S = \begin{cases} q^{s-1}l - q^{t-1}\psi((-1)^t\delta) & (s = 2t), \\ q^{s-1}l + q^t\psi((-1)^t\omega\delta) & (s = 2t+1). \end{cases}$$

For $\alpha \neq 0$, $\omega \neq 0$, we take first $s = 2t+1$. The sum in (2.2) is evaluated by means of (2.4), (2.6), and (2.8). We find that

$$\text{(3.4)} \qquad S = \begin{cases} q^{s-1}l + q^t\psi((-1)^t\omega\delta)(e(2\gamma) + e(-2\gamma)) & (\alpha\omega = \gamma^2), \\ q^{s-1}l & (\psi(\alpha\omega) = -1). \end{cases}$$

On the other hand, for $s = 2t$ we get

$$\text{(3.5)} \qquad S = q^{s-1}l + q^{t-1}\psi((-1)^t\delta)K(\alpha, \omega).$$

Thus the sum S defined in (1.3) has been evaluated explicitly except in the case $\alpha\omega \neq 0$, $s = 2t$; according to (3.5), the value of S depends on the Kloosterman sum $K(\alpha, \omega)$. We remark that if $\lambda_1 = \ldots = \lambda_s = 0$ (so that $\omega = 0$) then (3.1) and (3.2) reduce to the well-known results [**3**, pp. 47-48] for the number of solutions of the equation $\alpha_1\xi_1^2 + \ldots + \alpha_s\xi_s^2 = \alpha$.

4. Bounds for S. In view of (3.5) it is of some interest to find an estimate for S that will give some information in that case. If we put

$$T(\alpha, \beta) = \sum_{\xi} e(\alpha\xi^2 + 2\beta\xi) \qquad (\alpha \neq 0),$$

https://doi.org/10.4153/CJM-1953-036-x Published online by Cambridge University Press

we have

$$|T(\alpha, \beta)|^2 = \sum_{\xi, \eta} e(\alpha(\xi + \eta)^2 + 2\beta(\xi + \eta))e(-\alpha\eta^2 - 2\beta\eta)$$
$$= \sum_{\xi} e(\alpha\xi^2 + 2\beta\xi) \sum_{\eta} e(2\alpha\xi\eta).$$

By (1.2), the inner sum vanishes unless $\xi = 0$. Hence

$$|T(\alpha, \beta)|^2 = q. \tag{4.1}$$

Returning to (1.4) we have

$$qS = q^s l + \sum_{\beta \neq 0} e(-\alpha\beta) \prod_{i=1}^{s} T(\alpha_i \beta, \lambda_i).$$

Applying (4.1), this becomes

$$|S - q^{s-1} l| \leqslant (q-1) q^{\frac{1}{2}s-1}. \tag{4.2}$$

The estimate (4.2) has been obtained without using any property of the sum $K(\alpha, \omega)$. If the trivial estimate $|K(\alpha, \omega)| \leqslant q - 1$ is used in (3.5), we again get (4.2).

Now it can be shown by elementary methods (compare [**4**, p. 106]) that

$$|K(\alpha, \omega)| < 2q^{3/4}. \tag{4.3}$$

Substituting from (4.3) in (3.5) we find

$$|S - q^{s-1} l| < 2q^{t-\frac{1}{4}} \qquad (s = 2t), \tag{4.4}$$

which is somewhat sharper than (4.2). If in place of (4.3) we use Weil's result [**5**, p. 207]

$$|K(\alpha, \omega)| \leqslant 2q^{\frac{1}{2}},$$

then (4.4) becomes

$$|S - q^{s-1} l| < 2q^{t-\frac{1}{2}} \qquad (s = 2t). \tag{4.5}$$

5. Generalization and reciprocity formula. The results of §3 can be stated in more general terms if in place of (1.3) we consider the sum

$$S = S(\alpha, \lambda, Q) = \sum_{Q(\xi) = \alpha} e(2\lambda_1\xi_1 + \ldots + 2\lambda_s\xi_s), \tag{5.1}$$

where Q denotes a quadratic form

$$Q(u) = \sum_{1}^{s} \alpha_{ij} u_i u_j \qquad (\alpha_{ij} \in GF(q), \quad \delta = |\alpha_{ij}| \neq 0),$$

and the summation in (5.1) is over all ξ_i such that $Q(\xi_1, \ldots, \xi_s) = \alpha$. Since a quadratic form with coefficients in $GF(q)$ can be reduced to diagonal form by a linear transformation, it follows that the sum (5.1) can be reduced to the form (1.3). Thus the λ's undergo a linear transformation; however, the number l occurring in the formulae of §§3, 4 will have the same meaning as before, namely, $l = 1$ if all λ's vanish, $l = 0$ otherwise.

To compute the number ω in the general case we recall [**1**, p. 140, Theorem 2] that a quadratic form

$$f = \sum_1^{s+1} a_{ij} x_i x_j$$

in $s + 1$ variables can be transformed into

$$\sum_1^{s} a_{ij} x'_i x'_j + \frac{\Delta}{\delta} x'^2_{s+1} \qquad (x'_{s+1} = x_{s+1}),$$

where Δ is the discriminant of f and δ is the co-factor of $a_{s+1,\,s+1}$. Applying this result to

$$Q(\xi_1, \ldots, \xi_s) + 2(\lambda_1 \xi_1 + \ldots + \lambda_s \xi_s),$$

we evidently get

$$Q(\xi'_1, \ldots, \xi'_s) + \frac{\Delta}{\delta},$$

where it is clear that δ is the discriminant of Q and

$$\Delta = \begin{vmatrix} \alpha_{11} \ldots \alpha_{1s} & \lambda_1 \\ \cdots\cdots & \\ \alpha_{s1} \ldots \alpha_{ss} & \lambda_s \\ \lambda_1 \ \ldots \ \lambda_s & 0 \end{vmatrix}.$$

Consequently

$$\omega = Q'(\lambda_1, \ldots, \lambda_s), \tag{5.2}$$

where $Q'(u)$ denotes the quadratic form inverse to $Q(u)$. The results of §3 have been written in terms of δ, the discriminant in the quadratic case. We see that all the results of §§3, 4 can now be carried over to the general case and need not be restated.

The following remark may be of interest. Let $\lambda_1, \ldots, \lambda_s$ be assigned and define $\lambda'_1, \ldots, \lambda'_s$ by means of

$$\lambda_i = \sum_{j=1}^{s} \alpha_{ij} \lambda'_j. \tag{5.3}$$

By a well-known theorem, the linear transformation (5.3) carries Q into Q', that is,

$$Q(\lambda') = Q'(\lambda). \tag{5.4}$$

Now we have also

https://doi.org/10.4153/CJM-1953-036-x Published online by Cambridge University Press

$$qS(\alpha, \lambda, Q) = q^s l + \sum_{\beta \neq 0} e\left(-\alpha\beta - \frac{1}{\beta} Q'(\lambda)\right) \psi(\delta\beta^s) G^s(1),$$

and for the inverse form

$$qS(\alpha, \lambda', Q') = q^s l + \sum_{\beta \neq 0} e\left(-\alpha\beta - \frac{1}{\beta} Q(\lambda')\right) \psi(\delta^{-1}\beta^s) G^s(1).$$

Therefore, by (5.4) we have the following reciprocity formula:

$$S(\alpha, \lambda, Q) = S(\alpha, \lambda', Q'). \tag{5.5}$$

6. **The sum S_1.** A word may be added about the sum

$$S_1 = \sum_{\alpha_1\xi_1 + \ldots + \alpha_s\xi_s = \alpha} e(\lambda_1\xi_1^2 + \ldots + \lambda_s\xi_s^2). \tag{6.1}$$

Clearly we have

$$\begin{aligned} qS_1 &= \sum_{\xi_1, \ldots, \xi_s} \sum_{\beta} e(2\beta(\alpha_1\xi_1 + \ldots + \alpha_s\xi_s - \alpha) e(\lambda_1\xi_1^2 + \ldots + \lambda_s\xi_s^2) \\ &= \sum_{\beta} e(-2\alpha\beta) \prod_{i=1}^{s} \sum_{\xi} e(\lambda_i\xi^2 + 2\beta\alpha_i\xi). \end{aligned} \tag{6.2}$$

For simplicity we assume that no $\lambda_i = 0$. Then

$$\begin{aligned} \sum_{\xi} e(\lambda_i\xi^2 + 2\beta\alpha_i\xi) &= \sum_{\xi} e\left(\lambda_i\left(\xi + \frac{\beta\alpha_i}{\lambda_i}\right)^2\right) e\left(-\frac{\beta^2\alpha_i^2}{\lambda_i}\right) \\ &= G(\lambda_i) e(-\beta^2\alpha_i^2/\lambda_i). \end{aligned}$$

Substitution in (6.2) now leads to

$$qS_1 = \psi(\lambda) G^s(1) \sum_{\beta} e(-2\alpha\beta - \beta^2\mu), \tag{6.3}$$

where

$$\lambda = \lambda_1 \ldots \lambda_s, \quad \mu = \frac{\alpha_1^2}{\lambda_1} + \ldots + \frac{\alpha_s^2}{\lambda_s}. \tag{6.4}$$

If $\mu = 0$, (6.3) becomes

$$S_1 = \begin{cases} 0 & (\alpha \neq 0), \\ q^{s-1}\psi(\lambda) G^s(1) & (\alpha = 0). \end{cases} \tag{6.5}$$

For $\mu \neq 0$,

$$\sum_{\beta} e(-2\alpha\beta - \beta^2\mu) = e(\alpha^2/\mu) \sum_{\beta} e\left(-\mu\left(\beta + \frac{\alpha}{\mu}\right)^2\right),$$

and therefore,

$$S_1 = q^{-1}\psi(-\lambda\mu) G^{s+1}(1) e(\alpha^2/\mu) \qquad (\mu \neq 0). \tag{6.6}$$

By means of (6.5) and (6.6), S_1 is determined in all cases. We remark that an explicit formula for $G(1)$ is available, since it is expressible in terms of an ordinary Gauss sum.

7. Another weighted sum. Finally we consider the following sum which is closely related to some of the results obtained above:

$$S = \sum \psi(\xi_1) \ldots \psi(\xi_r), \tag{7.1}$$

where the summation is over all $\xi_i \neq 0$ such that

$$\alpha_1\xi_1 + \beta_1\xi_1^{-1} + \ldots + \alpha_r\xi_r + \beta_r\xi_r^{-1} = \alpha \qquad (\alpha_i \neq 0,\ \beta_i \neq 0). \tag{7.2}$$

We have

$$qS = \sum_{\beta} e(-\alpha\beta) \sum_{\xi_1,\ldots,\xi_r} e\left(\beta \sum_{i=1}^{r} (\alpha_i\xi_i + \beta_i\xi_i^{-1})\right) \psi(\xi_1) \ldots \psi(\xi_r)$$

$$= \sum_{\beta\neq 0} e(-\alpha\beta) \prod_{i=1}^{r} L(\beta\alpha_i, \beta\beta_i),$$

where $L(\alpha, \beta)$ is defined by (2.4). In view of (2.8) we have at once

$$S = 0 \qquad (\text{for all } \alpha) \tag{7.3}$$

if $\psi(\alpha_i\beta_i) = -1$ for at least one value of i. On the other hand, if $\alpha_i\beta_i = \gamma_i^2$, $i = 1, \ldots, r$, then we get, using the second of (2.8),

$$S = q^{-1}G^r(1)\psi(\beta_1 \ldots \beta_r) \sum_{\beta\neq 0} e(-\alpha\beta)\psi^r(\beta) \prod_{i=1}^{r} (e(2\beta\gamma_i) + e(-2\beta\gamma_i)). \tag{7.4}$$

Thus the sum (7.1) is evaluated in all cases by (7.3) and (7.4).

Note that if $1 \leqslant s \leqslant r$, and

$$S' = \sum \psi(\xi_1) \ldots \psi(\xi_s)$$

the summation extending over all $\xi_i \neq 0$ satisfying (7.2), then $S' = 0$ provided $\psi(\alpha_i\beta_i) = -1$ for at least one value of $i \leqslant s$.

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