Canad. J. Math. Vol. 67 (3), 2015 pp. 696–720
http://dx.doi.org/10.4153/CJM-2014-033-0

Geography of Irregular Gorenstein 3-folds

Dedicated to the memory of Professor Gang Xiao

Tong Zhang

Abstract. In this paper, we study the explicit geography problem of irregular Gorenstein minimal 3-folds of general type. We generalize the classical Noether–Castelnuovo type inequalities for irregular surfaces to irregular 3-folds according to the Albanese dimension.

1 Introduction

The geography of 3-folds of general type is a vast and important subject in the study of algebraic varieties. Much work has been done by Hunt, Chen, Catanese, Chen, Hacon, and others in the general case (see [6, 10–13, 22]), and by Ohno [31] and Barja [1] in the fibered case.

The purpose of this paper is to study the geography problem of irregular 3-folds of general type.

We work over an algebraically closed field of characteristic 0. A projective variety X is called *irregular* if $h^1(\mathcal{O}_X) > 0$, *i.e.,* X has a nontrivial Albanese map. Denote by $a(X) \subseteq \mathrm{Alb}(X)$ the image of X under its Albanese map. The Albanese dimension $\dim a(X)$ can vary from one to $\dim X$. We say that X is *of Albanese dimension* m if $\dim a(X) = m$. In particular, we say X is *of maximal Albanese dimension* if $\dim a(X) = \dim X$.

Let C be a projective curve of genus $g > 0$. One has

$$\deg(\omega_C) = 2\chi(\omega_C) \geq 0.$$

The above result has several 2-dimensional generalizations. For an irregular minimal surface S of general type (with ADE singularities), $\chi(\omega_S) > 0$. One has the Noether type, Castelnuovo type, and Severi inequalities for *irregular* surfaces proved respectively by Bombieri [5], Horikawa [20], and Pardini [32].

(a) Noether type inequality: $K_S^2 \geq 2\chi(\omega_S)$ if S is irregular (see [5]);
(b) Castelnuovo type inequality: $K_S^2 \geq 3\chi(\omega_S)$ if the Albanese fiber is not hyperelliptic of genus 2 or 3 (see [20]); [1]

Received by the editors November 27, 2013; revised August 11, 2014.
Published electronically March 2, 2015.
This work is supported by an NSERC Discovery Grant.
AMS subject classification: **14J30**.
Keywords: 3-fold, geography, irregular variety.

[1]Here we call it Castelnuovo type inequality because it is closely related to the classical Castelnuovo inequality $K_S^2 \geq 3p_g + q - 7$ when S has birational canonical map. Hence the line of slope 3 in the geography of surfaces of general type has this name.

(c) Severi inequality: $K_S^2 \geq 4\chi(\omega_S)$ if S is of maximal Albanese dimension (see [15, 32]).

The above results concern the geography of irregular surfaces, and they have played a very important role in surface theory. Also, in the recent work of Lu [27, 28] and Lopes–Pardini [29], results of such type have been applied to study hyperbolicity of irregular surfaces.

Having seen the importance of such results one can ask the following natural question.

Question 1.1 What are the Noether type, Castelnuovo type and Severi inequalities for irregular minimal Gorenstein 3-folds of general type?

Here we assume that the minimal 3-folds are Gorenstein so that $\chi(\omega_X) > 0$ (see [9, §2.1]), and we can have nontrivial inequalities $K_X^3 \geq a\chi(\omega_X)$ with $a > 0$ similar to the surface case.

In fact, several questions of the similar type have been raised before. In the early 1980's, Miles Reid asked the following question: what is the Noether inequality for 3-folds? Also, as an open problem in [12, §3.9], Chen conjectured as follows.

Conjecture 1.2 *For Gorenstein minimal 3-fold X of general type, there should be a Noether inequality in the form*

$$K_X^3 \geq a\chi(\omega_X) - b,$$

where $a(>1)$, b are both positive rational numbers.

As is mentioned in [12], any bound $a > 1$ is nontrivial and interesting. However, it might be more difficult than the inequality between K_X^3 and $p_g(X)$. One possible problem comes from the difficulty to understand $h^1(\mathcal{O}_X)$ and $h^2(\mathcal{O}_X)$. Another problem may be due to the non-smoothness of X. For example, when X is smooth and minimal, it is proved in [6] that $K_X^3 \geq \frac{2}{3}(2p_g(X) - 5)$. From this, Chen and Hacon [13] have proved that $a = \frac{8}{9}$. But if X is Gorenstein, it is still an open question whether $K_X^3 \geq \frac{2}{3}(2p_g(X) - 5)$ holds. See [6, Conjecture 4.4].

Recently, the Severi inequality was proved by Barja [2] and by the author [36] independently: Let X be an irregular minimal Gorenstein 3-fold of general type. If X is of maximal Albanese dimension, then

$$K_X^3 \geq 12\chi(\omega_X).$$

Our first purpose of this paper is to give a complete answer to Question 1.1 by proving the following Noether-Castelnuovo type inequalities.

Theorem 1.3 *Let X be an irregular minimal Gorenstein 3-fold of general type. If X has Albanese fiber dimension one, then*

$$K_X^3 \geq 4\chi(\omega_X).$$

Moreover, if the Albanese fiber is not hyperelliptic of genus ≤ 5, then

$$K_X^3 \geq 6\chi(\omega_X).$$

This theorem, combined with the Miyaoka–Yau inequality $K_X^3 \le 72\chi(\omega_X)$, will give the explicit geography of Gorenstein irregular 3-folds of Albanese dimension 2 or more.

Let us now denote by d the Albanese dimension of X. Then Theorem 1.3 just says that when $d = 2$, the coefficients before $\chi(\omega_X)$ are $2d! = 4$ and $3d! = 6$. Along this line, one can naturally consider the following conjecture of Noether–Castelnuovo type when X has Albanese dimension one (*i.e.*, $d = 1$).

Conjecture 1.4 *Let X be an irregular minimal Gorenstein 3-fold of general type with Albanese dimension one. Then* $K_X^3 \ge 2\chi(\omega_X)$. *If the Albanese fiber has large volume, then* $K_X^3 \ge 3\chi(\omega_X)$.

Note that the similar inequalities appear in the surface case. Here we consider the volume of the fiber instead of its geometric genus, simply because in this case the Albanese fiber has dimension 2 and the volume is always positive for a surface of general type, but the geometric genus is not.

By the sharp inequality $K_X^3 \ge \frac{2}{3}(2p_g(X) - 5)$ [6] in the smooth case, Conjecture 1.4 seems to be too optimistic. But surprisingly, the following theorem shows that the above conjecture is not too far from being true.

Theorem 1.5 *Let X be an irregular minimal Gorenstein 3-fold of general type with Albanese dimension one. Let* $f: X \to Y$ *be the induced Albanese fibration with a smooth general fiber F. Then* $K_X^3 \ge 2\chi(\omega_X)$, *unless one of the following holds:*

(i) $p_g(F) = 2$ *and* $K_F^2 = 1$. *In this case,* $K_X^3 \ge \frac{4}{3}\chi(\omega_X)$.
(ii) $p_g(F) = 3$ *and* $K_F^2 = 2$. *In this case,* $K_X^3 \ge \frac{12}{7}\chi(\omega_X)$.

Moreover, we have $K_X^3 \ge 3\chi(\omega_X)$ *provided that one of the following holds:*

(iii) $K_F^2 \ge 12$;
(iv) *F has no hyperelliptic pencil and* $K_F^2 \ge 9$.

Remark 1.6 The geography of non-Gorenstein 3-folds of general type is a very subtle topic. In particular, $\chi(\omega_X)$ can be zero or even negative. For example, there do exist examples of non-Gorenstein 3-folds of maximal Albanese dimension with $\chi(\omega_X) = 0$ (see [16]). In [10], Chen and Hacon have constructed a family of non-Gorenstein 3-folds of general type with $\chi(\omega_X)$ negative. In their paper, they obtained a similar type of inequality

$$K_X^3 \ge c\chi(\omega_X),$$

but with $c < 0$. One can also construct families of non-Gorenstein 3-folds of general type with Albanese dimension two and $\chi(\omega_X) < 0$ (see Example 6.2). In these cases, Theorem 1.3 holds trivially. We would like to point out that in [8], Chen and Chen proved that there exists an explicit effective lower bound for $\mathrm{Vol}(X)$.

Let us sketch the proofs of the above theorems. If X has Albanese dimension two, Pardini's method [32] on étale covering and limiting can be applied here, provided one has a good slope inequality for fibered 3-folds over surfaces, which is not known yet. In this paper, to overcome this difficulty, we prove the relative Noether inequalities in Theorem 5.2 for fibered 3-folds over surfaces. These inequalities are about K_X^3 and $h^0(K_X)$ up to some explicit error terms. Then by the generic vanishing theorem due to Green and Lazarsfeld [18], we know that $\chi(\omega_X)$ is bounded from above by $h^0(K_X)$ up to étale covering. Finally, by Pardini's limiting trick, we can prove Theorem 1.3.

If the Albanese dimension of X is one, we still have the étale covering method by Bombieri and Horikawa [5, 20]. But the generic vanishing does not help in this case. Alternatively, we will prove the relative Noether inequalities in Theorems 7.1 and 8.1, for fibered 3-folds over curves. This will imply Theorem 1.5 via the above covering method if the volume of the Albanese fiber is at least 4. Note that the slope inequality for fibered 3-folds over curves has been studied by Ohno [31] and Barja [1]. Finally, we will prove case by case when the volume of the Albanese fiber is at most 3.

This paper is organized as follows. In Section 3, we prove several basic results for fibered 3-folds. In Section 4, we list several results about linear systems on algebraic surfaces. In Section 5, we prove the relative Noether inequalities for fibered 3-folds over surfaces. In Section 6, we prove Theorem 1.3. In Sections 7 and 8, we consider the case of Albanese dimension one and prove Theorem 1.5.

After finishing the paper, J. Chen informed the author of a very recent paper [7] in which he and M. Chen proved that $K_X^3 \geq \frac{2}{3}(2h^0(K_X)-5)$ still holds in the Gorenstein case. In another very recent paper, Hu [21] showed that $K_X^3 \geq \frac{4}{3}\chi(\omega_X) - 2$ in this case. Also, the author has been informed by M. Barja that in [2, Remark 4.6], the first inequality of Theorem 1.3 is independently proved using a different method.

2 Notation

The following notation will be frequently used in this paper.

Let X be a projective variety and let L be a line bundle on X such that $h^0(L) \geq 2$. We denote by $\phi_L\colon X \dashrightarrow \mathbb{P}^{h^0(L)-1}$ the rational map induced by the complete linear series $|L|$. We say that ϕ_L is *generically finite* if $\dim \phi_L(X) = \dim X$. Otherwise, we say ϕ_L *factors through a fibration*. In particular, we say $|L|$ is *composed with a pencil* if the image of ϕ_L has dimension 1.

A $\mathbb{Q}$-Weil divisor D on a variety X of dimension n is called *pseudo-effective* if for any nef line bundles $A_1, \dots, A_{n-1}$ on X, we have

$$A_1 \cdots A_{n-1} D \geq 0.$$

Such divisors can be characterized as the limit of effective $\mathbb{Q}$-divisors.

Let $\alpha\colon X \to A$ be a morphism from X to an abelian variety A. Denote by $\mu_d\colon A \to A$ the multiplicative map of A by d. We have the diagram

$$\begin{array}{ccc} X_d & \xrightarrow{\phi_d} & X \\ \downarrow{\scriptstyle \alpha_d} & & \downarrow{\scriptstyle \alpha} \\ A & \xrightarrow{\mu_d} & A. \end{array}$$

Here $X_d = X \times_{\mu_d} A$ is the fiber product. We call X_d *the d-th lifting of X by α*. In particular, if α is the Albanese map of X, then we call X_d *the d-th Albanese lifting of X*. This construction was used by Pardini in [32]. We would like to remark here that X_d could be non-connected in general, but in this paper, since α is always the Albanese map of X, the induced map from $\mathrm{Pic}^0(A)$ to $\mathrm{Pic}^0(X_d)$ is always injective, which implies that X_d is connected.

In this paper, a *fibration* $f: X \to Y$ always means a surjective morphism with connected fibers.

3 Preliminaries

Let X be a projective n-fold. Here we assume that $n = 2, 3$. Let $f: X \to Y$ be a fibration from X to a smooth projective curve Y with a smooth general fiber F. Then for any nef line bundle L on X, we can find a unique integer e_L such that

- $L - e_L F$ is not nef;
- $L - eF$ is nef for any integer $e < e_L$.

We call this number the *minimum of L with respect to F*. In particular, $e_L > 0$. Another important fact is that if $h^0(L - e_L F) > 0$, then $|L - e_L F|$ has horizontal base locus (*cf.* [36, §2]). We have the following theorem.

Theorem 3.1 *Using the above notation, let L be a nef and effective line bundle on X. Then we have the quadruples*

$$\{(X_i, L_i, Z_i, a_i), \quad i = 0, 1, \dots, N\}$$

with the following properties:

(i) $(X_0, L_0, Z_0, a_0) = (X, L, 0, e_L)$.

(ii) *For any $i = 0, \dots, N-1$, $\pi_i: X_{i+1} \to X_i$ is a composition of blow-ups of X_i such that the proper transform of the movable part of $|L_i - a_i F_i|$ is base point free. Here $F_0 = F$, $F_{i+1} = \pi_i^* F_i$, and $a_i = e_{L_i}$ is the minimum of L_i with respect to F_i. Moreover, we have the decomposition*

$$|\pi_i^*(L_i - a_i F_i)| = |L_{i+1}| + Z_{i+1}$$

such that $|L_{i+1}|$ is base point free and $Z_{i+1}|_{F_{i+1}} > 0$.

(iii) *We have $h^0(L_0) \geq h^0(L_1) > \cdots > h^0(L_N) > h^0(L_N - a_N F_N) = 0$. Here $a_N = e_{L_N}$.*

Proof See [36, §2]. ∎

Remark 3.2 From the above construction, we have

$$h^0(L_0|_{F_0}) \geq h^0(L_1|_{F_1}) > h^0(L_2|_{F_2}) > \cdots > h^0(L_N|_{F_N}).$$

In general, we do *not* know if $h^0(L_0|_{F_0}) > h^0(L_1|_{F_1})$. But if $|L_0|_F|$ is base point free, then

$$h^0(L_0|_{F_0}) > h^0(L_1|_{F_1}).$$

This fact will be used in the proofs of Theorem 7.1 and 8.1.

Write $\rho_i = \pi_0 \circ \cdots \circ \pi_{i-1}: X_i \to X_0$ for $i = 1, \dots, N$. Fix a nef line bundle $P = P_0$ on X and denote

$$L_i' = L_i - a_i F_i, \quad r_i = h^0(L_i|_{F_i}), \quad d_i = (P_i|_{F_i})(L_i|_{F_i})^{n-2},$$

https://doi.org/10.4153/CJM-2014-033-0 Published online by Cambridge University Press

where $P_i = \rho_i^* P$. It is easy to see that we have

$$d_0 \geq d_1 \geq \cdots \geq d_N \geq 0.$$

Proposition 3.3 *Let $H^0(L) \to H^0(L|_F)$ be the restriction map. Denote by r the dimension of its image. Then for any $j = 0, \ldots, N$, we have the following numerical inequalities:*

$$h^0(L_0) \leq h^0(L'_j) + a_0 r + \sum_{i=1}^{j} a_i r_i \leq h^0(L'_j) + \sum_{i=0}^{j} a_i r_i; \tag{3.1}$$

$$P_0 L_0^2 \geq 2a_0 d_0 + \sum_{i=1}^{j} a_i(d_{i-1} + d_i) - 2d_0 \quad (n = 3). \tag{3.2}$$

Proof The proof is almost identical to [36, §2]. We sketch it here and point out the difference. When $i > 0$, by the following exact sequence

$$0 \longrightarrow H^0(L_i - F_i) \longrightarrow H^0(L_i) \longrightarrow H^0(L_i|_{F_i}),$$

we get

$$h^0(L_i - F_i) \geq h^0(L_i) - h^0(L_i|_{F_i}) = h^0(L_i) - r_i.$$

The only difference here from [36, §2] is when $i = 0$. In order to prove the inequality here, we only need to show that

$$h^0(L - iF) - h^0(L - (i+1)F) \leq h^0(L) - h^0(L - F)$$

for any $0 \leq i \leq a_0 - 1$.

In fact, the result holds if $a_0 = 1$. If $a_0 > 1$, by the exact sequence

$$0 \longrightarrow H^0(L - 2F) \longrightarrow H^0(L - F) \oplus H^0(L - F) \longrightarrow H^0(L),$$

we know that

$$h^0(L - F) - h^0(L - 2F) \leq h^0(L) - h^0(L - F) = r.$$

Therefore, we can finish the proof by induction. Finally, summing over $i = 0, \ldots, j$, we can get (3.1).

For (3.2), since $\pi_i^* L'_i = a_{i+1} F_{i+1} + L'_{i+1} + Z_{i+1}$, we have the following computation of 1-cycles:

$$\begin{aligned}(\pi_i^* L'_i)^2 - L'^2_{i+1} &= (\pi_i^* L'_i - L'_{i+1})(\pi_i^* L'_i + L'_{i+1}) \\ &= a_{i+1}(\pi_i^* L'_i + L'_{i+1})F_{i+1} + (\pi_i^* L'_i + L'_{i+1})Z_{i+1} \\ &= a_{i+1}(\pi_i^* L'_i + L'_{i+1})F_{i+1} + (\pi_i^* L'_i + F_{i+1})Z_{i+1} \\ &\quad + (L'_{i+1} + F_{i+1})Z_{i+1} - 2(\pi_i^* L'_i - L'_{i+1})F_{i+1}.\end{aligned}$$

Note that $\pi_i^* L'_i + F_{i+1}$ and $L'_{i+1} + F_{i+1}$ are both nef. Taking intersections with P_{i+1} for both sides, we can get

$$P_i L'^2_i - P_{i+1} L'^2_{i+1} = P_{i+1}(\pi_i^* L'_i)^2 - P_{i+1} L'^2_{i+1} \geq a_{i+1}(d_i + d_{i+1}) - 2(d_i - d_{i+1}).$$

Summing over $i = 0, \ldots, j-1$, we have

$$P_0 L'^2_0 - P_j L'^2_j \geq \sum_{i=1}^{j} a_i(d_{i-1} + d_i) - 2(d_0 - d_j).$$

Note that we also have

$$P_0L_0^2 - P_0L_0'^2 = 2a_0d_0, \quad P_jL_j'^2 + 2d_j \ge 0.$$

Hence (3.2) follows. ■

We also have the following lemma.

Lemma 3.4 *Under the above setting, for $n = 2, 3$, we have*

$$P_0L_0^{n-1} \ge (n-1)d_0(a_0 - 1) + d_0 \sum_{i=1}^{N} a_i.$$

Proof For $i = 0, \dots, N-1$, denote by $\tau_i = \pi_i \circ \cdots \circ \pi_{N-1}: X_N \to X_i$ the composition of blow-ups.

Write $b = a_1 + \cdots + a_N$ and $Z = \tau_1^* Z_1 + \cdots + \tau_{N-1}^* Z_{N-1} + Z_N$. We have the following numerical equivalence on X_N:

$$\tau_0^* L_0' \sim_{\text{num}} L_N' + bF_N + Z.$$

Since $L_0' + F_0$ and $L_N' + F_N$ are both nef, it follows that

$$\begin{aligned} P_0(L_0' + F_0)^{n-1} &= P_N(\tau_0^* L_0' + F_N)^{n-2}(L_N' + F_N + bF_N + Z) \\ &\ge bP_N(\tau_0^* L_0' + F_N)^{n-2} F_N \ge bd_0. \end{aligned}$$

Combining with

$$P_0L_0^{n-1} - P_0(L_0' + F_0)^{n-1} = (n-1)(a_0 - 1)d_0,$$

the proof is finished. ■

4 Linear Series on Algebraic Surfaces

In this section, we recall some basic results about linear series on algebraic surfaces. These results will be used to compare the numbers r_i and d_i. They will also serve as the first step of the induction process.

In this section, we always use the following assumptions:

(a) S is a smooth algebraic surface of general type with the smooth minimal model $\sigma: S \to S'$;

(b) $L \ge M$ are two nef line bundles on S such that $L \le K_S$.

We list the following results that will be frequently used in the sequel.

Proposition 4.1 *Suppose that ϕ_L is generically finite. Then*

- $LM \ge 2h^0(M) - 4$;
- $(\sigma^* K_{S'})L \ge 2h^0(L) - 2$ *if* $h^0(L) < p_g(S)$.

If we further assume that S has no hyperelliptic pencil, then

- $LM \ge 3h^0(M) - 7$,
- $(\sigma^* K_{S'})L \ge 3h^0(L) - 5$ *if* $h^0(L) < p_g(S)$.

Proof To prove this result, we can assume that $h^0(M) \geq 2$. If $|M|$ is not composed with a pencil, from a result in [33, Théorem 2], we know that

$$LM \geq M^2 \geq 2h^0(M) - 4.$$

In particular, if S has no hyperelliptic pencil, then by the Castelnuovo type inequality (*cf.* [3, Théorème 5.5]),

$$LM \geq M^2 \geq 3h^0(M) - 7.$$

If $|M|$ is composed with a pencil, we can write $M \sim_{\text{num}} rC + Z$. Here C is a general member of the pencil, $r \geq h^0(M) - 1$, and Z is the fixed part of $|M|$. Because ϕ_L is generically finite, $h^0(L|_C) \geq 2$. This implies that $LC \geq 2$, since S is of general type. We get

$$LM \geq rLC \geq 2h^0(M) - 2.$$

Note that if S has no hyperelliptic pencil, C will not be hyperelliptic and $LC \geq 3$. Hence, $LM \geq 3h^0(M) - 3$.

The second inequality is from [31, Lemma 2.3]. We only need to prove the last one. Since $h^0(L) < p_g(S)$, by the Hodge index theorem and the Castelnuovo inequality [3, Théorème 5.5], we have

$$\begin{aligned}((\sigma^* K_{S'})L)^2 &\geq L^2 K_{S'}^2 \geq (3h^0(L) - 7)(3p_g(S) - 7)\\ &\geq 9(h^0(L))^2 - 33h^0(L) + 28\\ &> 9(h^0(L))^2 - 36h^0(L) + 36 = (3h^0(L) - 6)^2.\end{aligned}$$

It implies that $(\sigma^* K_{S'})L \geq 3h^0(L) - 5$, which completes the proof. ∎

Proposition 4.2 *Assume that $|L|$ is composed with a pencil. Then*

$$(\sigma^* K_{S'})L \geq 2h^0(L) - 2,$$

except for the case when $K_{S'}^2 = 1$, $p_g(S) = 2$, and $q(S) = 0$. If we further assume that S has no hyperelliptic pencil, then

$$(\sigma^* K_{S'})L \geq 3h^0(L) - 3,$$

except for the case when $K_{S'}^2 = 2$, $p_g(S) = 2$, and $q(S) = 0$.

Proof The first inequality is just [31, Lemma 2.2]. For the second one, since $|L|$ is composed with a pencil, we can write $L \sim_{\text{num}} rC + Z$, where $Z \geq 0$ is the fixed part, C is a general member of the pencil, and $r \geq h^0(L) - 1$. Let $C' = \sigma(C)$. Then $p_a(C') \geq 3$ by our assumption.

If $K_{S'}C' \geq 3$, then

$$(\sigma^* K_{S'})L \geq rK_{S'}C' \geq 3h^0(L) - 3.$$

Now assume that $K_{S'}C' \leq 2$. By our assumption on L, we know that $C \leq L \leq K_S$, which implies that $C' \leq K_{S'}$. In particular, $C'^2 \leq K_{S'}^2$. By the Hodge index theorem, we get $C'^2 \leq 2$. Note that $p_a(C') \geq 3$. The genus formula forces $K_{S'}C' = C'^2 = 2$, which also implies that $K_{S'}^2 = 2$. By the Hodge index theorem again, we have $K_{S'} \sim_{\text{lin}} C'$. Hence $p_g(S) = h^0(C') = 2$, as $|C'|$ is a rational pencil. Moreover, we get $q(S) = 0$. Otherwise, S' is irregular and $K_{S'}^2 \geq 2p_g(S')$ by [14]. ∎

We refer to [1] for more results of the above type. We also give the following numerical result when S has a free pencil.

Theorem 4.3 *Suppose that S has a free hyperelliptic pencil of genus $g \geq 6$. Then*

$$(\sigma^* K_{S'})L \geq 3h^0(L) - (4g-4).$$

Proof Denote by C a general member of this pencil. Up to birational transformation, we can assume that $|L|$ is base point free. In particular, $LC \leq 2g-2$ should be an even number.

If $LC = 0$, then $L \sim_{\text{num}} aC$, where $a \geq h^0(L) - 1$. Then

$$(\sigma^* K_{S'})L = a(2g-2) > 3h^0(L) - 3.$$

If $LC \geq 6$, by [35, Theorem 1.1], one has

$$L^2 \geq \frac{4LC}{LC+2} h^0(L) - 2LC \geq 3h^0(L) - (4g-4).$$

If $2 \leq LC \leq 4$, resume the notation from Theorem 3.1, Proposition 3.3, and Lemma 3.4. Denote $L_0 = L$ and $P = \sigma^* K_{S'}$. We have

$$h^0(L_0) \leq \sum_{i=0}^{N} a_i r_i, \quad PL_0 \geq (a_0 - 1)d_0 + \sum_{i=1}^{N} a_i d_i.$$

Here, $d_i = 2g-2 \geq 10$ for all i and $r_i \leq h^0(L|_C) \leq 3$ by the Clifford's inequality. It follows that $d_i > 3r_i$. Hence,

$$PL_0 \geq \sum_{i=0}^{N} a_i d_i - d_0 > 3\sum_{i=0}^{N} a_i r_i - d_0 \geq 3h^0(L_0) - (2g-2),$$

which completes the proof. ∎

5 Relative Noether Inequalities

In this section, we will prove several relative Noether inequalities. The relative Noether inequality in [36] studies linear series on fibered varieties over curves, while the relative Noether inequalities in this section are devoted to studying fibered varieties whose fibers are curves.

Assumption 5.1 Throughout this section, we assume the following.

- X is a Gorenstein minimal projective 3-fold of general type.
- $f\colon X \to Y$ is a fibration of curves of genus g from X to a normal projective surface Y.
- G is a nef and big line bundle on Y such that $|G|$ is base point free. Write $B = f^*G$.
- If $p_g(X) > 0$, write

$$K_X = \sum_{i=1}^{I_0} H_i + V,$$

  where each H_i is an irreducible and reduced horizontal divisor (H_i and H_j might be the same) and V is the vertical part. Note that we have $I_0 \leq 2g-2$. Since $B|_{H_i}$ is big on H_i, we can find an integer $k > 0$ such that $(kB - K_X)|_{H_i}$ is pseudo-effective for each i. We can also assume that $kB - V$ is pseudo-effective by increasing k.

Theorem 5.2 *Assumption 5.1 holds.*

(i) *We have*

$$h^0(K_X) - \frac{K_X^3}{4} \le \frac{3K_X^2 B + 2K_X B^2}{4} + 2(2k+1).$$

(ii) *If X has no hyperelliptic fibration over surfaces, then*

$$h^0(K_X) - \frac{K_X^3}{6} \le \frac{3K_X^2 B + 2K_X B^2}{6} + \frac{7}{3}(2k+1).$$

(iii) *If f is hyperelliptic and $g \ge 6$, then*

$$h^0(K_X) - \frac{K_X^3}{6} \le \frac{3K_X^2 B + 2K_X B^2}{6} + \frac{4g-4}{3}(2k+1).$$

Proof To prove these results, one can assume that $h^0(K_X) > 0$.

Choose two very general members in $|B|$ and denote by $\sigma: X_0 \to X$ the blow-up of their intersection. Let F be its proper transform. Then we get a fibration $X_0 \to \mathbb{P}^1$ with general fiber F. Denote $L_0 = \sigma^* K_X$, $P_0 = \sigma^*(K_X + B)$ and $B_0 = \sigma^* B$.

Now, apply Theorem 3.1 to X_0, L_0, P_0 and use the notation from Proposition 3.3. We get

$$h^0(L_0) \le \sum_{i=0}^{N} a_i r_i, \quad P_0 L_0^2 \ge 2\sum_{i=0}^{N} a_i d_i - 2d_0.$$

The proofs of the three inequalities are quite similar. We will give the detailed proof of (i) and sketch the others.

To prove (i), note that $P_0|_F = K_F$. By the classical Noether inequality (since it might happen that $r_0 = p_g(F)$), Propositions 4.1 and 4.2 (when $r_i < p_g(F)$), we know that

$$r_i \le \frac{1}{2} d_i + 2.$$

Here, because B is movable and K_X is nef and big,

$$d_0 = (P_0|_F)(L_0|_F) = (K_X + B)K_X B > 0.$$

By Lemma 3.4, we have

$$\sum_{i=0}^{N} a_i \le \frac{P_0 L_0^2}{d_0} - a_0 + 2 \le \frac{P_0 L_0^2}{d_0} + 1.$$

Combine the above inequalities and we get

$$h^0(L_0) - \frac{P_0 L_0^2}{4} \le 2\sum_{i=0}^{N} a_i + \frac{d_0}{2} \le \frac{2P_0 L_0^2}{d_0} + 2 + \frac{d_0}{2}.$$

On the other hand, note that $kB - V$ and $(kB - K_X)|_{H_i}$ are pseudo-effective. We can get

$$(K_X + B)K_X V \le k(K_X + B)K_X B,$$

$$\sum_{i=1}^{I_0} (K_X + B)K_X H_i \le k\sum_{i=1}^{I_0} (K_X + B)BH_i \le k(K_X + B)K_X B.$$

As a result, we have

$$\frac{P_0 L_0^2}{d_0} = \frac{K_X^2(K_X + B)}{(K_X + B)K_X B} = \frac{(K_X + B)K_X V}{(K_X + B)K_X B} + \sum_{i=1}^{I_0} \frac{(K_X + B)K_X H_i}{(K_X + B)K_X B} \le 2k.$$

Therefore,

$$\begin{aligned} h^0(K_X) &\le \frac{P_0L_0^2}{4} + \frac{2P_0L_0^2}{d_0} + \frac{d_0}{2} + 2 \\ &= \frac{K_X^3 + K_X^2B}{4} + 4k + \frac{K_X^2B + K_XB^2}{2} + 2 \\ &= \frac{K_X^3}{4} + \frac{3K_X^2B + 2K_XB^2}{4} + 2(2k+1). \end{aligned}$$

This completes the proof of (i).

For (ii), by the assumption, F has no hyperelliptic pencil. By the classical Castelnuovo inequality (when $r_0 = p_g(F)$), Propositions 4.1 and 4.2 (when $r_i < p_g(F)$), we have

$$r_i \le \frac{1}{3}d_i + \frac{7}{3}.$$

Combining with Lemma 3.4 gives

$$h^0(L_0) - \frac{P_0L_0^2}{6} \le \frac{7}{3}\sum_{i=0}^{N} a_i + \frac{d_0}{3} \le \frac{7P_0L_0^2}{3d_0} + \frac{7}{3} + \frac{d_0}{3}.$$

Hence,

$$\begin{aligned} h^0(K_X) &\le \frac{P_0L_0^2}{6} + \frac{7P_0L_0^2}{3d_0} + \frac{7}{3} + \frac{d_0}{3} \\ &\le \frac{K_X^3 + K_X^2B}{6} + \frac{7}{3}(2k+1) + \frac{K_X^2B + K_XB^2}{3} \\ &= \frac{K_X^3}{6} + \frac{3K_X^2B + 2K_XB^2}{6} + \frac{7}{3}(2k+1). \end{aligned}$$

To prove (iii), note that F has a free of genus g induced by f. By Theorem 4.3, we have

$$r_i \le \frac{1}{3}d_i + \frac{4g-4}{3}.$$

Follow the above proof almost verbatim, and we can get

$$\begin{aligned} h^0(K_X) &\le \frac{K_X^3 + K_X^2B}{6} + \frac{4g-4}{3}(2k+1) + \frac{K_X^2B + K_XB^2}{3} \\ &= \frac{K_X^3}{6} + \frac{3K_X^2B + 2K_XB^2}{6} + \frac{4g-4}{3}(2k+1). \end{aligned}$$

We leave the detailed proof to the interested reader. ∎

6 Proof of Theorem 1.3

Note that the strategy here has been used by Pardini [32] in the proof of Severi inequality for surfaces.

We first generalize a theorem that was used in [36].

Theorem 6.1 ([36]) *Let X be a projective, minimal, normal, and irregular variety. Denote by $a(X)$ its Albanese image. For each $d \in \mathbb{N}$, let X_d be the d-th Albanese lifting of X. Then for each $i = 0, \dots, \dim a(X) - 1$, we have*

$$\lim_{d\to\infty} \frac{h^i(\mathcal{O}_{X_d})}{d^{2m}} = 0.$$

Here, $m = h^1(\mathcal{O}_X)$.

Proof Since the Albanese lifting is étale, X_d is also minimal. Furthermore, X_d has only terminal singularities, which are rational. Hence it suffices to assume that X is smooth. In this case, the result is just [36, Theorem 4.1]. Also see [19, Remark 1.4] for the generic vanishing theorem for singular varieties with rational singularities in characteristic 0. ■

Go back to the 3-fold case. Now assume that X has Albanese dimension two. Let X_d be the d-th Albanese lifting of X. We have the diagram

$$\begin{array}{ccc} X_d & \xrightarrow{\phi_d} & X \\ {\scriptstyle \alpha_d}\downarrow & & \downarrow{\scriptstyle \alpha=\mathrm{Alb}_X} \\ A & \xrightarrow{\mu_d} & A. \end{array}$$

Let $m = h^1(\mathcal{O}_X)$. We have

$$h^0(K_{X_d}) \geq \chi(\omega_{X_d}) + h^2(\mathcal{O}_{X_d}) - h^1(\mathcal{O}_{X_d}) + 1 \geq d^{2m}\chi(\omega_X) - h^1(\mathcal{O}_{X_d}).$$

By Theorem 6.1, it follows that

$$h^0(K_{X_d}) \geq d^{2m}\chi(\omega_X) + o(d^{2m}).$$

Note that in order to prove Theorem 1.3, we can assume that $\chi(\omega_X) > 0$. Thus one can find an integer $d > 0$ such that $h^0(K_{X_d}) > 0$. Also note that Theorem 1.3 is true up to étale covers. Without loss of generality, let us assume that $h^0(K_X) > 0$.

Let

$$X \xrightarrow{g_0} Y \xrightarrow{h_0} A \quad (\text{resp. } X_d \xrightarrow{g_d} Y_d \xrightarrow{h_d} A)$$

be the Stein factorization of α (resp. α_d). Let H be a sufficiently ample line bundle on A and $L_d = h_d^* H$ for all d. Write $B_d = g_d^* L_d$ for all d. Then we have ([4, Chapter 2, Proposition 3.5])

$$d^2 B_d \sim_{\text{num}} \phi_d^* B_0.$$

Resume the notation from the previous section. We can write

$$K_{X_d} = \phi_d^* K_X = \sum_{i=1}^{I_0} \phi_d^* H_i + \phi_d^* V.$$

By pulling back from X to X_d, one can check that

$$(k g_d^* B_0 - K_{X_d})|_{\phi_d^* H_i}, \quad \text{and} \quad k\phi_d^* B_0 - \phi_d^* V$$

are both pseudo-effective. Apply the above numerical equivalence and we get

$$(kd^2 B_d - K_{X_d})|_{\phi_d^* H_i}, \quad kd^2 B_d - \phi_d^* V$$

https://doi.org/10.4153/CJM-2014-033-0 Published online by Cambridge University Press

are both pseudo-effective. Now by Theorem 5.2(i),

$$h^0(K_{X_d}) \le \frac{K_{X_d}^3}{4} + \frac{3K_{X_d}^2 B_d + 2K_{X_d}B_d^2}{4} + 2(2kd^2+1)$$
$$= \frac{d^{2m}K_X^2}{4} + \frac{3d^{2m-2}K_X^2 B_0 + 2d^{2m-4}K_X B_0^2}{4} + 2(2kd^2+1).$$

Let $d \to \infty$. We can prove that $K_X^3 \ge 4\chi(\omega_X)$.

When the Albanese fiber is hyperelliptic of genus $g \ge 6$, using the same approach as above and by Theorem 5.2(iii), we can prove

$$h^0(K_{X_d}) \le \frac{d^{2m}K_X^2}{6} + \frac{3d^{2m-2}K_X^2 B_0 + 2d^{2m-4}K_X B_0^2}{6} + \frac{4g-4}{3}(2kd^2+1).$$

Let $d \to \infty$ and we will have $K_X^3 \ge 6\chi(\omega_X)$.

Now we consider the case when the Albanese fiber is not hyperelliptic.

We first show that, up to étale covers, we have $h^0(K_X - B_0) > 0$, *i.e.*, $|B_0| \subset |K_X|$. This will imply that the Albanese fibration factors through the canonical map of X, because the map given by $|B_0|$ is the same as the Albanese fibration.

Choose a very general member $M \in |B_d|$. Since $h^0(K_{X_d}) > 0$, by adjunction, $h^0(K_M) > 0$. Apply the classical Noether inequality, and we get

$$h^0(K_{X_d}|_M) \le h^0(K_M) \le \frac{1}{2}K_M^2 + 2 = \frac{1}{2}(K_{X_d} + B_d)^2 B_d + 2 \sim o(d^{2m}).$$

Therefore, up to étale covers, we can assume that

$$h^0(K_X) - h^0(K_X|_M) > 0$$

for a general member $M \in |B_0|$, which means that $h^0(K_X - B_0) > 0$.

Second, we claim that X *cannot have hyperelliptic pencils*. Otherwise, suppose there is a hyperelliptic pencil on X. We would have the following two possibilities. Either this pencil is contracted by ϕ_{K_X} or it is not. If it is contracted by ϕ_{K_X}, it would also be contracted by the Albanese map, which is impossible, since the Albanese pencil is nonhyperelliptic. The second case is still impossible, because if this pencil is not contracted by ϕ_{K_X}, then its image under ϕ_{K_X} is a $\mathbb{P}^1$ pencil. Since the Albanese image of X cannot have any $\mathbb{P}^1$ pencil, by the factorization of the Albanese map, this pencil has to be contracted by the Albanese map, which contradicts our assumption again.

Similar to the above claim, we can prove that X_d has no hyperelliptic pencil. Thus by Theorem 5.2(ii), we have

$$h^0(K_{X_d}) \le \frac{d^{2m}K_X^3}{6} + \frac{3d^{2m-2}K_X^2 B_0 + 2d^{2m-4}K_X B_0^2}{6} + \frac{7}{3}(2kd^2+1).$$

As before, we can get $K_X^3 \ge 6\chi(\omega_X)$ by letting $d \to \infty$.

We remark here that if X is $\mathbb{Q}$-Gorenstein but not Gorenstein, $\chi(\omega_X)$ might be negative by looking at the following example. Note that the construction has appeared in many places in the literature (*e.g.*, [30, Remark 8.7], [16, Example 1.13], etc.).

Example 6.2 Fix an elliptic curve E. Take two curves C_i $(i = 1, 2)$ with genus $g_i \ge 2$ and involutions τ_i such that $C_1/\langle\tau_1\rangle = E$ and $C_2/\langle\tau_2\rangle = \mathbb{P}^1$. Let the 3-fold X be the quotient of $C_1 \times C_1 \times C_2$ by the diagonal involution. Hence, X is $\mathbb{Q}$-Gorenstein but not Gorenstein. Let $\alpha: X \to E \times E \times \mathbb{P}^1$ be the induced cover. Then

$$\alpha_* \mathcal{O}_X = \mathcal{O}_E \boxtimes \mathcal{O}_E \boxtimes \mathcal{O}_{\mathbb{P}^1} \oplus L_1 \boxtimes L_1 \boxtimes \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_E \boxtimes L_1 \boxtimes L_2 \oplus L_1 \boxtimes \mathcal{O}_E \boxtimes L_2,$$

where L_1 and L_2 are determined by the branch locus of $C_1 \to E$ and $C_2 \to \mathbb{P}^1$. It is easy to see that $\chi(\mathcal{O}_X) > 0$. Hence $\chi(\omega_X) < 0$. It is also easy to see that X has Albanese dimension two.

7 Proof of Theorem 1.5: Part 1

In this section, we will prove Theorem 1.5(i)–(iii).

We have the following relative Noether inequality for fibered 3-folds.

Theorem 7.1 *Let X be a Gorenstein minimal 3-fold of general type, let Y be a smooth curve, and let $f: X \to Y$ be a fibration with a smooth general fiber F. Then*

$$h^0(\omega_{X/Y}) \leq \Big(\frac{1}{4} + \frac{1}{K_F^2}\Big)\omega_{X/Y}^3 + \frac{K_F^2 + 4}{2},$$

except for the case when $K_F^2 = 1$, $p_g(F) = 2$, and $q(F) = 0$.

Proof We know in this case that $\omega_{X/Y}$ is nef. Resume the notations in Theorem 3.1 and Proposition 3.3. Denote $L_0 = P = \omega_{X/Y}$. It follows that

$$h^0(L_0) \leq \sum_{i=0}^{N} a_i r_i,$$

$$L_0^3 \geq 2a_0 d_0 + \sum_{i=1}^{N} a_i(d_{i-1} + d_i) - 2d_0.$$

Hence

$$h^0(L_0) - \frac{L_0^3}{4} \leq \frac{d_0}{2} + \Big(r_0 - \frac{1}{2}d_0\Big)a_0 + \sum_{i=1}^{N}\Big(r_i - \frac{1}{2}d_i - \frac{d_{i-1} - d_i}{4}\Big)a_i.$$

By the classical Noether inequality, Propositions 4.1, 4.2, and Remark 3.2, we can always get

$$r_0 \leq \frac{1}{2}d_0 + 2, \quad r_i \leq \frac{1}{2}d_i + \frac{d_{i-1} - d_i}{4} + 1 \quad (i > 0)$$

except when $r_0 = r_1$ and $d_1 \leq 2r_1 - 3$.

If we are not in the exceptional case, then

$$h^0(L_0) - \frac{L_0^3}{4} \leq \frac{d_0}{2} + 2a_0 + \sum_{i=1}^{N} a_i.$$

By Lemma 3.4,

$$2a_0 + \sum_{i=1}^{N} a_i \leq \frac{L_0^3}{d_0} + 2 \leq \frac{L_0^3}{d_0} + 2.$$

Hence,

$$h^0(L_0) \leq \Big(\frac{1}{4} + \frac{1}{d_0}\Big)L_0^3 + \frac{d_0 + 4}{2}.$$

https://doi.org/10.4153/CJM-2014-033-0 Published online by Cambridge University Press

Finally, let us consider the exceptional case. Recall that we have $r_0 = r_1$ and $d_1 \le 2r_1 - 3$. By Proposition 4.2, $|L_1|_{F_1}|$ defines a generically finite morphism. Thus from Proposition 4.1, we know that $(L_1|_{F_1})^2 \ge 2r_1 - 4$. Note that $\pi_0^* K_F > L_1|_{F_1}$. By the Hodge index theorem,

$$(2r_1 - 3)^2 \ge d_1^2 = ((\pi_0^* K_F)(L_1|_{F_1}))^2 > (L_1|_{F_1})^2 K_F^2,$$

which implies that $K_F^2 \le 2r_0 - 2$. On the other hand, $|K_F|$ is not base point free, thus $K_F^2 \ge 2r_0 - 3$. It also implies that $d_1 = 2r_1 - 3$ and $(L_1|_{F_1})^2 = 2r_1 - 4$. Therefore, we have two possibilities:

(a) $d_1 = 2r_1 - 3$ and $K_F^2 = d_1$;
(b) $d_1 = 2r_1 - 3$ and $K_F^2 = d_1 + 1$.

In Case (a), choose a blow up $\pi\colon X' \to X$ such that the movable part $|M|$ of $|\pi^* \omega_{X/Y}|$ is base point free. Write $F' = \pi^* F$. Abusing the notation, we denote the new $L_0 = M$ and $P = \pi^* \omega_{X/Y}$. Then we will have a new sequence of r_i's and d_i's. Under this new setting, $r_{i-1} > r_i$ for each $i > 0$. Running the same process as before, we can get

$$h^0(\omega_{X/Y}) = h^0(M) \le \Big(\frac{1}{4} + \frac{1}{d_M}\Big)\omega_{X/Y}^3 + \frac{d_M + 4}{2},$$

where $d_M = (\pi^* K_F)(M|_{F'})$. We need to show that $d_M = K_F^2$. In fact, $d_M = K_F(\pi(M)|_F) \le K_F^2$. On the other hand, since L_1 comes from the movable part of $|\omega_{X/Y} - a_0 F|$, it implies that $\pi_0(L_1) \le \pi(M)$. In particular, it means that

$$K_F^2 = d_1 = K_F(\pi_0(L_1)|_F) \le K_F(\pi(M)|_F) = d_M.$$

Hence $d_M = K_F^2$.

We claim that Case (b) does not occur. Write

$$|K_F| = |V| + Z,$$

where $|V|$ is the movable part of $|K_F|$ and $Z \ge 0$. Since $r_1 = r_0$, we see that $\pi_0(L_1)|_F = V$, and so $d_1 = K_F V$. Since $K_F^2 > d_1$, it means that $K_F Z > 0$ and $Z > 0$. By the 2-connectedness of the canonical divisor, we have $VZ \ge 2$. Note that $V^2 \ge 2r_0 - 4$. This implies that

$$2r_0 - 2 = K_F^2 = V^2 + VZ + K_F Z \ge 2r_0 - 2 + K_F Z,$$

i.e., $K_F Z = 0$ and $K_F V = K_F^2$. This contradicts $K_F^2 > d_1$. ∎

Remark 7.2 Recall from [35] that if $f\colon S \to C$ is a relative minimal fibered surface of genus $g \ge 2$. One has

$$h^0(\omega_{S/C}) \le \frac{g}{4g-4}\omega_{S/C}^2 + g = \Big(\frac{1}{4} + \frac{1}{2\deg\omega_F}\Big)\omega_{S/C}^2 + \frac{\deg\omega_F + 2}{2}.$$

Theorem 7.1 is a natural generalization of the above result. Moreover, based on several results concerning linear series on surfaces in positive characteristic (*cf.* [25, 26]), our method can also be used to study the fibered 3-folds in positive characteristic.

It is also interesting to compare Theorem 7.1 with the slope inequalities proved by Ohno [31] and Barja [1] for fibered 3-folds over curves. In their papers, Xiao's method on the Harder–Narasimhan filtration [34] plays a very important role, and it works only in characteristic zero.

By the same technique, we can also show the following Noether type inequality for fibered 3-folds over curves in order for independent interests.

Theorem 7.3 *Let X be a Gorenstein and minimal fibered 3-fold of general type fibered over a smooth curve Y with a smooth general fiber F. Then*

$$p_g(X) \leq \Big(\frac{1}{4} + \frac{1}{K_F^2}\Big) K_X^3 + \frac{K_F^2 + 4}{2}$$

except for the case when $K_F^2 = 1$, $p_g(F) = 2$, and $q(F) = 0$.

Proof We only need to replace $\omega_{X/Y}$ by K_X in the proof of Theorem 7.1. ∎

From now on, we assume that $f\colon X \to Y$ is the induced fibration by the Albanese map from X to a projective curve Y with a smooth general fiber F. One has $g(Y) = h^1(\mathcal{O}_X)$.

Remark 7.4 Suppose we can prove that

$$K_X^3 \geq a_F \chi(\omega_X) + b_F,$$

where a_F and b_F only depend on the numerical invariants of F. Then we can get

$$K_X^3 \geq a_F \chi(\omega_X).$$

This philosophy has been applied by Bombieri [5], Horikawa [20], and Yuan and Zhang [35]. In fact, since $g(Y) = h^1(\mathcal{O}_X) > 0$, using the degree d étale base change $\pi\colon Y' \to Y$, we can get a new fibration $f'\colon X' \to Y'$, where $X' = X \times_Y Y'$. We still have

$$K_{X'}^3 \geq a_F \chi(\omega_{X'}) + b_F.$$

Note that

$$K_{X'}^3 = dK_X^3, \quad \chi(\omega_{X'}) = d\chi(\omega_X).$$

The conclusion will follow after we let $d \to \infty$. This remark will also be used in the next section.

Proposition 7.5 *If $p_g(F) = 0$, then $K_X^3 \geq 6\chi(\omega_X)$.*

Proof In this situation, $p_g(X) = 0$. Hence,

$$\chi(\omega_X) = p_g(X) - h^2(\mathcal{O}_X) + h^1(\mathcal{O}_X) - 1 \leq g(Y) - 1.$$

From the nefness of $\omega_{X/Y}$, we get $\omega_{X/Y}^3 \geq 0$. Since $\omega_{X/Y} = K_X - (2g(Y) - 2)F$, we get

$$K_X^3 \geq 6(g(Y) - 1)K_F^2 \geq 6(g(Y) - 1) \geq 6\chi(\omega_X).$$ ∎

Proposition 7.6 *If $p_g(F) > 0$ and $(p_g(F), K_F^2) \neq (2,1)$, then*

$$\chi(\omega_X) \leq \Big(\frac{1}{4} + \frac{1}{K_F^2}\Big) K_X^3.$$

https://doi.org/10.4153/CJM-2014-033-0 Published online by Cambridge University Press

Proof From Theorem 7.1, we get

$$h^0(\omega_{X/Y}) \le \Big(\frac{1}{4}+\frac{1}{K_F^2}\Big)\big(K_X^3 - 6K_F^2(g(Y)-1)\big) + \frac{K_F^2+4}{2}$$
$$= \Big(\frac{1}{4}+\frac{1}{K_F^2}\Big)K_X^3 - 6\Big(\frac{K_F^2}{4}+1\Big)\big(g(Y)-1\big) + \frac{K_F^2+4}{2}.$$

On the other hand, by [23, 24] $f_*\omega_{X/Y}$ and $R^1f_*\omega_{X/Y}$ are both semipositive. Following [31, Lemma 2.4, 2.5], we have

$$h^0(\omega_{X/Y}) \ge \deg f_*\omega_{X/Y} - p_g(F)\big(g(Y)-1\big)$$
$$\ge \deg f_*\omega_{X/Y} - \deg R^1f_*\omega_{X/Y} - p_g(F)\big(g(Y)-1\big)$$
$$= \chi(\omega_X) - \big(\chi(\mathcal{O}_F)+p_g(F)\big)\big(g(Y)-1\big).$$

By applying Remark 7.4, to prove the conclusion, it suffices to prove that

$$6\Big(\frac{K_F^2}{4}+1\Big) \ge \chi(\mathcal{O}_F)+p_g(F).$$

It is easy to see that this inequality follows from the classical Noether inequality, since

$$6\Big(\frac{K_F^2}{4}+1\Big) > K_F^2+6 \ge 2p_g(F)+2 > \chi(\mathcal{O}_F)+p_g(F).$$

This finishes the proof. ∎

From the above proposition, we see that Theorem 1.5 holds if $K_F^2 \ge 4$. In the following, we will consider the case when $K_F^2 \le 3$.

Proposition 7.7 *We have $K_X^3 \ge 2\chi(\omega_X)$ in the following two cases:*

(i) $p_g(F) = 2$, $K_F^2 = 2, 3$;
(ii) $p_g(F) = 3$, $K_F^2 = 3$.

Proof We first prove that

$$\omega_{X/Y}^3 \ge 2h^0(\omega_{X/Y}) - 6.$$

Suppose we have proven the above result. As before, we still have

$$h^0(\omega_{X/Y}) \ge \chi(\omega_X) - \big(\chi(\mathcal{O}_F)+p_g(F)\big)(g(Y)-1),$$
$$\omega_{X/Y}^3 = K_X^3 - 6K_F^2\big(g(Y)-1\big).$$

It is easy to check that

$$6K_F^2 \ge 2(2p_g(F)+1) \ge 2\big(\chi(\mathcal{O}_F)+p_g(F)\big)$$

in these cases. Hence it will imply that $K_X^3 \ge 2\chi(\omega_X)$ by Remark 7.4.

To prove that $\omega_{X/Y}^3 \ge 2h^0(\omega_{X/Y}) - 6$, we can assume that $h^0(\omega_{X/Y}) \ge 4$. Hence we have the relative canonical map $\phi_{\omega_{X/Y}}$. Note that $p_g(F) \le 3$, and we have

$$0 \longrightarrow H^0\big(\omega_{X/Y}(-F)\big) \longrightarrow H^0\big(\omega_{X/Y}\big) \longrightarrow H^0(K_F).$$

Thus $h^0(\omega_{X/Y}(-F)) > 0$.

Choose a blow-up $\pi: X' \to X$ such that the movable part $|M|$ of $|\pi^*\omega_{X/Y}|$ is base point free. Write $F' = \pi^*F$ and S as a general member of $|M|$.

If $\dim \phi_{\omega_{X/Y}}(X) = 1$, since $h^0(\omega_{X/Y}(-F)) > 0$ and F is a free pencil, we know that the pencil induced by $\phi_{\omega_{X/Y}}$ is the same as f. Hence we can write $M \sim_{\text{num}} aF' + Z$, where $Z \geq 0$. Also, because Y is an irrational curve that dominates the Albanese image of X, $a \geq h^0(\omega_{X/Y})$. Hence

$$\omega_{X/Y}^3 \geq a(\pi^*\omega_{X/Y})^2 F' = aK_F^2 \geq 2h^0(\omega_{X/Y}).$$

If $\dim \phi_{\omega_{X/Y}}(X) = 2$, denote by C' a general member of the induced pencil by ϕ_M. Then $M|_S$ is a free pencil and $M|_S \sim_{\text{num}} aC'$, where $a \geq h^0(\omega_{X/Y}) - 2$. Hence,

$$\omega_{X/Y}^3 \geq \big((\pi^*\omega_{X/Y})C'\big)\big(h^0(\omega_{X/Y}) - 2\big).$$

On the other hand, since $h^0(\omega_{X/Y}(-F)) > 0$, we know that $M|_{F'}$ is also a free pencil with the same numerical type as C'. In particular, we can find a general F' such that $C' \subset F'$. Then $(\pi^*\omega_{X/Y})C' = K_F C$, where $C = \pi(C')$. Note that $g(C) \geq 2$ and $K_F^2 \geq 2$. By the Hodge index theorem, we can get $K_F C \geq 2$, so

$$\omega_{X/Y}^3 \geq 2(h^0(\omega_{X/Y}) - 2).$$

If $\dim \phi_{\omega_{X/Y}}(X) = 3$, then $\dim \phi_M(S) = 2$. By Proposition 4.1,

$$(M|_S)^2 \geq 2h^0(M|_S) - 4,$$

which implies

$$\omega_{X/Y}^3 \geq M^3 \geq 2h^0(M|_S) - 4 \geq 2h^0(\omega_{X/Y}) - 6,$$

which completes the proof. ∎

In fact, the above method has been applied by Chen [11] for the study of the canonical linear system. Here we use this method for the relative canonical linear system.

Proposition 7.8 *If $p_g(F) = 3$ and $K_F^2 = 2$, then $K_X^3 \geq \frac{12}{7}\chi(\omega_X)$.*

Proof In this case, since $K_F^2 < 2p_g(F)$ by [14], $h^1(\mathcal{O}_F) = 0$ and $\chi(\mathcal{O}_F) = 4$. Apply the same method as in Proposition 7.7. We can still get

$$\omega_{X/Y}^3 \geq 2h^0(\omega_{X/Y}) - 6,$$
$$h^0(\omega_{X/Y}) \geq \chi(\omega_X) - 7(g(Y) - 1),$$
$$\omega_{X/Y}^3 = K_X^3 - 12(g(Y) - 1).$$

Then the result follows from Remark 7.4. ∎

Remark 7.9 In fact, from the above inequalities, we can also get

$$K_X^3 \geq 2\chi(\omega_X) - 2h^1(\mathcal{O}_X) - 4.$$

But here $h^1(\mathcal{O}_X)$ is still involved.

Proposition 7.10 *If $p_g(F) = 2$ and $K_F^2 = 1$, then $K_X^3 \geq \frac{4}{3}\chi(\omega_X)$.*

Proof By a very recent result of Hu (*cf.* [21]), we know that in this case,

$$K_X^3 \geq \frac{4}{3}\chi(\omega_X) - 2.$$

Therefore, the result follows from Remark 7.4. ∎

Proposition 7.11 *If $p_g(F) = 1$, then $K_X^3 \geq 2\chi(\omega_X)$.*

Proof We separate the proof into two cases, since the proofs are quite different.

Case 1. $K_F^2 > 1$.

First, let us assume that $p_g(X) \geq 2$. Then the canonical maps of X will factor through f. So we can write $K_X \sim_{\text{num}} rF + Z$, where $r \geq p_g(X)$ and $Z \geq 0$. It follows that

$$\begin{aligned} K_X^3 &= K_X^2(\omega_{X/Y} + (2g(Y)-2)F) = (2g(Y)-2)K_F^2 + \omega_{X/Y}K_X^2 \\ &\geq (2g(Y)-2)K_F^2 + r\omega_{X/Y}K_XF = (r + 2g(Y) - 2)K_F^2 \geq 2\chi(\omega_X). \end{aligned}$$

Second, if $p_g(X) \leq 1$ and $h^1(\mathcal{O}_X) > 1$, then $\chi(\omega_X) \leq h^1(\mathcal{O}_X)$. We have

$$K_X^3 \geq 6(g(Y)-1)K_F^2 \geq 2\chi(\omega_X).$$

The only missing case is when $p_g(X) \leq 1$ and $h^1(\mathcal{O}_X) = 1$. In this case, $\chi(\omega_X) \leq 1$ and $g(Y) = 1$. $\chi(\omega_X) = 0$ is absurd. So we can assume that $\chi(\omega_X) = 1$. Now let $\mu: Y \to Y$ be any nontrivial étale map and let $X' = X \times_\mu Y$. We have $\chi(\omega_{X'}) > 1$. So either $p_g(X') \geq 2$ or $h^1(\mathcal{O}_{X'}) \geq 2$. If X' has Albanese dimension ≥ 2, then by Theorem 1.3, $K_{X'}^3 \geq 4\chi(\omega_{X'})$. If not, we are in one of the first two cases and $K_{X'}^3 \geq 2\chi(\omega_{X'})$. In either situation, this will imply that $K_X^3 \geq 2\chi(\omega_X)$.

Case 2. $K_F^2 = 1$.

Here we prove this result by studying the linear system $|2K_X|$.

Since $p_g(F) = 1$ and $K_F^2 = 1$, $h^1(\mathcal{O}_F) = 0$, $h^0(2K_F) = 3$, and $|2K_F|$ is base point free (see [17]). Hence ϕ_{2K_F} is a generically finite morphism of degree 4.

As before, we choose a blow-up $\pi: X' \to X$ such that the movable part $|M|$ of $|\pi^*(2K_X)|$ is base point free. Note that by the plurigenus formula of Reid,

$$h^0(M) = h^0(2K_X) \geq \frac{1}{2}K_X^3 - 3\chi(\mathcal{O}_X).$$

Denote $F' = \pi^*F$. Consider the following restriction map:

$$\text{res}: H^0(X', M) \to H^0(F', M|_{F'}).$$

Denote by r the dimension of its image. So $1 \leq r \leq 3$.

If $r = 1$, then $\phi_M(X')$ is a curve and ϕ_M factors through the fibration $X' \to Y$. In this case, $M \sim_{\text{num}} aF' + Z$ with $a \geq h^0(M)$. Hence we have

$$2K_X^3 \geq M(\pi^*K_X)^2 \geq aK_F^2 \geq \frac{1}{2}K_X^3 - 3\chi(\mathcal{O}_X),$$

i.e., $K^3 \geq 2\chi(\omega_X)$.

If $r = 2$, write $L_0 = M$ and $P = \pi^*(2K_X)$. Resume the notation from Theorem 3.1 and Proposition 3.3. Similar to Theorem 7.1, we have

$$h^0(L_0) - \frac{PL_0^2}{4} \leq \frac{d_0}{2} + \Big(r_0 - \frac{1}{2}d_0\Big)a_0 + \sum_{i=1}^{N}\Big(r_i - \frac{1}{2}d_i - \frac{d_{i-1}-d_i}{4}\Big)a_i.$$

Here $r_0 = h^0(M|_{F'})$ and $d_0 = 2(\pi^*K_F)(M|_{F'})$. Also, by Proposition 3.3, the above inequality still holds if we replace r_0 by r.

We claim that $d_0 = 4$ in this case. In fact, we know that $d_0 \leq 4K_F^2 = 4$. If $r_0 = 3$, then $M|_{F'} = \pi^*(2K_F)$ and $d_0 = 4K_F^2 = 4$. If not, then $r_0 = r = 2$. By [11, Lemma 2.5], we know that $(\pi^*K_F)(M|_{F'}) \geq 2$, which still gives $d_0 \geq 4$. Hence the claim is true, and we have $r - \frac{1}{2}d_0 \leq 0$.

For $i > 0$, we know that $r_i < r_0 = 3$ by Remark 3.2. Moreover, by [11, Lemma 2.5] again, $d_i \geq 4$ if $r_i = 2$, and $r_{i-1} \geq 2$ if $r_i = 1$, which implies $d_{i-1} \geq 4$. From this, one can check that for any $i > 0$,

$$r_i - \frac{1}{2}d_i - \frac{d_{i-1}-d_i}{4} \leq 0.$$

Therefore, we have

$$h^0(2K_X) - 2K_X^3 \leq h^0(L_0) - \frac{PL_0^2}{4} \leq 2.$$

If $r = 3$, then $\phi_M|_F = \phi_{2K_F}$. Thus, ϕ_M is generically finite of degree 4. Choose a general member $S \in |M|$. We have

$$(M|_S)^2 \geq 4\big(h^0(M|_S) - 2\big).$$

Hence,

$$K_X^3 \geq \frac{1}{8}M^3 \geq \frac{1}{2}\big(h^0(M|_S) - 2\big) \geq \frac{1}{2}\big(h^0(2K_X) - 3\big).$$

As a result, if $r \geq 2$, we always have $h^0(2K_X) - 2K_X^3 \leq 3$. Applying the plurigenus formula and Remark 7.4 to the above two cases, we can complete the entire proof. ∎

Now the proof of Theorem 1.5 is straightforward.

Proof of Theorem 1.5 (i)–(iii) From Proposition 7.6, we know that $K_X^2 \geq 2\chi(\omega_X)$ holds when $K_F^2 \geq 4$, and $K_X^2 \geq 3\chi(\omega_X)$ holds when $K_F^2 \geq 12$. If $K_F^2 \leq 4$, by Noether inequality, $p_g(F) \leq 3$. In this case, the result just comes from Propositions 7.5, 7.7, 7.8, 7.10, and 7.11. ∎

8 Proof of Theorem 1.5: Part 2

In this section, we prove the rest of Theorem 1.5. We first give a better version of the relative Noether inequality for fibered 3-folds over curves.

Theorem 8.1 *Let X be a Gorenstein minimal 3-fold of general type, let Y be a smooth curve, and let $f\colon X \to Y$ be a fibration with a smooth general fiber F. Suppose that F has no hyperelliptic pencil and $(p_g(F), K_F^2) \neq (2,2)$. Then*

$$h^0(\omega_{X/Y}) \leq \Big(\frac{1}{6} + \frac{3}{2K_F^2}\Big)\omega_{X/Y}^3 + \frac{K_F^2 + 7}{3}.$$

https://doi.org/10.4153/CJM-2014-033-0 Published online by Cambridge University Press

Proof Retain the notation from Theorem 3.1 and Proposition 3.3. Denote $L_0 = P = \omega_{X/Y}$. It follows that

$$h^0(L_0) \le \sum_{i=0}^{N} a_i r_i, \quad L_0^3 \ge 2a_0 d_0 + \sum_{i=1}^{N} a_i(d_{i-1} + d_i) - 2d_0.$$

Hence,

$$h^0(L_0) - \frac{L_0^3}{6} \le \frac{d_0}{3} + \Big(r_0 - \frac{1}{3}d_0\Big)a_0 + \sum_{i=1}^{N}\Big(r_i - \frac{1}{3}d_i - \frac{d_{i-1} - d_i}{6}\Big)a_i.$$

By the Castelnuovo inequality, we can always get $r_0 \le \frac{1}{3}d_0 + \frac{7}{3}$. Recall that by Remark 3.2, $r_0 \ge r_1 > \cdots > r_N$. We will prove in Lemma 8.2 that if $r_0 > r_1$, then

$$r_i \le \frac{1}{3}d_i + \frac{d_{i-1} - d_i}{6} + \frac{3}{2}, \quad (i > 0).$$

Assume the above result for now. It follows that

$$h^0(L_0) - \frac{L_0^3}{6} \le \frac{d_0}{3} + \frac{7}{3}a_0 + \frac{3}{2}\sum_{i=1}^{N} a_i.$$

By Lemma 3.4,

$$\frac{7}{3}a_0 + \frac{3}{2}\sum_{i=1}^{N} a_i \le \frac{3L_0^3}{2d_0} + 3 - \frac{2}{3}a_0 \le \frac{3L_0^3}{2d_0} + \frac{7}{3}.$$

Hence

$$h^0(L_0) \le \Big(\frac{1}{6} + \frac{3}{2d_0}\Big)L_0^3 + \frac{d_0 + 7}{3}.$$

Now we assume that $r_0 = r_1$. It implies that $|K_F|$ has base locus by Remark 3.2. Hence by the Castelnuovo inequality,

$$K_F^2 \ge 3p_g(F) - 6.$$

We have three exceptional cases.

Case 1. Suppose $K_F^2 \ge 3p_g(F) - 4$. We claim we still have

$$r_i \le \frac{1}{3}d_i + \frac{d_{i-1} - d_i}{6} + \frac{3}{2}, \quad (i > 0).$$

This claim is true for $i \ge 2$ by Lemma 8.2. We only need prove it for $i = 1$.

If $|K_F|$ is composed with a pencil, by Proposition 4.2, we have $r_1 \le \frac{1}{3}d_1 + 1$. Hence the claim holds.

If ϕ_{K_F} is generically finite, then by Proposition 4.1,

$$(L_1|_{F_1})^2 \ge 3r_1 - 7.$$

Therefore, by the Hodge index theorem, $d_1 \ge \sqrt{K_F^2(L_1|_{F_1})^2}$, which gives us

$$r_1 \le \frac{1}{3}d_1 + \frac{5}{3}.$$

By our assumptions on K_F^2, we know that $d_0 \ge 3r_0 - 4 \ge d_1 + 1$. One can directly check that the claim holds in this case.

Case 2. Suppose $K_F^2 \le 3p_g(F) - 5$ and $|K_F|$ has only isolated base points. Note that in this case, since $h^0(L_1|_{F_1}) = h^0(K_F)$, $|L_1|_{F_1}|$ is just the proper transform of $|K_F|$. Hence,

$$(\pi_0^* K_F)(L_1|_{F_1}) = K_F^2.$$

https://doi.org/10.4153/CJM-2014-033-0 Published online by Cambridge University Press

Again, choose a blow-up $\pi: X' \to X$ such that the movable part $|M|$ of $|\pi^*\omega_{X/Y}|$ is base point free. Let $F' = \pi^* F$. Denote the new $L_0 = M$ and $P = \pi^*\omega_{X/Y}$. We will have a new sequence of r_i's and d_i's. Under this new setting, for each $i > 0$, we have $r_{i-1} > r_i$. By Lemma 8.2,

$$r_i \le \frac{1}{3}d_i + \frac{d_{i-1} - d_i}{6} + \frac{3}{2}, \quad (i > 0),$$

in this new setting. Run the same process as the non-exceptional case, and we will have

$$h^0(\omega_{X/Y}) \le \Big(\frac{1}{6} + \frac{3}{2d_M}\Big)\omega^3_{X/Y} + \frac{d_M + 7}{3},$$

where $d_M = (\pi^* K_F)(M|_{F'})$. We only need to show that $d_M = K_F^2$. This is quite similar to Theorem 7.1. In fact, we have $\pi_0(L_1) \le \pi(M)$ by the same reason as in Theorem 7.1. It implies that $|M|_{F'}|$ is the proper transform of $|K_F|$. By our assumption, $|K_F|$ has no fixed part. Thus, $|\pi(M)|_F| = |K_F|$ and $d_M = K_F^2$.

Case 3. Suppose $K_F^2 \le 3p_g(F) - 5$ and $|K_F|$ has a fixed part; *i.e.*, $|K_F| = |V| + Z$, where $Z > 0$. In this case, $|L_1|_{F_1}|$ is the proper transform of $|V|$.

Let $\pi: X' \to X$ and $|M|$ be the same as in Case 2. We still have

$$h^0(\omega_{X/Y}) \le \Big(\frac{1}{6} + \frac{3}{2d_M}\Big)\omega^3_{X/Y} + \frac{d_M + 7}{3},$$

where $d_M = (\pi^* K_F)(M|_{F'})$. Using a similar argument, we can prove that $|\pi(M)|_F| = |V|$. As before, we only need to show that $K_F V = K_F^2$.

By the 2-connectedness of the canonical divisor, $VZ \ge 2$. Note that $V^2 \ge 3p_g(F) - 7$ by Proposition 4.1. Thus

$$3p_g(F) - 5 \ge K_F^2 \ge K_F V = V^2 + VZ \ge 3p_g(F) - 5.$$

This implies that $K_F V = 3p_g(F) - 5 = K_F^2$ and completes the proof. ∎

As before, we need to show the following lemma.

Lemma 8.2 *For any $i > 0$, if $r_{i-1} > r_i$, we have*

$$r_i \le \frac{1}{3}d_i + \frac{d_{i-1} - d_i}{6} + \frac{3}{2}.$$

Proof The lemma holds if $r_i \le \frac{1}{3}d_i + \frac{4}{3}$. If not, then by Proposition 4.1 and 4.2,

$$r_i = \frac{1}{3}d_i + \frac{5}{3}.$$

In this case, $d_{i-1} - d_i \ge 1$. Otherwise, we would have

$$r_{i-1} \ge r_i + 1 = \frac{1}{3}d_{i-1} + \frac{8}{3}.$$

This contradicts Propositions 4.1 and 4.2. ∎

From now on, we assume that X is an irregular minimal Gorenstein 3-fold of general type, $f: X \to Y$ is the fibration over a smooth curve Y induced by the Albanese map, and F is a smooth general fiber.

Proposition 8.3 *If* $p_g(F) > 0$ *and* $(p_g(F), K_F^2) \neq (2,2)$, *then*

$$\chi(\omega_X) \leq \Big(\frac{1}{6} + \frac{3}{2K_F^2}\Big) K_X^3.$$

Proof The proof is similar to Proposition 7.6. We sketch it here. From Theorem 8.1, we get

$$h^0(\omega_{X/Y}) \leq \Big(\frac{1}{6} + \frac{3}{2K_F^2}\Big) K_X^3 - (K_F^2 + 9)(g(Y) - 1) + \frac{K_F^2 + 7}{3}.$$

We still have

$$h^0(\omega_{X/Y}) \geq \chi(\omega_X) - (\chi(\mathcal{O}_F) + p_g(F))(g(Y) - 1).$$

To prove the conclusion, by applying Remark 7.4, it suffices to prove that

$$K_F^2 + 9 \geq \chi(\mathcal{O}_F) + p_g(F),$$

which follows from the Noether inequality. ■

Proof of Theorem 1.5(iv) If $p_g(F) = 0$, then the theorem is true by Proposition 7.5. If $p_g(F) > 0$, by Proposition 8.3, the theorem holds if $K_F^2 \geq 9$. ■

Remark 8.4 By the Castelnuovo inequality $K_F^2 \geq 3p_g(F) - 7$, we know that $K_F^2 \geq 9$ provided that $p_g(F) \geq 6$. In particular, it means that under the same assumption as in Theorem 1.5, we have $K_X^3 \geq 6\chi(\omega_X)$ provided that $p_g(F) \geq 6$.

As Theorem 1.5, one might guess that $K_F^2 \geq 3$ will "almost" imply that $K_X^3 \geq 3\chi(\omega_X)$. But it would be probably optimistic. It is true when $p_g(F) = 0$ by Proposition 7.5. It is also true when $p_g(F) = 1$.

Proposition 8.5 *If* $p_g(F) = 1$ *and* $K_F^2 > 2$, *then* $K_X^3 \geq 3\chi(\omega_X)$.

Proof The proof is very similar to Proposition 7.11. We omit it here. ■

Acknowledgments The author is very grateful to Miguel Barja, Jungkai Chen, Meng Chen, Xi Chen, Zhi Jiang, and Xinyi Yuan for their interest and valuable discussions related to this paper. He has benefited a lot from several discussions with Steven Lu about and beyond this subject since December 2012. The author thanks Chenyang Xu for his hospitality when the author visited BICMR. Special thanks also goes to ECNU, NUS, USTC for their hospitality. Last but not the least, the author wants to thank the referee for carefully reading this paper and giving lots of suggestions that improved the paper dramatically.

References

[1] M. A. Barja, *Lower bounds of the slope of fibred threefolds*. Internat. J. Math. **11**(2000), no. 4, 461–491.

[2] ———, *Generalized Clifford-Severi inequality and the volume of irregular varieties*. arxiv:1303.3045v2

[3] A. Beauville, *L'application canonique pour les surfaces de type général*. Invent. Math. **55**(1979), no. 2, 121–140. http://dx.doi.org/10.1007/BF01390086

[4] C. Birkenhake and H. Lange, *Complex abelian varieties*. Grundlehren der Mathematischen Wissenschaften, 302, Springer-Verlag, Berlin, 1992.
[5] E. Bombieri, *Canonical models of surfaces of general type*. Inst. Hautes Études Sci. Publ. Math. **42**(1973), 171–219.
[6] F. Catanese, M. Chen, and D.-Q. Zhang, *The Noether inequality for smooth minimal 3-folds*. Math. Res. Lett. **13**(2006), no. 4, 653–666. http://dx.doi.org/10.4310/MRL.2006.v13.n4.a13
[7] J. A. Chen and M. Chen, *The Noether inequality for Gorenstein minimal 3-folds*. arxiv:1310.7709
[8] ———, *Explicit birational geometry of threefolds of general type, I*. Ann. Sci. Éc Norm. Supér. **43**(2010), 365–394.
[9] J. A. Chen, M. Chen, and D.-Q. Zhang, *On the 5-canonical system of 3-folds of general type*. J. Reine Angew. Math. **603**(2007), 161–181.
[10] J. A. Chen and C. D. Hacon, *On the geography of threefolds of general type*. J. Algebra **321**(2009), no. 9, 2500–2507. http://dx.doi.org/10.1016/j.jalgebra.2009.01.012
[11] M. Chen, *Inequalities of Noether type for 3-folds of general type*. J. Math. Soc. Japan **56**(2004), no. 4, 1131–1155. http://dx.doi.org/10.2969/jmsj/1190905452
[12] ———, *Minimal threefolds of small slope and the Noether inequality for canonically polarized threefolds*. Math. Res. Lett. **11**(2004), no. 5–6, 833–852. http://dx.doi.org/10.4310/MRL.2004.v11.n6.a9
[13] M. Chen and C. D. Hacon, *On the geography of Gorenstein minimal 3-folds of general type*. Asian J. Math. **10**(2006), no. 4, 757–763. http://dx.doi.org/10.4310/AJM.2006.v10.n4.a8
[14] O. Debarre, *Inégalités numériques pour les surfaces de type général.* Bull. Soc. Math. France **110**(1982), no. 3, 319–346.
[15] F. di Severi, *La serie canonica e la teoria delle serie principali de gruppi di punti sopra una superifcie algebrica*. Comment. Math. Helv. **4**(1932), no. 1, 268–326. http://dx.doi.org/10.1007/BF01202721
[16] L. Ein and R. Lazarsfeld, *Singularities of theta divisors and the birational geometry of irregular varieties*. J. Amer. Math. Soc. **10**(1997), no. 1, 243–258. http://dx.doi.org/10.1090/S0894-0347-97-00223-3
[17] P. Francia, *On the base points of the bicanonical system*. In: Problems in the theory of surfaces and their classification (Cortona, 1988), Sympos. Math., 32, Academic Press, London, 1991, pp. 141–150.
[18] M. Green and R. Lazarsfeld, *Deformation theory, generic vanishing theorems, and some conjectures of Enriques, Catanese and Beauville*. Invent. Math. **90**(1987), no. 2, 389–407. http://dx.doi.org/10.1007/BF01388711
[19] C. Hacon and S. Kovács, *Generic vanishing fails for singular varieties and in characteristic* $p > 0$. arxiv:1212.5105
[20] E. Horikawa, *Algebraic surfaces of general type with small* c_1^2. *V*. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28**(1981), no. 3, 745–755.
[21] Y. Hu, *Inequality of Noether type for smooth minimal 3-folds of general type*. arxiv:1309.4618
[22] B. Hunt, *Complex manifold geography in dimension* 2 *and* 3. J. Differential Geom. **30**(1989), no. 1, 51–153.
[23] J. Kollár, *Higher direct images of dualizing sheaves. I*. Ann. of Math. (2) **123**(1986), no. 1, 11–42. http://dx.doi.org/10.2307/1971351
[24] ———, *Higher direct images of dualizing sheaves. II*. Ann. of Math. (2) **124**(1986), no. 1, 171–202. http://dx.doi.org/10.2307/1971390
[25] C. Liedtke, *Algebraic surfaces in positive characteristic*. In: Birational geometry, rational curves, and arithmetic. Springer, New York, 2013, pp. 229–292. http://dx.doi.org/10.1007/978-1-4614-6482-2_11
[26] ———, *Algebraic surfaces of general type with small* c_1^2 *in positive characteristic*. Nagoya Math. J. **191**(2008), 111–134.
[27] S. S. Y. Lu, *On surfaces of general type with maximal Albanese dimension*. J. Reine Angew. Math. **641**(2010), 163–175.
[28] ———, *Holomorphic curves on irregular varieties of general type starting from surfaces*. In: Affine algebraic geometry, CRM Proc. Lecture Notes, 54, American Mathematical Society, Providence, RI, 2011, pp. 205–220.
[29] ———, *A uniform bound on the canonical degree of Albanese defective curves on surfaces*. Bull. Lond. Math. Soc. **44**(2012), no. 6, 1182–1188. http://dx.doi.org/10.1112/blms/bds043
[30] Y. Miyaoka, *The Chern classes and Kodaira dimension of a minimal variety*. In: Algebraic Geometry, Sendai, 1985, Adv. Stud. Pure Math., 10, North-Holland, Amsterdam, 1987, pp. 449–476.

[31] K. Ohno, *Some inequalities for minimal fibrations of surfaces of general type over curves*. J. Math. Soc. Japan **44**(1992), no. 4, 643–666. http://dx.doi.org/10.2969/jmsj/04440643
[32] R. Pardini, *The Severi inequality $K^2 \geq 4\chi$ for surfaces of maximal Albanese dimension*. Invent. Math. **159**(2005), no. 3, 669–672. http://dx.doi.org/10.1007/s00222-004-0399-7
[33] D.-K. Shin, *Noether inequality for a nef and big divisor on a surface*. Commun. Korean Math. Soc. **23**(2008), no. 1, 11–18. http://dx.doi.org/10.4134/CKMS.2008.23.1.011
[34] G. Xiao, *Fibered algebraic surfaces with low slope*. Math. Ann. **276**(1987), no. 3, 449–466. http://dx.doi.org/10.1007/BF01450841
[35] X. Yuan and T. Zhang, *Relative Noether inequality on fibered surfaces*. Adv. Math. **259**(2014), 89–115. http://dx.doi.org/10.1016/j.aim.2014.03.018
[36] T. Zhang, *Severi inequality for varieties of maximal Albanese dimension*. Math. Ann. **359**(2014), no. 3–4, 1097–1114. http://dx.doi.org/10.1007/s00208-014-1025-7

Department of Mathematics, University of Alberta, Edmonton, AB T6G 2G1
e-mail: tzhang5@ualberta.ca