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MAXIMAL SUBSETS OF PAIRWISE NONCOMMUTING ELEMENTS OF SOME p-GROUPS OF MAXIMAL CLASS

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Abstract

Let G be a group. A subset X of G is a set of pairwise noncommuting elements if $xy \neq yx$ for any two distinct elements x and y in X. If $|X| \geq |Y|$ for any other set of pairwise noncommuting elements Y in G, then X is said to be a maximal subset of pairwise noncommuting elements. In this paper we determine the cardinality of a maximal subset of pairwise noncommuting elements for some p-groups of maximal class. Specifically, we determine this cardinality for all 2-groups and 3-groups of maximal class.

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1. Introduction

Let G be a nonabelian group and let X be a maximal subset of pairwise noncommuting elements of G. The cardinality of such a subset is denoted by $\omega(G)$. Also $\omega(G)$ is the maximal clique size in the noncommuting graph of a group G. Let $Z(G)$ be the center of G. The noncommuting graph of a group G is a graph with $G \setminus Z(G)$ as the vertices and join all distinct vertices x and y such that $xy \neq yx$. By a famous result of Neumann [8], answering a question of Erdős, the finiteness of $\omega(G)$ in G is equivalent to the finiteness of the factor group $G/Z(G)$. Chin [4] has obtained upper and lower bounds for $\omega(G)$ for an extra-special p-group G, where p is an odd prime number. For $p = 2$, it has been shown by Isaacs (see [3, p. 40]) that $\omega(G) = 2n + 1$ for any extra-special group G of order 2^{2n+1}. In [1, 2], $\omega(\mathrm{GL}(n, q))$ is given for $n \in \{2, 3\}$.

Let G be a p-group of maximal class and order p^n, where p is a prime number. In this paper we find $\omega(G)$ when G satisfies some extra conditions. On the other hand, for these groups of maximal class we show that $\omega(G) = p^{n-2} + 1$ or $\omega(G) = p^{n-2} + p + 1$ (Theorems 3.4 and 3.7). Then by the above observation we determine $\omega(G)$ for all 2-groups and 3-groups G of maximal class. In particular, we show that $\omega(G) = 2^{n-2} + 1$ for any 2-group G of maximal class and order 2^n (Corollary 3.10). Also for a 3-group G of maximal class and order 3^n we show that $\omega(G) = 3^{n-2} + 1$

when G possesses an abelian maximal subgroup, and $\omega(G) = 3^{n-2} + 4$ when G possesses no abelian maximal subgroups (Corollary 3.11).

Throughout this paper we use the following notation. The letter p denotes a prime number. The terms of the lower central series of G are denoted by $\gamma_i = \gamma_i(G)$. A group G is called an AC-group if the centralizer of every noncentral element of G is abelian and $C_G(x)$ is the centralizer of an element x in a group G.

2. Some basic results

In this section we give some basic results that are needed for the main results of the paper.

LEMMA 2.1. *The following conditions on a group G are equivalent:*

(i) *G is an AC-group;*
(ii) *if $[x, y] = 1$ then $C_G(x) = C_G(y)$, where $x, y \in G \setminus Z(G)$.*

PROOF. This is straightforward. See also [9, Lemma 3.2]. □

LEMMA 2.2. *Let G be an AC-group.*

(i) *If $a, b \in G \setminus Z(G)$ with distinct centralizers, then $C_G(a) \cap C_G(b) = Z(G)$.*
(ii) *If $G = \bigcup_{i=1}^{k} C_G(a_i)$, where $C_G(a_i)$ and $C_G(a_j)$ are distinct for $1 \leq i < j \leq k$, then $\{a_1 \cdots a_k\}$ is a maximal set of pairwise noncommuting elements in G.*

PROOF. (i) We see that $Z(G) \leq C_G(a) \cap C_G(b)$. If $Z(G) < C_G(a) \cap C_G(b)$, then there exists an element x in $C_G(a) \cap C_G(b)$ such that $x \notin Z(G)$. This means that $C_G(a) = C_G(x)$ and $C_G(b) = C_G(x)$ by Lemma 2.1(ii), which is impossible.

(ii) By Lemma 2.1(ii), $\{a_1, a_2, \ldots, a_k\}$ is a set of pairwise noncommuting elements. Suppose to the contrary that $\{b_1, b_2, \ldots, b_t\}$ is another set of noncommuting elements of G with $t > k$. Then we see that there exist positive integers r, s and i with $1 \leq r < s \leq t$ and $1 \leq i \leq k$ such that $b_r, b_s \in C_G(a_i)$. This yields $C_G(b_r) = C_G(b_s)$ by Lemma 2.1(ii), or equivalently $b_r b_s = b_s b_r$, which is a contradiction. □

LEMMA 2.3. *Let G be a finite group of order p^n with the central quotient of order p^2, where p is a prime number. Then $\omega(G) = p + 1$.*

PROOF. First we show that G is an AC-group. Suppose that a is a noncentral element of G. So $Z(G) < C_G(a)$. Therefore $|C_G(a)| = p^{n-1}$. Since $C_G(a) = \langle Z(G), a\rangle$, we see that $C_G(a)$ is abelian and so G is an AC-group. Now since G is finite we may write $G = \bigcup_{i=1}^{k} C_G(a_i)$, where $C_G(a_i)$ and $C_G(a_j)$ are distinct for $1 \leq i < j \leq k$. Therefore $X = \{a_1, a_2, \ldots, a_k\}$ is a maximal subset of pairwise noncommuting elements of G by Lemma 2.2(ii). Thus by Lemma 2.2(i),

$$|G| = \sum_{i=1}^{k} (|C_G(a_i)| - |Z(G)|) + |Z(G)|.$$

This yields $p^n = k \times (p^{n-1} - p^{n-2}) + p^{n-2}$ and so $k = p + 1$. □

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3. Main results

Let G be a p-group of maximal class and order p^n $(n \geq 4)$, where p is a prime. Following [7], we define the 2-step centralizer K_i in G to be the centralizer in G of $\gamma_i(G)/\gamma_{i+2}(G)$ for $2 \leq i \leq n-2$ and define $P_i = P_i(G)$ by $P_0 = G$, $P_1 = K_2$, $P_i = \gamma_i(G)$ for $2 \leq i \leq n$. The degree of commutativity $l = l(G)$ of G is defined to be the maximum integer such that $[P_i, P_j] \leq P_{i+j+l}$ for all $i, j \geq 1$ if P_1 is not abelian and $l = n-3$ if P_1 is abelian.

In this section we determine $\omega(G)$ for any p-group G of maximal class and order p^n, with positive degree of commutativity when $[P_1, P_3] = 1$. Then we deduce $\omega(G)$ for all 2-groups and 3-groups of maximal class

LEMMA 3.1. *Let G be a p-group of maximal class which possesses an abelian maximal subgroup. Then P_1 is abelian.*

PROOF. Let M be an abelian maximal subgroup of G. Then $[M, \gamma_2(G)] = 1$. This implies that $P_1 = M$ by using the definition of P_1. □

LEMMA 3.2 [7, Corollary 3.2.7]. *Let G be a p-group of maximal class. The degree of commutativity of G is positive if and only if the 2-step centralizers of G are all equal.*

THEOREM 3.3. *Let G be a p-group of maximal class and order $p^n (n \geq 4)$ with positive degree of commutativity and let $s \in G \setminus P_1$, $s_1 \in P_1 \setminus P_2$ and $s_i = [s_{i-1}, s]$ for $1 \leq i \leq n-1$. Then:*

(i) *$G = \langle s, s_1 \rangle$, $P_i = \langle s_i, \ldots, s_{n-1} \rangle$, $|P_i| = p^{n-i}$ for $1 \leq i \leq n-1$ and $P_{n-1}(G) = Z(G)$ is of order p;*
(ii) *$\mathcal{C}_G(s) = \langle s \rangle P_{n-1}$, $s^p \in P_{n-1}$ and $|\mathcal{C}_G(s)| = p^2$. So*

$$\mathcal{C}_G(s) = \{s^i s_{n-1}^j : 0 \leq i, j \leq p-1\};$$

(iii) *$\mathcal{C}_G(s) \cap (G \setminus P_1) = \{s^i s_{n-1}^j : 1 \leq i \leq p-1, 0 \leq j \leq p-1\}$;*
(iv) *$\mathcal{C}_G(s) \cap P_1 = Z(G)$;*
(v) *if $s, s' \in G \setminus P_1$ and $[s, s'] \neq 1$, then $\mathcal{C}_G(s) \cap \mathcal{C}_G(s') = Z(G)$.*

PROOF. (i) This is obvious by [7, Lemma 3.2.4].

(ii) This follows from Lemma 3.2 and [6, Hilfssatz III 14.13].

(iii) and (iv) are evident.

(v) We have $Z(G) \leq \mathcal{C}_G(s) \cap \mathcal{C}_G(s') < \mathcal{C}_G(s)$. Also by (i) and (ii), we see that $|Z(G)| = p$ and $|\mathcal{C}_G(s)| = p^2$ which completes the proof. □

THEOREM 3.4. *Let G be a p-group of maximal class and order p^n $(n \geq 4)$ with positive degree of commutativity which possesses an abelian maximal subgroup. Then:*

(i) *G is an AC-group;*
(ii) *$\omega(G) = p^{n-2} + 1$.*

PROOF. (i) By Lemma 3.1, P_1 is abelian and so $\mathcal{C}_G(x) = P_1$ for any $x \in P_1 \setminus Z(G)$. Moreover, if $x \in G \setminus P_1$, then by Theorem 3.3(ii), $|\mathcal{C}_G(x)| = p^2$ as desired.

(ii) We may write $G = \bigcup_{i=1}^{m} \mathcal{C}_G(a_i)$, where $\mathcal{C}_G(a_i)$ and $\mathcal{C}_G(a_i)$ are distinct for $1 \leq i < j < m$ and $a_i \notin Z(G)$. Therefore $\{a_1, \ldots, a_m\}$ is a maximal subset of pairwise noncommuting elements of G by Lemma 2.2(ii). Now let $s_1 \in P_1 \setminus P_2$ as in Theorem 3.3. Then we may assume that $s_1 \in \mathcal{C}_G(a_1)$ and so $\mathcal{C}_G(s_1) = \mathcal{C}_G(a_1)$ by Lemma 2.1. Moreover, $\mathcal{C}_G(s_1) = P_1$ by considering the proof of (i). Therefore $|\mathcal{C}_G(a_1)| = |P_1| = p^{n-1}$. So $a_i \notin P_1$ for $2 \leq i \leq m$. Therefore $|\mathcal{C}_G(a_i)| = p^2$ for $2 \leq i \leq m$ by Theorem 3.3(ii). On the other hand,

$$|G| = |\mathcal{C}_G(a_1)| + \sum_{i=2}^{m} (|\mathcal{C}_G(a_i)| - |Z(G)|)$$

by Lemma 2.2(i). This means that $p^n = p^{n-1} + (m-1)(p^2 - p)$ which yields $\omega(G) = p^{n-2} + 1$. □

Lemma 3.5. *Let G be a p-group of maximal class and order p^n $(n \geq 4)$ with positive degree of commutativity and $[P_1, P_3] = 1$ such that G possesses no abelian maximal subgroup. Then P_1 is not abelian and $\omega(P_1) = p + 1$.*

Proof. Note that P_1 is not abelian since P_1 is a maximal subgroup of G. Also, $P_3 \leq Z(P_1)$ by the fact that $[P_1, P_3] = 1$. Therefore $|P_1/Z(P_1)| = p^2$ by Theorem 3.3(i). Now the result follows from Lemma 2.3. □

Lemma 3.6. *By the assumption of Lemma 3.5, if $Y = \{y_1, \ldots y_t\}$ is a maximal subset of pairwise noncommuting elements in $G \setminus P_1$, then $t = p^{n-2}$.*

Proof. On setting $A_i = \mathcal{C}_G(y_i) \cap (G \setminus P_1)$ for $1 \leq i \leq t$, we see that $|A_i| = p(p-1)$ and $A_i \cap A_j = \emptyset$ for $1 \leq i < j \leq t$ by Theorem 3.3(iii), (v). We claim that $G \setminus P_1 = A_1 \cup \cdots \cup A_t$. For otherwise if $y \in G \setminus P_1$ and $y \notin A_i$ for $1 \leq i \leq t$, then $y \notin \mathcal{C}_G(y_i)$. This implies that $\{y, y_1, \ldots, y_t\}$ is a subset of pairwise noncommuting elements in $G \setminus P_1$, which is a contradiction. Therefore $G \setminus P_1 = A_1 \cup \cdots \cup A_t$ is a partition for $G \setminus P_1$. Hence $|G \setminus P_1| = \sum_{i=1}^{t} |A_i|$ and so $p^n - p^{n-1} = p(p-1)t$, and consequently $t = p^{n-2}$. □

Theorem 3.7. *Let G be a p-group of maximal class and order p^n $(n \geq 4)$ with positive degree of commutativity and $[P_1, P_3] = 1$ such that G possesses no abelian maximal subgroup. Then $\omega(G) = p^{n-2} + p + 1$.*

Proof. Let $Y = \{y_1, \ldots, y_t\}$ and $X = \{x_1, \ldots, x_{p+1}\}$ be maximal subsets of pairwise noncommuting elements in $G \setminus P_1$ and P_1, respectively. Obviously, by Lemma 3.6, $t = p^{n-2}$. First we see that $[x_i, y_j] \neq 1$ for $1 \leq i \leq p+1$ and $1 \leq j \leq t$. For otherwise $x_i \in \mathcal{C}_G(y_j)$ and so $x_i \in \mathcal{C}_G(y_j) \cap P_1$ or, equivalently, $x_i \in Z(G)$ by Theorem 3.3(iv), which is impossible. Therefore $X \cup Y$ is a subset of pairwise noncommuting elements in G and so $\omega(G) \geq t + p + 1 = p^{n-2} + p + 1$. Now let $\omega(G) = m$ and $\{a_1, \ldots, a_m\}$ be a maximal subset of pairwise noncommuting elements in G. We may assume that $\{a_1, \ldots, a_k\} \subseteq G \setminus P_1$ and $\{a_{k+1}, \ldots, a_m\} \subseteq P_1$. By Lemma 3.5, we see that $m - k \leq \omega(P_1) = p + 1$. Also, by Lemma 3.6, we have $k \leq p^{n-2}$. Consequently $\omega(G) = m - k + k \leq p + 1 + p^{n-2}$ as desired. □

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Now we determine $\omega(G)$ for all 2-groups and 3-groups G of maximal class by using the following two theorems.

THEOREM 3.8 [7, Theorem 3.4.1]. *Let G be a 2-group of maximal class. Then P_1 is cyclic.*

THEOREM 3.9 [7, Theorem 3.4.3]. *Let G be a 3-group of maximal class. Then G has degree of commutativity $l \geq n-4$.*

COROLLARY 3.10. *If G is a 2-group of maximal class and order 2^n, then $\omega(G) = 2^{n-2}+1$.*

PROOF. This is evident for $n=3$ by Lemma 2.3. Now since P_1 is abelian and $n \geq 4$, the degree of commutativity of G is $n-3$. Therefore we can complete the proof by Theorem 3.4. □

COROLLARY 3.11. *Let G be a 3-group of maximal class and order 3^n.*

(i) *If G possesses an abelian maximal subgroup, then $\omega(G) = 3^{n-2}+1$.*
(ii) *If G possesses no abelian maximal subgroup, then $\omega(G) = 3^{n-2}+4$.*

PROOF. This is obvious for $n=3$ by Lemma 2.3. Also for $n=4$, we see that $\omega(G)=10$ by using GAP [5]. Now we may assume that $n \geq 5$. Moreover, G has degree of commutativity $l \geq n-4$ by Theorem 3.9 and so $[P_1, P_3]=1$. Therefore we can complete the proof by using Theorems 3.4 and 3.7. □

References

[1] A. Abdollahi, A. Akbari and H. R. Maimani, 'Non-commuting graph of a group', *J. Algbera* **298**(2) (2006), 468–492.
[2] A. Azad and C. E. Praeger, 'Maximal subsets of pairwise non-commuting elements of three-dimensional general linear groups', *Bull. Aust. Math. Soc.* **80**(1) (2009), 91–104.
[3] E. A. Bertram, 'Some applications of graph theory to finite groups', *Discrete Math.* **44**(1) (1983), 31–43.
[4] A. M. Y. Chin, 'On non-commuting sets in an extraspecial p-group', *J. Group Theory* **8**(2) (2005), 189–194.
[5] The GAP Group, 'GAP – Groups, Algorithms, and Programming', Version 4.4.12, 2008, http://www.gap-system.org.
[6] B. Huppert, *Endliche Gruppen, I* (Springer, Berlin, 1967).
[7] C. R. Leedham-Green and S. McKay, *The Structure of Groups of Prime Power Order*, London Mathematical Society Monographs, New Series, 27 (Oxford University Press, Oxford, 2002).
[8] B. H. Neumann, 'A problem of Paul Erdős on groups', *J. Aust. Math. Soc. Ser. A* **21**(4) (1976), 467–472.
[9] D. M. Rocke, 'p-groups with abelian centralizers', *Proc. Lond. Math. Soc.* **30**(3) (1975), 55–57.

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