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A characterization of generalized Hall planes

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We prove that a translation plane π of odd order is a generalized Hall plane if and only if π is derived from a translation plane of semi-translation class 1-3a . Also, a derivable translation plane of even order and class 1-3a derives a generalized Hall plane. We also show that the generalized Hall planes of Kirkpatrick form a subclass of the class of planes derived from the Dickson semifield planes.

1. Introduction and background

Kirkpatrick [6] defines generalized Hall planes as follows:

A translation plane π *is a generalized Hall plane if and only if* π *admits a collineation group* G *which fixes a Baer subplane* π_0 *pointwise and acts simply transitively on the points of* $l_\infty - \pi_0 \cap l_\infty$.

Kirkpatrick (Theorem 1, [6]) shows that a generalized Hall plane of odd order admits a coordinatization so that the corresponding quasifield is a right two dimensional vector space over $GF(q)$, q a prime power, where $GF(q)$ coordinatizes π_0 . Furthermore, Kirkpatrick defines a class of quasifields that coordinatize generalized Hall planes of odd order and which properly contains the Hall quasifields.

Originally the Hall planes were defined by constructing a

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coordinatizing quasifield (see [2], pp. 364-365). Albert [1] has shown the finite Hall planes to be precisely the planes derived from the desarguesian planes of square order.

We show analogously that a translation plane π of odd order is a generalized Hall plane if and only if π is derivable from a translation plane of semi-translation class 1-3a . Furthermore, if the generalized Hall planes of even order can be coordinatized by a quasifield which is a right vector space over $GF(q)$ a similar result holds.

The author [5] considers translation planes derived from semifield planes. We show that the planes of Kirkpatrick (see [6], Section 3) are among the planes obtained by deriving the Dickson semifield planes. The results used in the following sections will be listed for convenience.

We shall write $y = f(x)$ instead of $\{(x, y) \mid y = f(x)\}$ to indicate an affine line. The line at infinity will be denoted by l_∞ and $x = 0 \cap l_\infty$ will be denoted by (∞) .

We shall also assume that the reader is somewhat familiar with Ostrom's development of "derivation". We refer the reader to [7] and [8] for the basic background material.

RESULT I (Ostrom [7], Theorem 6). Let π be an affine plane of order q^2 and let M be a set of $q + 1$ points on l_∞ on π^* (the projective extension of π). Suppose that for every pair of distinct points P and Q such that $PQ \cap l_\infty \in M$ there exists a projective subplane of π^* which contains P, Q and M . Then the affine parts of the proper subplanes of π^* which contain M and the affine parts of the lines of π^* which do not intersect M form the lines of a new affine plane $\bar{\pi}$ (called the plane derived from π) containing the same points as π .

RESULT II (Ostrom [7], Theorem 7 and Corollary). Let σ be a permutation of the points of π inducing a collineation of π^* which carries M into itself. Then σ induces a collineation of $\bar{\pi}^*$ which carries $\bar{M}$ into itself. Moreover, if σ is a translation of π , then σ induces a translation of $\bar{\pi}$.

The term "coordinate system" shall mean a Hall coordinate system (see for example [7], pp. 9-10).

RESULT III (Ostrom [7] - a strong form of Theorem 9). Let π be an affine plane of order q^2 coordinatized by a system Q such that

(1) Q contains a subfield F of order q ,

(2) addition is associative and commutative,

(3) Q is a right two dimensional vector space over F ,

(4) Q is linear with respect to F .

Then π is derivable and

$$\{(x, y) \mid x = a\alpha + c,\ y = a\beta + b,\ a \neq 0,\ b,\ c \text{ fixed in } Q \text{ and for all } \alpha, \beta \in F\}$$

is the set of points of an affine Baer subplane.

RESULT IV (Ostrom [7], Theorem 10). Let π be an affine plane coordinatized by a system Q as in Result III and let $t \in Q - F$. Then $\bar{\pi}$ (the plane derived from π) can be coordinatized by a system $\bar{Q}$ such that a point with coordinates $(x, y) = (tx_1+x_2, ty_1+y_2)$ in Q ; $x_i, y_i \in F$, $i = 1,2$, has coordinates $(\bar{x}, \bar{y}) = (tx_1+y_1, tx_2+y_2)$ in $\bar{Q}$.

Let $(S, +) = (\mathrm{GF}(q^2), +)$. Let t be a fixed element of $S - \mathrm{GF}(q)$. The multiplication of the Dickson semifields $(S, +, \cdot)$ is defined as follows:

$t\alpha = t\cdot\alpha$, $(t\alpha+\beta)\cdot(t\delta+\gamma) = t(\alpha\gamma+\beta^\sigma\delta) + (\alpha^\eta\delta^\rho g+\beta\gamma)$ where σ, η, ρ are automorphisms of $\mathrm{GF}(q)$, g a non-square in $\mathrm{GF}(q)$, for all $\alpha, \beta, \delta, \gamma \in \mathrm{GF}(q)$.

It easily follows that the Dickson semifield planes are derivable.

RESULT V (Johnson [5], Theorem (3.4)(1)). The planes derived from the Dickson semifields may be coordinatized by a right quasifield $(S, +, *)$ such that $t * \alpha = t\alpha$,

$$(t\alpha+\delta)*(t\beta+\gamma) = t\left(\delta-\alpha\beta^{-1}\gamma\right)^{\sigma^{-1}}\beta + \left(\delta-\alpha\beta^{-1}\gamma\right)^{\sigma^{-1}}\gamma + \alpha^\eta\beta^{-\sigma\rho}g$$

for $\beta \neq 0$, $\alpha, \beta, \delta, \gamma \in \mathrm{GF}(q)$, δ, η, ρ automorphisms of F , and g a nonsquare in $\mathrm{GF}(q)$. Also, $(t\alpha+\delta) * \gamma = t(\alpha\gamma) + \delta\gamma$.

DEFINITION 1.1. Let π be a projective plane of order q^2 and π_0

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a subplane of order q. Let p be a point of π_0 and L a line of π such that $L \cap \pi_0$ is a line of π_0. π is said to be (p, L, π_0)-transitive if the stabilizer of π_0 in the group of all (p, L)-collineations of π induces a collineation group of π_0 such that π_0 is (p, L)-transitive (see for example, [3], p. 137).

DEFINITION 1.2. A projective plane π is a semi-translation plane with respect to a line L if and only if there is a Baer subplane π_0 containing the line L such that π is (p, L, π_0)-transitive for all points $p \in L \cap \pi_0$ (see [3], p. 1372 for another definition).

DEFINITION 1.3. A semi-translation plane π with respect to l_∞ and subplane π_0 is of class 1-3a if and only if π is (p_∞, L, π_0)-transitive for all lines L of π_0 such that $L \text{ I } p_\infty$, p_∞ a fixed point of L_∞ and (p, L, π_0)-transitive for all points $p \text{ I } L_\infty \cap \pi_0$ for all lines of π_0 incident with p_∞ (see [3], (2.16), p. 1380).

RESULT VI (Johnson [4], Lemmas (3.1), (3.2)). Let π be a semi-translation plane with respect to l_∞ and Baer subplane π_0. Assume π_0 is coordinatized by $GF(q)$. If π is $((0), x = 0, \pi_0)$- and $((\infty), x = 0, \pi_0)$-transitive then $c(\alpha m) = (c\alpha)m$, and $c(\alpha+m) = c\alpha + cm$ for all c, m in a coordinate system for π and for all $\alpha \in GF(q)$.

2. Translation planes of class 1-3a

A translation plane π which contains a Baer subplane π_0 is a semi-translation plane. If π is of class 1-3a, π_0 desarguesian, and coordinates are chosen so that π_0 is coordinatized by $GF(q)$, $p_\infty = (\infty)$ (see Definition 1.3), then clearly π is $((\infty), x = 0, \pi_0)$- and $((0), x = 0, \pi_0)$-transitive. By Result VI and the ordinary properties of a coordinatizing quasifield Q it follows that Q is a right two

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dimensional vector space over $GF(q)$. By Result III, π is derivable.

THEOREM 2.1. *A translation plane* π *containing a desarguesian Baer subplane of semi-translation class* 1-3a *is derivable and derives a generalized Hall plane.*

Proof. We choose coordinates as above and as in Result IV. By Result III (we also use here the fact that "derivation" is involutory) the $x = 0$ of π appears as a Baer subplane $\bar{\pi}_0$ in the plane $\bar{\pi}$ derived from π . By Result II, $\bar{\pi}$ is a translation plane and admits a collineation group inherited from the $\left((\infty),\ x = 0,\ \pi_0\right)$- and $\left((0),\ x = 0,\ \pi_0\right)$-collineation groups of π .

The group of $\bar{\pi}$ generated by the inherited groups clearly fixes $\bar{\pi}_0$ pointwise. Since the original group is simply transitive on the points of $l_\infty - \pi_0 \cap l_\infty$ and since the set of lines on these points is not altered by derivation (see Result I) it follows that the inherited group is simply transitive on $\bar{l}_\infty - \bar{\pi}_0 \cap \bar{l}_\infty$. Thus, the derived plane $\bar{\pi}$ is a generalized Hall plane.

3. Generalized Hall planes of odd order

Let $\bar{\pi}$ be a generalized Hall plane of odd order with Baer subplane $\bar{\pi}_0$. By Kirkpatrick's Theorem 1 [6], there is a coordinatizing quasifield $\bar{Q}$ for $\bar{\pi}$ such that $\bar{Q}$ is a right two dimensional vector space over $F = GF(q)$ where F coordinatizes $\bar{\pi}_0$.

By Result III, $\bar{\pi}$ is derivable and by Result II the derived plane π is a translation plane.

Let $\bar{G}$ denote the group acting simply transitively on $\bar{l}_\infty - \bar{\pi}_0 \cap \bar{l}_\infty$. Clearly, $\bar{G}$ induces an automorphism group $\bar{G}_a$ on $\bar{Q}$ which fixes F elementwise. Let $\{1,\ t\}$ be a basis for Q over F . Then $\bar{G}_a = \{\sigma_{\alpha,\beta};\ \alpha \neq 0,\ \beta \in F\}$ where $\sigma_{\alpha,\beta}$ is defined by $t\sigma_{\alpha,\beta} = t\alpha + \beta$.

It follows that the group $\bar{G}$ is generated by the mappings: $(tx_1+x_2,\ ty_1+y_2) \to (tx_1\alpha+x_2,\ ty_1\alpha+y_2)$ and

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$(tx_1+x_2,\ ty_1+y_2) \to (tx_1+x_2+x_1\beta,\ ty_1+y_2+y_2\beta)$ for all $\alpha \neq 0$, $\beta \in F$. By Results II and IV, the mappings induce collineations of π represented by $(tx_1+y_1,\ tx_2+y_2) \to (tx_1\alpha+y_1\alpha,\ tx_2+y_2)$ and $(tx_1+y_1,\ tx_2+y_2) \to \left(tx_1+y_1,\ t(x_2+x_1\beta)+(y_2+y_1\beta)\right)$ or rather $(x,\ y) \to (x\alpha,\ y)$ and $(x,\ y) \to (x,\ x\beta+y)$. These mappings clearly represent $\left((0),\ x = 0,\ \pi_0\right)$- and $\left((\infty),\ x = 0,\ \pi_0\right)$-transitivity, respectively. Thus, we have the following:

THEOREM 3.1. *Generalized Hall planes of odd order are derivable and derive translation planes of semi-translation class* 1-3a.

4. The planes derived from the Dickson semifields

The known translation planes of class 1-3a are semifield planes. The author has studied planes derived from semifield planes in [5]. In particular, the planes of [5], Theorem (3.1), (1), (3) and (4) are generalized Hall planes.

Kirkpatrick's generalized Hall systems are defined as follows (see Section 3, [6]).

Let $(Q,\ +) = \left(GF(q^2),\ +\right)$, q odd, and θ, φ automorphisms of $GF(q)$ and ν a nonsquare of $GF(q)$.

Define $(z\alpha+\beta)z = z\beta^{\theta} + \alpha^{\varphi}\nu$ for all $z \in Q - GF(q)$; $\alpha, \beta \in GF(q)$.

Notice that if $\{1,\ t\}$ is a basis for Q over $GF(q)$ and $z = tz_1 + z_2$; $z_i \in GF(q)$, then, for $\beta \neq 0$,

$$
\begin{aligned}
(t\alpha+\delta)(t\beta+\gamma) &= \left[(t\beta+\gamma)\left(\beta^{-1}\alpha\right)+\left(\delta-\beta^{-1}\alpha\gamma\right)\right](t\beta+\gamma)\\
&= (t\beta+\gamma)\left(\delta-\beta^{-1}\alpha\gamma\right)^{\theta} + \left(\beta^{-1}\alpha\right)^{\varphi}\nu\\
&= t\left(\delta-\alpha\beta^{-1}\gamma\right)^{\theta}\beta + \left(\delta-\alpha\beta^{-1}\gamma\right)^{\theta}\gamma + \alpha^{\varphi}\beta^{-\varphi}\nu ,
\end{aligned}
$$

which is precisely the $*$-multiplication with $\theta = \sigma^{-1}$, $\varphi = \eta$, $\varphi\theta = \rho$ of Result V.

Thus, we have the following:

THEOREM 4.1. Kirkpatrick's *generalized Hall planes form a proper subclass of the planes derived from the Dickson semifields.*

Note added in proof on 22 September 1971. T.G. Ostrom has pointed out to the author that *derivable* translation planes of class 1-3a are semifield planes.

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