

COIN TOSSING AND SUM SETS

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Abstract

We consider the distribution μ of numbers whose binary digits are generated from infinitely many tosses of a biased coin. It is shown that, if E has positive μ measure, then some n -fold sum of E with itself must contain an interval. This contrasts with the known result that all convolution powers of μ are singular.

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1. Introduction

Let X be a random variable in the unit interval the digits of whose binary expansion are determined by tossing a biased coin (0 with probability p , 1 with probability $1 - p$, $p \neq 0, 1/2, 1$). Then it is well known that every n -fold sum of independent copies of X has a purely singular distribution: for a proof, note that the characteristic function of X does not vanish at infinity. By way of counterpoint we show here that if E is any Borel set within which there is a positive probability of locating X , then some n -fold sum of E must contain an interval.

Our result is perhaps the most natural probabilistic way to exhibit a phenomenon previously discussed in the context of Banach algebras of measures [2], [3]. Following [3] we call a measure μ *basic* provided that

$$\mu(E) > 0 \Rightarrow \text{Gp}(E) = \mathbb{R},$$

where $Gp(E)$ is the subgroup of the additive reals generated by the elements of E , Lebesgue measure is obviously basic, but the algebraic property of interest is the existence of basic measures all of whose convolution powers are singular. Cantor measure μ_C (or, more accurately, Lebesgue’s singular measure on Cantor’s middle third set!) was shown in [2] to be of this type, the case of (most) Riesz products was covered in [3], and the result is extended here to coin-tossing distributions.

We follow the pattern of proof of [2] where the inequality

$$\lambda(E + F) \geq \mu_C(E)^\alpha \mu_C(F)^\alpha,$$

(λ Lebesgue measure, μ_C Cantor measure, $\alpha = \log 3/\log 4$) was established. Our main theorem gives analogous inequalities for n -fold sums involving the coin-tossing distributions. Related inequalities for different classes of measures can be found in [1], [5].

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2. Main result

For $0 < p < 1$, let μ_p denote the infinite convolution

$$\mu_p = \bigstar_{n=1}^\infty (p\delta_0 + (1 - p)\delta_{2^{-n}}),$$

where δ_x denotes the positive unit mass at x . Thus μ_p is the distribution of the random variable X described in the introduction. Let λ denote Lebesgue measure on the real line (so that λ restricted to $[0, 1]$ is $\mu_{1/2}$).

THEOREM 1. *Let $a = [\max(p, 1 - p)]^{-1}$. Suppose that $a \geq 2^{1/n}$ and let $\alpha = \alpha_n = \log 2/n \log a$. Suppose that E_1, E_2, \dots, E_n are Borel subsets of \mathbb{R} . Then*

$$\lambda(E_1 + E_2 + \dots + E_n) \geq \mu_p(E_1)^\alpha \mu_p(E_2)^\alpha \dots \mu_p(E_n)^\alpha.$$

We will show, in the next section, how to reduce the proof of Theorem 1 to a purely combinatorial result which will be established in Section 4. For now, let us show how the corollary follows from the theorem. In fact, given p ($0 < p < 1$), we have $a \geq 1$, and hence there exists a positive integer n such that $a \geq 2^{1/n}$. Suppose now that E is a Borel set of positive μ_p -measure and apply the theorem with $E_1 = E_2 = \dots = E_n = E$. This shows that the n -fold sum, $(n)E$, of E with itself has positive λ -measure. By a classical theorem of Steinhaus (cf. [4], page 143) the sum of $(n)E$ with itself must contain an open interval. It is now clear that the group generated by E covers the entire line, in other words, that μ_p is basic.

REMARK. The key property in the preceding argument is the fact that some n -fold sum of every set of positive μ_p -measure has positive Lebesgue measure. It is appropriate to note the extraordinary fact (cf. [2]) that there exist basic measures supported by (closed) null sets all of whose n -fold sums are Lebesgue null.

3. Reduction step

Our object in this section is to eliminate the measure theory from the proof of Theorem 1. In the first stage we shift attention to measures having finite support.

Fix p, a, n as in the statement of Theorem 1. For each positive integer k let

$$S_k = \left\{ \sum_{i=1}^k \varepsilon_i 2^{-i} : \varepsilon_i = 0, 1 \right\},$$

$$\mu_k = \prod_{i=1}^k (p\delta_0 + (1-p)\delta_{2^{-i}}),$$

and let λ_k be defined on the n -fold sum, $(n)S_k$, of S_k with itself by the property

$$\lambda_k\{x\} = 2^{-k} \quad (x \in (n)S).$$

Eventually we will prove

THEOREM 2. *Let k be a positive integer and suppose that B_1, B_2, \dots, B_n are subsets of S_k . Then*

$$\lambda_k(B_1 + B_2 + \dots + B_n) \geq \mu_k(B_1)^\alpha \mu_k(B_2)^\alpha \dots \mu_k(B_n)^\alpha.$$

The first task is to prove

LEMMA 1. *Theorem 2 implies Theorem 1.*

PROOF. Let us start with another reduction by noting that it will suffice to prove Theorem 1 for closed sets. For suppose we have that limited form of the result and are given Borel subsets E_1, E_2, \dots, E_n of $[0, 1]$. Fix $\eta > 0$ and choose $\varepsilon > 0$ such that $(1 + \varepsilon)^{n\alpha} \leq 1 + \eta$. By regularity of μ_p we may choose closed sets F_1, F_2, \dots, F_n such that $F_i \subseteq E_i$ and

$$\mu_p(F_i) \geq (1 + \varepsilon)^{-1} \mu_p(E_i), \quad \text{for } i = 1, \dots, n.$$

Then

$$\begin{aligned} \mu_p(E_1)^\alpha \dots \mu_p(E_n)^\alpha &\leq (1 + \varepsilon)^{n\alpha} \mu_p(F_1)^\alpha \dots \mu_p(F_n)^\alpha \\ &\leq (1 + \eta) \mu_p(F_1)^\alpha \dots \mu_p(F_n)^\alpha \\ &\leq (1 + \eta) \lambda(F_1 + \dots + F_n) \\ &\leq (1 + \eta) \lambda(E_1 + \dots + E_n). \end{aligned}$$

Since η was an arbitrary positive number, we have verified the opening remark of the proof.

Observe next that we may write any closed set F_j in $[0, 1]$ as an intersection

$$F_i = \bigcap_{k=1}^{\infty} F_i^k,$$

where

$$F_i^k = B_i^k + [0, 2^{-k}].$$

A simple compactness argument shows that

$$F_1 + F_2 + \dots + F_n = \bigcap_{k=1}^{\infty} (F_1^k + F_2^k + \dots + F_n^k).$$

We have, as $k \rightarrow \infty$,

$$\mu_p(F_i^k) \rightarrow \mu_p(F_i), \quad \lambda(F_1^k + F_2^k + \dots + F_n^k) \rightarrow \lambda(F_1 + F_2 + \dots + F_n).$$

Moreover,

$$\mu_p(F_i^k) = \mu_k(B_i^k), \quad \lambda(F_1^k + F_2^k + \dots + F_n^k) \geq \lambda_k(B_1^k + B_2^k + \dots + B_n^k),$$

and so the statement of the lemma is true.

Now we must set about proving Theorem 2. We shall use an inductive argument to reduce it to a purely combinatorial result. This reduction is similar to the argument of Lemma 2.6 of [2]. That proof is unfortunately somewhat garbled so we take the opportunity to note that the sets D_n^0, D_n^1 defined there should be given as

$$D_n^0 = \left\{ \sum_{k=1}^{n-1} \epsilon_k 3^{-k} : \epsilon_k = 0, 2 \right\}, \quad D_n^1 = \left\{ \sum_{k=1}^{n-1} \epsilon_k 3^{-k} + 2 \cdot 3^{-n} : \epsilon_k = 0, 2 \right\}.$$

Then the induction works “for the tail” in similar fashion to the following argument.

REDUCTION OF THEOREM 2. Suppose that we have the result of Theorem 2 for some positive integer k . Let B_1, \dots, B_n be subsets of S_{k+1} . For $j = 1, \dots, n$, we write

$$B_{j,0} = \left\{ \sum_{i=1}^{k+1} \epsilon_j 2^{-i} \in B_j : \epsilon_{k+1} = 0 \right\},$$

$$B_{j,1} = \left\{ \sum_{i=1}^{k+1} \epsilon_i 2^{-i} \in B_j : \epsilon_{k+1} = 1 \right\}.$$

Note that the sum set, $B_1 + B_2 + \dots + B_n$, is a union of sum sets each of the form

$$(1) \quad B_{1,j(1)} + B_{2,j(2)} + \dots + B_{n,j(n)},$$

where $j(i) \in \{0, 1\}$. Let us write \mathcal{C} for the collection of all those sum sets given as in (1) for which $j(1) + j(2) + \dots + j(n)$ is even and \mathcal{D} for those sum sets for which $j(1) + j(2) + \dots + j(n)$ is odd. Then every set in \mathcal{C} is disjoint from every set in \mathcal{D} . It follows that

$$(2) \quad \lambda_{k+1}(B_1 + B_2 + \dots + B_n) \geq \max\{\lambda_{k+1}(C) : C \in \mathcal{C}\} + \max\{\lambda_{k+1}(D) : D \in \mathcal{D}\}.$$

Let us use a prime to denote projection from S_{k+1} to S_k . Thus we write

$$(3) \quad B'_{1,j(i)} = \left\{ \sum_{m=1}^k \varepsilon_m 2^{-m} : \sum_{m=1}^k \varepsilon_m 2^{-m} + j(i)2^{-k-1} \in B_{i,j(i)} \right\};$$

also

$$(4) \quad C' = B'_{1,j(1)} + B'_{2,j(2)} + \dots + B'_{n,j(n)},$$

provided that

$$(5) \quad C = B_{1,j(1)} + B_{2,j(2)} + \dots + B_{n,j(n)}.$$

Using the notation introduced in (3), (4), (5), we may rewrite (2) as the inequality

$$(6) \quad \lambda_{k+1}(B_1 + B_2 + \dots + B_n) \geq \frac{1}{2} \max\{\lambda_k(C') : C \in \mathcal{C}\} + \frac{1}{2} \max\{\lambda_k(D') : D \in \mathcal{D}\}.$$

The inductive hypothesis will enable us to replace a term such as $\lambda_k(C')$ in (6) by an expression of the form

$$(7) \quad \mu_k(B'_{1,j(1)})^\alpha \mu_k(B'_{2,j(2)})^\alpha \dots \mu_k(B'_{n,j(n)})^\alpha.$$

Now observe that

$$(8) \quad \mu_{k+1}(B_{1,j(i)}) = p\mu_k(B'_{i,j(i)}) \quad \text{or} \quad (1-p)\mu_k(B'_{i,j(i)}),$$

with the first or second alternative occurring according to whether $j(i)$ is even or odd. Moreover,

$$(9) \quad \mu_{k+1}(B_{i,0}) = x_i\mu_{k+1}(B_i), \quad \mu_{k+1}(B_{i,1}) = (1-x_i)\mu_{k+1}(B_i),$$

for some $0 \leq x_i \leq 1$. Combining (8) and (9) we find that

$$(10) \quad \mu_k(B'_{i,0}) = \left(\frac{x_i}{p}\right)\mu_{k+1}(B_i), \quad \mu_k(B'_{i,1}) = \left(\frac{1-x_i}{1-p}\right)\mu_{k+1}(B_i).$$

Now we may substitute (10) and (7) to see that (7) can be rewritten as an expression of the form

$$(11) \quad \mu_{k+1}(B_1)^\alpha \mu_{k+1}(B_2)^\alpha \dots \mu_{k+1}(B_n)^\alpha (y_1 y_2 \dots y_n),$$

where each y_i is of the form $(x_i/p)^\alpha$ or $((1-x_i)/(1-p))^\alpha$, the choice being determined by the parity of the sequence $j(1), j(2), \dots, j(n)$ which is in turn determined by C . Accordingly, let us relabel $y_1 y_2 \dots y_n$ as $y(C)$ and deduce

from (6), using the inductive hypothesis, that

$$\begin{aligned} &\lambda_{k+1}(B_1 + B_2 + \dots + B_n) \\ &\geq \left[\frac{1}{2} \max_j \{ y(C) : C \in \mathcal{C} \} + \frac{1}{2} \max \{ y(D) : D \in \mathcal{D} \} \right] \\ &\quad \cdot \mu_{k+1}(B_1)^\alpha \mu_{k+1}(B_2)^\alpha \dots \mu_{k+1}(B_n)^\alpha. \end{aligned}$$

It is immediately clear that the inductive step will be accomplished once we can prove that

$$(12) \quad \max \{ y(C) : C \in \mathcal{C} \} + \max \{ y(D) : D \in \mathcal{D} \} \geq 2.$$

This is the purely combinatorial theorem which we shall isolate and prove in the next section.

It remains to ground the induction by checking the case $k = 1$. Each $B_i \subseteq \{0, \frac{1}{2}\}$. If, in fact, $B_i = \{0, \frac{1}{2}\}$ for some i , then we see that $\lambda_1(B_1 + B_2 + \dots + B_n) \geq 1 \geq \mu_1(B_1)^\alpha \dots \mu_1(B_n)^\alpha$. We may suppose then that each B_i is a singleton and that $\lambda_i(B_1 + B_2 + \dots + B_n) = \frac{1}{2}$. We must verify that $[\max(p, (1 - p))]^{n\alpha} \leq 1/2$. But this is the requirement that $a^{n\alpha} \geq 2$, and we have already chosen $\alpha = \log 2/n \log a$, so the case $k = 1$ is indeed true.

4. Combinatorial result

It remains to prove the combinatorial theorem which corresponds to assertion (12) of the last section.

THEOREM 3. *Suppose that $0 < p < 1$ and that n is a positive integer. Let $a = [\max(p, (1 - p))]^{-1}$ and let $\alpha = \alpha_n = \log 2/n \log a$. Suppose that $a \geq 2^{1/n}$ and that $0 \leq x_j \leq 1$ for $i = 1, \dots, n$. Let Y be the set of all products $y_1 \dots y_j \dots y_n$ of n terms in which the j th term y_j satisfies one or the other of*

$$(i) \ y_j = (x_j/p)^\alpha, \quad (ii) \ y_j = \left[(1 - x_j)/(1 - p) \right]^\alpha.$$

Write Y as the disjoint union $Y_0 \cup Y_1$, where Y_0 is the set of all products in which the second choice is made an even number of times and Y_1 is the set of products in which the second choice is made an odd number of times.

Then

$$\max Y_0 + \max Y_1 \geq 2.$$

We have another reduction in mind.

LEMMA 2. *It will suffice to prove that, for $1 \geq x \geq p$, we have*

$$G(x) = \left(\frac{x}{p}\right)^{(n-1)\alpha} \left(\left(\frac{x}{p}\right)^\alpha + \left[\frac{(1-x)}{(1-p)} \right]^\alpha \right) \geq 2.$$

PROOF OF LEMMA. In tackling Theorem 3 we can (and do) assume without loss of generality that

$$(13) \quad x_1 \geq x_2 \geq \dots \geq x_n.$$

The first case to be considered occurs when we also have $x_n \geq p$. Now the product $(x_1/p)^\alpha(x_2/p)^\alpha \dots (x_n/p)^\alpha$ belongs to Y_0 , and the product $(x_1/p)^\alpha(x_2/p)^\alpha \dots ((x_{n-1})/p)^\alpha((1-x_n)/(1-p))^\alpha$ belongs to Y_1 . Therefore

$$(14) \quad \max Y_0 + \max Y_1 \geq \left(\frac{x_1}{p}\right)^\alpha \dots \left(\frac{x_{n-1}}{p}\right)^\alpha \left[\left(\frac{x_n}{p}\right)^\alpha + \left(\frac{1-x_n}{1-p}\right)^\alpha \right].$$

Combining (13) and (14) and setting $x = x_n$, we obtain

$$(15) \quad \max Y_0 + \max Y_1 \geq G(x),$$

where $x \geq p$.

A similar argument is available when we have $p \geq x_1$, because we interchange the rôles of $p, 1-p; x_i, 1-x_i$. This gives

$$(16) \quad \max Y_0 + \max Y_1 \geq \left(\frac{1-x}{1-p}\right)^{(n-1)\alpha} \left[\left(\frac{1-x}{1-p}\right)^\alpha + \left(\frac{x}{p}\right)^\alpha \right],$$

where $x = x_1$, and thus $1-x \geq 1-p$.

Let us now consider the case where

$$(17) \quad x_1 \geq \dots \geq x_k \geq p \geq x_{k+1} \geq \dots \geq x_n.$$

There are two sub-cases. Suppose first that

$$(18) \quad x_k/p \geq (1-x_{k+1})/(1-p).$$

Of the two products

$$(19) \quad \left(\frac{x_1}{p}\right)^\alpha \dots \left(\frac{x_k}{p}\right)^\alpha \left(\frac{1-x_{k+1}}{1-p}\right)^\alpha \dots \left(\frac{1-x_n}{1-p}\right)^\alpha$$

and

$$(20) \quad \left(\frac{x_1}{p}\right)^\alpha \dots \left(\frac{x_k}{p}\right)^\alpha \left(\frac{x_{k+1}}{p}\right)^\alpha \left(\frac{1-x_{k+2}}{1-p}\right)^\alpha \dots \left(\frac{1-x_n}{1-p}\right)^\alpha,$$

one belongs to Y_0 and the other to Y_1 . Hence their sum gives a lower estimate for $\max Y_0 + \max Y_1$. Using (17) and (18) together, we see that each term in the product labelled (19) exceeds $((1-x_{k+1})/(1-p))^\alpha$. A similar argument shows that the product labelled (20) exceeds $((1-x_{k+1})/(1-p))^{(n-1)\alpha}(x_{k+1}/p)$. Accordingly we set $x = x_{k+1}$ and obtain (16) once more (with, of course, $1-x \geq 1-p$). The other sub-case occurs when

$$(21) \quad \frac{x_k}{p} \leq \frac{1-x_{k+1}}{1-p}.$$

This time we consider the products

$$\left(\frac{x_1}{p}\right)^\alpha \dots \left(\frac{x_k}{p}\right)^\alpha \left(\frac{1-x_{k+1}}{1-p}\right)^\alpha \dots \left(\frac{1-x_n}{1-p}\right)^\alpha$$

and

$$\left(\frac{x_1}{p}\right)^\alpha \dots \left(\frac{x_{k-1}}{p}\right)^\alpha \left(\frac{1-x_k}{1-p}\right)^\alpha \dots \left(\frac{1-x_n}{1-p}\right)^\alpha.$$

We use (17) and (21) to see that the first exceeds $(x_k/p)^{n\alpha}$ while the second exceeds $(x_k/p)^{(n-1)\alpha}((1-x_k)/(1-p))$. We set $x = x_k$ to obtain (15) with $x \geq p$. This completes the proof of the lemma.

It remains to check that the inequality mentioned in the statement of Lemma 2 really does hold. This is the first occasion on which the condition $a \geq 2^{1/n}$ bites, so let us recall the standing assumptions.

LEMMA 3. *Suppose that $0 < p < 1$, that $a [\max(p, (1-p))]^{-1}$, that n is a positive integer such that $a \geq 2^{1/n}$ and that $\alpha = \log 2/n \log a$.*

Then

$$F(t) = t^{n\alpha} + t^{(n-1)\alpha} \left(\frac{1-pt}{1-p}\right)^\alpha \geq 2, \quad \text{for } 1 \leq t \leq 1/p.$$

PROOF. $F(1) = 2$, $F(p^{-1}) = p^{-n\alpha} \geq a^{n\alpha} = 2$. Accordingly it will suffice to check that $F'(t) = 0$ implies $F(t) \geq 2$. In fact,

$$\begin{aligned} F'(t) &= n\alpha t^{n\alpha-1} + (n-1)\alpha t^{(n-1)\alpha-1} \left(\frac{1-pt}{1-p}\right)^\alpha \\ &\quad - \alpha p t^{(n-1)\alpha} \left(\frac{1-pt}{1-p}\right)^{\alpha-1} (1-p)^{-1} \\ &= \alpha t^{(n-1)\alpha-1} \left[n t^\alpha + \left(\frac{1-pt}{1-p}\right)^{\alpha-1} \{n-1-(n-1)pt-pt\} (1-p)^{-1} \right], \end{aligned}$$

Thus $F'(t) = 0$ entails

$$(22) \quad n t^\alpha = \left(\frac{1-pt}{1-p}\right)^{\alpha-1} \left(\frac{npt-n+1}{1-p}\right).$$

When (22) holds, we have

$$\begin{aligned} F(t) &= t^{n\alpha} + t^{(n-1)\alpha} n t^\alpha (1-pt)(npt-n+1)^{-1} \\ &= t^{n\alpha} (npt-n+1)^{-1} \{npt-n+1+n-npt\} \\ &= p^{-n\alpha} (pt)^{n\alpha} (npt-n+1)^{-1}. \end{aligned}$$

Since $p^{-n\alpha} \geq 2$, it will suffice to prove that

$$(23) \quad (pt)^{n\alpha} \geq npt - (n - 1).$$

Let us write $s = pt$, so that (23) becomes

$$(24) \quad s^{n\alpha} \geq ns - (n - 1), \quad \text{for } p \leq s \leq 1.$$

The condition $a \geq 2^{1/n}$ implies that $\alpha \leq 1$, so to prove (24) it will be enough to prove that

$$(25) \quad H(s) = s^n - ns + (n - 1) \geq 0, \quad \text{for } p \leq s \leq 1.$$

But $H'(s) = ns^{n-1} - n \leq 0$ and $H(1) = 0$, so (25) holds. this completes the proof of the lemma and hence of Theorem 3 and the results depending upon it.

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