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A MEAN VALUE THEOREM FOR EXPONENTIAL SUMS

M. N. HUXLEY

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Abstract

The exponential sum $S(x) = \sum e(f(m+x))$ has mean square size $O(M)$, when m runs through M consecutive integers, $f(x)$ satisfies bounds on the second and third derivatives, and x runs from 0 to 1.

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Let $f(x)$ be a real function satisfying

$$|f^{(r)}(x)| \le B^r T/M^r \tag{1}$$

in the range $M \le x \le 2M$, for suitable parameters B and $T(\ge M)$. The exponential sum problem (see Graham and Kolesnik [2]) is to find sufficient conditions on $f(x)$ to ensure that the exponential sum

$$S = \sum_{m=M}^{2M-1} e(f(m))$$

(where $e(x) = \exp 2\pi i x$) has order of magnitude $S = O(M^{1/2}T^{\epsilon})$ for any $\epsilon > 0$; the order of magnitude constant may depend on B and on ϵ. It is well-known that S has root mean square size $\sqrt{M}$, in the sense that for $a \ge 1$,

$$\int_a^{a+1} \left| \sum_M^{2M-1} e(tf(m)) \right|^2 dt = M + O\left(\frac{BM^2 \log M}{aT}\right), \tag{2}$$

provided that $|f'(x)| \ge T/BM$; the error term may be improved to $O(M^2/aT)$ using a theorem of Montgomery and Vaughan [1]. In this note we prove another mean value

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theorem. We write $S(x) = \sum_M^{2M-1} e(f(m+x))$. This sum is approximately periodic, since if we write more generally

$$S(a, N, x) = \sum_{m=a}^{a+N-1} e(f(m+x)),$$

then there is a 'Weyl shift' identity for integers a and b:

(3) $S(a, N, x+b) = S(a+b, N, x) = S(a, N, x) + S(a+N, b, x) - S(a, b, x)$.

In particular, $S(x+1) = S(x) + O(1)$. The Fourier expansion of the periodic function which equals $S(x)$ for $0 < x < 1$ gives the Poisson summation formula for the original sum S.

THEOREM. *Let $f(x)$ satisfy* (1) *for $r = 2$ and 3, and suppose that $f''(x)$ and $f^{(3)}(x)$ do not change sign, and that $|f''(x)| \geq T/B^2M^2$, for $M \leq x \leq 2M$. Then*

$$\int_0^1 |S(x)|^2\,dx = M + O\left(\frac{B^2M^2 \log M}{T} + \frac{B^9M^4}{T^2}\right).$$

PROOF. We expand and integrate term by term. We have

$$\int_0^1 |S(x)|^2\,dx = \int_0^1 \sum_m \sum_n e(f(m+x) - f(n+x))\,dx.$$

The terms with $m = n$ each give 1. For $m \neq n$,

$$\begin{aligned}
(4)\ \int_0^1 e(f(m+x) - f(n+x))dx &= \left[\frac{e(f(m+x) - f(n+x))}{2\pi i(f'(m+x) - f'(n+x))}\right]_0^1 \\
&\quad + \int_0^1 \frac{e(f(m+x) - f(n+x))(f''(m+x) - f''(n+x))}{2\pi i(f'(m+x) - f'(n+x))^2}dx \\
&= \frac{e(f(m+1) - f(n+1))}{2\pi i(f'(m+1) - f'(n+1))} - \frac{e(f(m) - f(n))}{2\pi i(f'(m) - f'(n))} \\
&\quad + \left[\frac{e(f(m+x) - f(n+x))(f''(m+x) - f''(n+x))}{(2\pi i)^2(f'(m+x) - f'(n+x))^3}\right]_0^1 \\
&\quad + \int_0^1 \frac{e(f(m+x)-f(n+x))}{(2\pi i)^2}\left(\frac{3(f''(m+x)-f''(n+x))^2}{(f'(m+x)-f'(n+x))^4} - \frac{f^{(3)}(m+x)-f^{(3)}(n+x)}{(f'(m+x)-f'(n+x))^3}\right)dx.
\end{aligned}$$

Since

$$f^{(r)}(m+x) - f^{(r)}(n+x) = (m-n)f^{(r+1)}(\xi)$$

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for some ξ between M and $2M$, the absolute value of the integrand on the right of (4) is

$$\leq \frac{|m-n|}{4\pi^2}\left(\max|f^{(3)}|\right)\frac{3|f''(m+x)-f''(n+x)|}{f'(m+x)-f'(n+x))^4}$$
$$+\frac{1}{4\pi^2}\max|f^{(3)}|\max_{0\leq x\leq 1}\frac{1}{|f'(m+x)-f'(n+x)|^3}$$
$$\leq \frac{B^3T}{4\pi^2M^3}\left(|m-n|\left|\frac{d}{dx}\frac{1}{|f'(m+x)-f'(n+x)|^3}\right|+\max_{0\leq x\leq 1}\frac{1}{|f'(m+x)-f'(n+x)|^3}\right).$$

Since f'' and $f^{(3)}(x)$ do not change sign, we see that $|f'(m+x)-f'(n+x)|^{-3}$ is monotone in x, and we may integrate over x to

$$\leq \frac{B^3T}{2\pi^2M^3}\max_{0\leq x\leq 1}\frac{|m-n|}{|f'(m+x)-f'(n+x)|^3}\leq\frac{B^9M^3}{2\pi^2(m-n)^2T^2}.$$

The sum over distinct m and n of $1/(m-n)^2$ is $O(M)$. The second set of integrated terms on the right of (4) has the same order of magnitude.

The first set of integrated terms cancels when summed over m and n, except for terms with $m=M, n=M, m+1=2M$ or $n+1=2M$. The uncancelled terms are

$$O\left(\sum_{n=M+1}^{2M-1}\left(\frac{1}{|f'(n)-f'(M)|}+\frac{1}{|f'(2M)-f'(n)|}\right)\right)$$
$$=O\left(\frac{B^2M^2}{T}\sum_{n=M+1}^{2M-1}\left(\frac{1}{n-M}+\frac{1}{2M-n}\right)\right)=O\left(\frac{B^2M^2\log M}{T}\right).$$

which completes the proof.

For M close to T, the bound becomes trivial. However, for $M=O(T^{1-\epsilon})$ we can deduce $\int_0^1|S(x)|^2dx=(1+o(1))M$ by an interative method, which actually establishes, for b and N positive integers less than M,

$$\int_0^b|S(a,N,x)|^2dx=(1+o(1))bN+O\left(\frac{bM^2\log M}{T}\right). \tag{5}$$

The orders of magnitude of the error terms depend on B and on ϵ. The method of the theorem gives (5) for large integers b. For smaller integers b we take a multiple d of the integer b, so large that (5) is true with d in place of b. We use the Weyl shift (3) to relate the integral from 0 to b to the integral from cb to $(c+1)b$, for each c from 0 to $d/b-1$. The two short correction sums in (3) are themselves estimated in mean square using the theorem or its generalisation (5). We find that (5) holds for a shorter range of b. This process of taking integer multiples of the length b can be iterated until we extend the validity of (5) to all integers $b \geq 1$.

A different way to extend the theorem is to assume further differentiability, and integrate by parts several times. This reduces the error term $O(M^4/T^2)$.

Although (2) holds with arbitrary bounded coefficients in the terms of the sum, our theorem can be generalised only to coefficients $g(m)$ which are values of a differentiable function satisfying conditions analogous to (1), like the weight functions $g(x)$ which are of practical use in the Poisson summation formula.

It seems difficult to replace the range of integration 0 to 1 by a shorter range. The method above only gives a non-trivial estimate for the Fourier coefficient $\int_0^1 |S(x)|^2 e(hx)dx$ when the integer h is small.

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References

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School of Mathematics
University of Wales College of Cardiff
23 Senghenydd Road
Cardiff, Wales CF2 4YH
UK