STABLE LEFT AND RIGHT BOUSFIELD LOCALISATIONS

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Abstract. We study left and right Bousfield localisations of stable model categories which preserve stability. This follows the lead of the two key examples: localisations of spectra with respect to a homology theory and A-torsion modules over a ring R with A a perfect R-algebra. We exploit stability to see that the resulting model structures are technically far better behaved than the general case. We can give explicit sets of generating cofibrations, show that these localisations preserve properness and give a complete characterisation of when they preserve monoidal structures. We apply these results to obtain convenient assumptions under which a stable model category is spectral. We then use Morita theory to gain an insight into the nature of right localisation and its homotopy category. We finish with a correspondence between left and right localisation.

1. Introduction. Localisations of homotopy theories are one of the most useful techniques in the tool kit of an algebraic topologist. Bousfield introduced this concept by studying topological spaces up to E_*-equivalence for E_* a homology theory. This became known as *left Bousfield localisation*. Later, the dual concept known as *cellularisation*, or *right Bousfield localisation*, was developed by Farjoun in [**8**]. A particularly interesting example of right localisation is given by Dwyer and Greenlees in [**6**]. Specifically, they consider A-torsion modules in the case of A a perfect algebra over the ring R.

As these two notions of localisation and cellularisation were studied, it became clear that there were advantages to phrasing these notions in the language of model categories. It was therefore natural to ask if localisation or cellularisation can be performed in a general model category. A good answer to this was given by Hirschhorn in the book [**10**], which discusses general existence questions as well as studying technical properties of left and right localisations. Left and right localisations are dual notions, but the main results of the book are not dual. This creates some very interesting differences in the behaviour of left and right localisations.

In this paper, we focus on stable model categories and stability-preserving left and right localisations. A localisation of a stable model category is not necessarily again

stable. We will briefly discuss some examples of this behaviour involving Postnikov decomposition of spectra at the end of Section 2. Because of these examples, most of the technical results in the literature on localisation do not take stability into account. However, the most interesting examples from stable homotopy theory, namely E_*-localisation and A-torsion R-modules, are localisations where the result is still stable. We are going to isolate this phenomenon, giving a new approach to localisation. Following this method has numerous immediate benefits, such as properness being preserved by localisation and the existence of convenient generating sets of cofibrations unlike in the general case. Moreover, our approach can be viewed as an improvement on the existence results of left and right localisations as the stable case requires fewer technical assumptions on the original model category.

We then exploit our new description of the generating sets to see that monoidal structures interact very well with localisations of stable model categories. We then return to the motivating example of spectra and see that left localisations behave extremely well and are easily made stable and monoidal. We also obtain an even simpler set of generating cofibrations and acyclic cofibrations. Analogously, we can use our tools to deduce that the category of A-torsion modules is a monoidal model category.

One further interesting consequence of the stable setting is that we are now able to prove that any stable, proper and cellular model category is Quillen equivalent to a spectral model category. Since we now know that stable left localisation preserves properness, we are able to combine existing results to obtain a sleeker and more tractable answer than previous results along these lines.

We continue by using Morita theory to show that for a set of homotopically compact objects K, right localisation with respect to K is Quillen equivalent to modules over the endomorphism ringoid spectrum of K. This shows that the K-colocal homotopy category of $\mathcal{C}$ is the smallest localising subcategory of the homotopy category of $\mathcal{C}$ containing K. We also provide an explicit description of colocalisation in this case.

We further show that for any left localisation there is a corresponding right localisation governing the acyclics of this left localisation and vice versa. This allows us to restate the telescope conjecture in chromatic homotopy theory in terms of right localisations.

Our results regarding properness, existence and monoidality of left and right localisations as well as their applications show that stable localisations of stable model categories have vast advantages over the general case. Furthermore, we have shown that right localisations are not to be dreaded and hope that our work will encourage others to use this powerful technique.

This paper is organised as follows. In Section 2, we establish the notions of left and right Bousfield localisations of model categories. We then discuss some standard examples, namely localisation of spectra with respect to a homology theory and A-torsion R-modules where A is a perfect complex over the commutative ring R.

In Section 3, we recall some definitions in the context of model categories, namely stability, framings, properness and cofibrant generation. These technical definitions will play a crucial role in our work.

Section 4 contains the first key results concerning left Bousfield localisation $L_S\mathcal{C}$. We define what it means for a set of maps S to be stable and then show that under the assumption of stability of S, localisation preserves stability and properness. Furthermore, we give a simple set of generating cofibrations and acyclic cofibrations

for $L_S\mathcal{C}$. Section 5 deals with analogous results for the dual case of right Bousfield localisation $R_K\mathcal{C}$, where K is a set of objects of $\mathcal{C}$.

The following pair of Sections 6 and 7 examines the interaction of left and right localisations with monoidal structures. More specifically, for a monoidal model category $\mathcal{C}$, we give necessary and sufficient conditions on S and K so that $L_S\mathcal{C}$ and $R_K\mathcal{C}$ are again monoidal model categories and prove some universal properties. We also apply our results to the leading examples of spectra and A-torsion R-modules.

Section 8 uses the fact that stable left localisations preserve properness to obtain convenient conditions under which a stable model category is Quillen equivalent to a spectral one.

In Section 9, we use the Morita theory of Schwede and Shipley to gain further insight into right localisations when the object set K consists of homotopically compact objects. In particular, we generalise the results of Dwyer and Greenlees [6] to a large class of well-behaved monoidal model categories. Thus, for such a set of objects K, we find a set of maps S such that right localisation at K is Quillen equivalent to a left localisation at S.

Finally, in Section 10, we update the important correspondence between cellularisations and acyclicisations to the language of left and right localisations by comparing colocal objects to acyclic objects, leading to an alternative description of the telescope conjecture.

2. Examples of left and right Bousfield localisations. Let E_* be a generalised homology theory. In the 1970s, Bousfield considered the resulting homotopy categories of spaces and spectra after inverting E_*-isomorphisms rather than π_*-isomorphisms. These homotopy categories are especially sensitive with respect to phenomena related to E_*. To talk about these constructions in a set-theoretically rigid manner, they were increasingly placed in a model category context in the subsequent decades. We recall some definitions and results in this section.

DEFINITION 2.1. A map $f : X \to Y$ of simplicial sets or spectra is an E-equivalence if $E_*(f)$ is an isomorphism. A simplicial set or a spectrum Z is E-local if

$$f^* : [Y, Z] \to [X, Z]$$

is an isomorphism for all E-equivalences $f : X \to Y$. A simplicial set or spectrum A is E-acyclic if $[A, Z]$ consists of only the trivial map, for all E-local Z. An E-equivalence from X to an E-local object Z is called an E-localisation.

These definitions then give rise to the following (see Bousfield [2] and [3]).

THEOREM 2.2. *Let E be a homology theory and $\mathcal{C}$ be the category of simplicial sets or spectra. Then, there is a model structure $L_E\mathcal{C}$ on $\mathcal{C}$ such that*

- *the weak equivalences are the E_*-isomorphisms,*
- *the cofibrations are the cofibrations of $\mathcal{C}$,*
- *the fibrations are those maps with the right lifting property with respect to cofibrations that are also E_*-isomorphisms.*

A map of simplicial sets or spectra is called an *E-acyclic cofibration* if it is a cofibration that is an E_*-isomorphism. Similarly, an *E-acyclic fibration* is a fibration that is an E_*-isomorphism.

This result can be seen as a special case of a more general result by Hirschhorn. For $X, Y \in \mathcal{C}$, we let $\mathrm{Map}_\mathcal{C}(X, Y)$ denote the homotopy function object, which is a simplicial set (see Hirschhorn [**10**, chapter 17] and Section 3).

DEFINITION 2.3. Let S be a set of maps in $\mathcal{C}$. Then, an object $Z \in \mathcal{C}$ is S-local if

$$\mathrm{Map}_\mathcal{C}(s, Z) : \mathrm{Map}_\mathcal{C}(B, Z) \longrightarrow \mathrm{Map}_\mathcal{C}(A, Z)$$

is a weak equivalence in simplicial sets for any $s : A \longrightarrow B$ in S. A map $f : X \longrightarrow Y \in \mathcal{C}$ is an S-equivalence if

$$\mathrm{Map}_\mathcal{C}(f, Z) : \mathrm{Map}_\mathcal{C}(Y, Z) \longrightarrow \mathrm{Map}_\mathcal{C}(X, Z)$$

is a weak equivalence for any S-local $Z \in \mathcal{C}$. An object $W \in \mathcal{C}$ is S-acyclic if

$$\mathrm{Map}_\mathcal{C}(W, Z) \simeq *$$

for all S-local $Z \in \mathcal{C}$.

A *left Bousfield localisation* of a model category $\mathcal{C}$ with respect to a class of maps S is a new model structure $L_S\mathcal{C}$ on $\mathcal{C}$ such that

- the weak equivalences of $L_S\mathcal{C}$ are the S-equivalences,
- the cofibrations of $L_S\mathcal{C}$ are the cofibrations of $\mathcal{C}$,
- the fibrations of $L_S\mathcal{C}$ are those maps that have the right lifting property with respect to cofibrations that are also S-equivalences.

Hirschhorn proves that with some minor assumptions on $\mathcal{C}$, $L_S\mathcal{C}$ exists if S is a set. In the case of homological localisation as in Theorem 2.2, the class S is initially the class of E_*-isomorphisms, which is not a set. Hence, the key to proving the existence of homological localisations is to show that there is a *set S* whose S-equivalences are exactly the E_*-isomorphisms.

For example, this has been done for spectra, specifically for $\mathbb{S}$-modules in the sense of EKMM [**7**]. In their Section 8.1, they show that there is a set $\mathcal{J}_E$ of generating E-acyclic cofibrations. That is, a morphism of spectra is a fibration in the E-local model structure if and only if it has the right lifting property with respect to all elements of $\mathcal{J}_E$. This implies that $L_E = L_{\mathcal{J}_E}$ as both localisations then possess the same fibrant objects and in particular the same local objects. Similar results exist for symmetric spectra, sequential spectra and orthogonal spectra and their equivariant counterparts.

We now turn to right Bousfield localisation. First, we note that Hirschhorn's existence theorem for right localisations [**10**, Theorem 5.1.1] is not entirely dual to the left local analogue as it starts with a set of objects rather than a set of maps. Thus, we always word right localisations in terms of a set (or class) of objects.

DEFINITION 2.4. Let $\mathcal{C}$ be a model category and K a class of objects of $\mathcal{C}$. We say that a map $f : A \longrightarrow B$ of $\mathcal{C}$ is a K-coequivalence if

$$\mathrm{Map}_\mathcal{C}(X, f) : \mathrm{Map}_\mathcal{C}(X, A) \longrightarrow \mathrm{Map}_\mathcal{C}(X, B)$$

is a weak equivalence of simplicial sets for each $X \in K$. An object $Z \in \mathcal{C}$ is K-colocal if

$$\mathrm{Map}_\mathcal{C}(Z, f) : \mathrm{Map}_\mathcal{C}(Z, A) \longrightarrow \mathrm{Map}_\mathcal{C}(Z, B)$$

is a weak equivalence for any K-coequivalence f. An object $A \in \mathcal{C}$ is K-coacyclic if $\mathrm{Map}_{\mathcal{C}}(W, A) \simeq *$ for any K-colocal W.

There are many other similar names for these terms, in particular Hirschhorn [10, Definition 5.1.3] uses the term K-colocal equivalences for K-coequivalences.

A *right Bousfield localisation* of $\mathcal{C}$ with respect to K is a model structure $R_K\mathcal{C}$ on $\mathcal{C}$ such that

- the weak equivalences are K-coequivalences
- the fibrations in $R_K\mathcal{C}$ are the fibrations in $\mathcal{C}$
- the cofibrations in $R_K\mathcal{C}$ are those morphisms that have the left lifting property with respect to fibrations that are K-coequivalences.

When K is a set rather than an arbitrary class, Hirschhorn showed in [10, Theorem 5.1.1] that, under some assumptions on $\mathcal{C}$, $R_K\mathcal{C}$ exists. This is discussed in more detail in Section 5.

An algebraic example of the right Bousfield localisation of modules over a ring R was discussed by Dwyer and Greenlees in [6]. A *perfect R-module* A is isomorphic to a differential graded R-module of finite length which is finitely generated projective in every degree. This is equivalent to A being small, meaning that $R\mathrm{Hom}_R(A, -)$, the derived functor of $\mathrm{Hom}_R(A, -)$, commutes with arbitrary coproducts.

Dwyer and Greenlees consider the right localisation of the category of R-modules with respect to $K = \{A\}$, where A is perfect. In their paper, they call the thus arising $\{A\}$-coequivalences 'E-equivalences', referring to the functor $E(-) = R\mathrm{Hom}_R(A, -)$. The $\{A\}$-colocal objects are referred to as 'A-torsion modules'. For example, in the case of $R = \mathbb{Z}$ and $A = (\mathbb{Z} \xrightarrow{p} \mathbb{Z}) \simeq \mathbb{Z}/p$, an R-module X is $\mathbb{Z}/p$-torsion if and only if it has p-primary torsion homology groups.

In [6], Dwyer and Greenlees also compare this version of right localisation with a dual notion of left localisation. In the same set-up, they consider left localisation with respect to the class S of $R\mathrm{Hom}_R(A, -)$-isomorphisms. They call the resulting S-local R-modules 'A-complete'. In their Theorem 2.1, they show that the derived categories of A-torsion and A-complete modules are equivalent. We will provide a generalisation of this type of result in Section 9.

The localisations discussed in [6] and the E_*-localisation of spectra are examples of localisations which preserve stability. Not all localisations have this property: there are left (and right) localisations of stable model categories which are not themselves stable. Two standard examples come from the Postnikov decomposition of the category of spectra. Consider the left Bousfield localisation of spectra where we add the boundary inclusion map $S^n_+ \to D^{n+1}_+$ to the set of weak equivalences. A fibrant object X in this homotopy category satisfies $\pi_i(X) = 0$ for $i \geqslant n$; hence, the localisation is not stable. The second example is the right Bousfield localisation of spectra at the object S^n_+. The resulting homotopy category is the homotopy category of $(n-1)$-connected spectra; hence, this localisation is also not stable.

3. Some model category techniques.   We will recall some technical facts about stable model categories. The homotopy category of any pointed model category can be equipped with an adjoint functor pair

$$\Sigma : \mathrm{Ho}(\mathcal{C}) \rightleftarrows \mathrm{Ho}(\mathcal{C}) : \Omega,$$

where Σ is called the *suspension functor* and Ω the *loop functor*. Let $X \in \mathrm{Ho}(\mathcal{C})$ be fibrant and cofibrant in $\mathcal{C}$. We factor the map

$$X \longrightarrow *$$

into a cofibration and a weak equivalence

$$X \rightarrowtail X \xrightarrow{\sim} *.$$

The suspension ΣX of X is defined as the pushout of the diagram

$$CX \leftarrowtail X \rightarrowtail CX.$$

Dually, the loops on X are defined as the pullback of

$$PX \twoheadrightarrow X \twoheadleftarrow PX,$$

where

$$* \xrightarrow{\sim} PX \twoheadrightarrow X$$

is a factorisation of $* \longrightarrow X$ into a weak equivalence and a fibration. For example, in the case of topological spaces, this gives the usual loop and suspension functors. For chain complexes of R-modules, denoted $\mathrm{Ch}(R)$, the suspension and loop functors are degree shifts of chain complexes.

DEFINITION 3.1. A model category $\mathcal{C}$ is stable if Σ and Ω are inverse equivalences of categories.

Thus, topological spaces are not stable whereas $\mathrm{Ch}(R)$ is.

An alternative description of Σ and Ω uses the technique of *framings* which is a generalisation of the notion of a simplicial model category. Recall that a simplicial model category is a model category that is enriched, tensored and cotensored over the model category of simplicial sets satisfying some adjunction properties. Furthermore, these functors are supposed to be compatible with the respective model structures on the model category $\mathcal{C}$ and simplicial sets sSet_*. Goerss and Jardine give an excellent introduction to this notion in [**9**, Section 2.3]. Not every model category can be given the structure of a simplicial model category, but framings at least give a similar structure up to homotopy. For details, see Hovey [**11**, Chapter 5], Hirschhorn [**10**, Chapter 16] or the authors' work [**1**, Section 3].

Let $\mathcal{C}$ be a pointed model category and $A \in \mathcal{C}$ a fixed object. Framings give adjoint Quillen functor pairs

$$A \otimes (-) : \mathrm{sSet}_* \rightleftarrows \mathcal{C} : \mathrm{Map}_l(A, -),$$
$$A^{(-)} : \mathrm{sSet}_*^{op} \rightleftarrows \mathcal{C} : \mathrm{Map}_r(-, A).$$

Unfortunately, the construction is not rigid enough to equip any model category with the structure of a simplicial model category. The reason for this is that for two fixed objects A and B the above defined 'left mapping space' $\mathrm{Map}_l(A, B)$ and 'right mapping space' $\mathrm{Map}_r(A, B)$ only agree up to a zigzag of weak equivalences. However, the above

functors possess total derived functors, giving rise to an adjunction of two variables

$$- \otimes^L - : \mathrm{Ho}(\mathcal{C}) \times \mathrm{Ho}(\mathrm{sSet}) \longrightarrow \mathrm{Ho}(\mathcal{C}),$$
$$R\,\mathrm{Map}(-,-) : \mathrm{Ho}(\mathcal{C})^{op} \times \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathrm{sSet}_*),$$
$$R(-)^{(-)} : \mathrm{Ho}(\mathrm{sSet})^{op} \times \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{C}).$$

THEOREM 3.2 (*Hovey*). *Let $\mathcal{C}$ be a pointed model category. Then its homotopy category* $\mathrm{Ho}(\mathcal{C})$ *is a* $\mathrm{Ho}(\mathrm{sSet}_*)$-*module category.*

In particular, the homotopy function complex $\mathrm{Map}_{\mathcal{C}}$ is weakly equivalent to $R\,\mathrm{Map}$. Hence, we will abuse notation and only write Map instead of $R\,\mathrm{Map}$ or $\mathrm{Map}_{\mathcal{C}}$. The suspension and loop functors can also be described using framings (see Hovey [**11**, chapter 6]).

LEMMA 3.3. *Let* $\mathbb{S}^1 \in \mathrm{sSet}_*$ *denote the simplicial circle. Then*

$$\Sigma X \cong X \otimes^L \mathbb{S}^1 \ and \ \Omega X \cong (RX)^{\mathbb{S}^1}.$$

$\square$

Another model category notion relevant to this paper is properness. This definition does not seem important at first sight but is crucial to many of the results about the existence of a localisation.

DEFINITION 3.4. A model category is left proper if every pushout of a weak equivalence along a cofibration is again a weak equivalence. Dually, it is said to be right proper if every pullback of a weak equivalence along a fibration is again a weak equivalence. It is proper if it is both left and right proper.

Recall that a model category $\mathcal{C}$ is said to be cofibrantly generated if there are *sets* of maps (rather than classes) that generate the cofibrations and acyclic cofibrations of $\mathcal{C}$. More precisely,

DEFINITION 3.5. A model category $\mathcal{C}$ is cofibrantly generated if there exist sets of maps I and J such that

- a morphism in $\mathcal{C}$ is a fibration if and only if it has the right lifting property with respect to all elements in I,
- a morphism in $\mathcal{C}$ is an acyclic fibration if and only if it has the right lifting property with respect to all elements in J.

Furthermore, I and J have to satisfy the small object argument, that is, the domains of the elements of I (and J) are small relative to I (respectively J).

For details of smallness and the small object argument, see Hirschhorn [**10**, Section 10.5.14]. The concept of cofibrant generation is crucial to some statements about model categories and in general allows many proofs to be greatly simplified.

A *cellular model category* is a cofibrantly generated model category where the generating cofibrations and acyclic cofibrations satisfy some more restrictive properties regarding smallness (see [**10**, Definition 12.1.1]). Not every cofibrantly generated model category is cellular, but many naturally occurring model categories are. Examples include simplicial sets, topological spaces, chain complexes of R-modules, sequential spectra, symmetric spectra, orthogonal spectra and EKMM $\mathbb{S}$-modules.

4. Stable left localisation. In this section, we introduce the notion of the left Bousfield localisation with respect to a 'stable' class of morphisms. We then show that in this framework, the left Bousfield localisation of a stable model category remains stable. We will see that if $\mathcal{C}$ is a stable model category and S is a stable class of maps, then $L_S\mathcal{C}$ (provided it exists) is right proper whenever $\mathcal{C}$ is. Furthermore, if $\mathcal{C}$ is cellular and proper, we can specify a very convenient set of generating cofibrations and acyclic cofibrations for $L_S\mathcal{C}$.

In Section 2, we defined the notion of S-local objects and S-equivalences for a class of maps $S \subset \mathcal{C}$. Note that elements $s \in S$ are automatically S-equivalences, although the converse does not have to be true. For example, any weak equivalence in $\mathcal{C}$ is an S-equivalence.

By Hirschhorn [**10**, Theorem 4.1.1], in well-behaved cases the S-local model structure on $\mathcal{C}$ exists. In particular, this result requires S to be a set.

THEOREM 4.1 *(Hirschhorn). Let $\mathcal{C}$ be a left proper, cellular model category. Let S be a set of maps in $\mathcal{C}$. Then there is a model structure $L_S\mathcal{C}$ on the underlying category $\mathcal{C}$ such that*

- *weak equivalences in $L_S\mathcal{C}$ are S-equivalences,*
- *the cofibrations in $L_S\mathcal{C}$ are the cofibrations in $\mathcal{C}$.*

The fibrations in this model structure are called S-fibrations.

Note that fibrant replacement U_S in $L_S\mathcal{C}$ is a localisation, that is, an S-equivalence

$$X \longrightarrow U_S(X),$$

where $U_S(X)$ is S-local. It is important to distinguish between fibrant in $\mathcal{C}$, S-fibrant and S-local. The first two are model category conditions, the third is a condition on the homotopy type of an object. Note that an object is S-fibrant if and only if it is S-local and fibrant in $\mathcal{C}$.

The functors Σ and Ω interact well with homotopy function complexes since all three can be defined via framings. In particular, we have weak equivalences as below:

$$\mathrm{Map}(\Sigma X, Y) \simeq \mathrm{Map}(X, \Omega Y) \simeq \Omega \mathrm{Map}(X, Y).$$

Combining this adjunction with Definition 2.3, we obtain the following pair of facts:

- The class of S-equivalences is closed under Σ.
- The class of S-local objects is closed under Ω.

DEFINITION 4.2. Let S be a class of maps in $\mathcal{C}$. We say that S is stable if the collection of S-local objects is closed under Σ.

EXAMPLE 4.3. Let $\mathcal{C}$ be either the category of pointed simplicial sets or the category of spectra. Let S be the class of E_*-isomorphisms for a generalised homology theory E. Then S is not a set in either of these two cases, but it is stable and $L_S\mathcal{C}$ exists.

A simple adjunction argument shows the following.

LEMMA 4.4. *If $\mathcal{C}$ is a stable model category, then a class of maps S is stable if and only if the collection of S-equivalences is closed under Ω. In particular, if the class S is closed under Ω, then the class S is stable.* $\square$

REMARK 4.5. The definitions of S-equivalences and S-local objects are given in terms of homotopy function complexes, denoted $\mathrm{Map}(-, -)$. However, since we work

in a stable context we can rewrite these definitions into more familiar forms involving $[-, -]_*^{\mathcal{C}}$, the graded set of maps in the homotopy category of $\mathcal{C}$.

By Hirschhorn [**10**, Theorem 17.7.2], there is a natural isomorphism

$$\pi_0 \operatorname{Map}(X, Y) \cong [X, Y]^{\mathcal{C}}.$$

It follows that

$$\pi_n \operatorname{Map}(X, Y) \cong [X, Y]_n^{\mathcal{C}} \text{ for } n \geqslant 0.$$

Similarly,

$$\pi_n \operatorname{Map}(\Omega^k X, Y) \cong [X, Y]_{n-k}^{\mathcal{C}} \text{ for } n, k \geqslant 0.$$

It follows that $f \colon X \to Y$ is an S-equivalence if and only if the map

$$[f, Z]_*^{\mathcal{C}} \colon [Y, Z]_*^{\mathcal{C}} \longrightarrow [X, Z]_*^{\mathcal{C}}$$

is an isomorphism of graded abelian groups for every S-local Z.

PROPOSITION 4.6. *Let $\mathcal{C}$ be a stable model category, let S be a class of maps and assume that $L_S\mathcal{C}$ exists. Then, $L_S\mathcal{C}$ is a stable model category if and only if S is a stable class of maps.*

Proof. The homotopy category of $L_S\mathcal{C}$ is equivalent to the full subcategory of $\operatorname{Ho}(\mathcal{C})$ with object class given by the S-local objects. In Section 3, we defined the functor Ω in terms of framings. In particular, the restriction of the functor

$$\Omega \colon \operatorname{Ho}(\mathcal{C}) \to \operatorname{Ho}(\mathcal{C})$$

to $\operatorname{Ho}(L_S\mathcal{C})$ is naturally isomorphic to the desuspension functor on $\operatorname{Ho}(L_S\mathcal{C})$ coming from framings on the model category $L_S\mathcal{C}$. We thus see that

$$\Omega \colon \operatorname{Ho}(L_S\mathcal{C}) \to \operatorname{Ho}(L_S\mathcal{C})$$

is a fully faithful functor as it is the restriction of an equivalence to a full subcategory. We must show that it is essentially surjective. Consider some S-local X, then the suspension ΣX of X is also S-local as S is a stable class of maps. Hence, ΣX is in $\operatorname{Ho}(L_S\mathcal{C})$ and the unit of the adjunction (Σ, Ω) on $\operatorname{Ho}(\mathcal{C})$ gives an isomorphism

$$X \to \Omega\Sigma X$$

in $\operatorname{Ho}(\mathcal{C})$ and hence in $\operatorname{Ho}(L_S\mathcal{C})$.

For the converse, assume that $L_S\mathcal{C}$ is stable, and consider some S-local X. Then

$$\Omega \colon \operatorname{Ho}(L_S\mathcal{C}) \to \operatorname{Ho}(L_S\mathcal{C})$$

is an essentially surjective functor. Hence, there is some S-local Y such that ΩY is isomorphic to X in $\operatorname{Ho}(L_S\mathcal{C})$. It follows that ΩY is isomorphic to X in $\operatorname{Ho}(\mathcal{C})$. Then by stability of $\mathcal{C}$, $\Sigma\Omega Y \cong Y$ is isomorphic to ΣX in $\operatorname{Ho}(\mathcal{C})$. Since Y is S-local, it follows that ΣX must also be S-local; hence, S is a stable class of maps. $\square$

Therefore, for a stable class S, the homotopy category of $\operatorname{Ho}(L_S\mathcal{C})$ is triangulated, which is the vital ingredient of the next proposition. By Hirschhorn [**10**, Proposition

3.4.4] we know that $L_S\mathcal{C}$ is left proper if $\mathcal{C}$ is left proper. However, we now also have the following.

PROPOSITION 4.7. *Let $\mathcal{C}$ be a stable, right proper model category and S a stable class of maps. If $L_S\mathcal{C}$ exists, then it is right proper.*

Proof. We consider the following pullback square:

$$
\begin{array}{ccc}
X' & \xrightarrow{p'} & Y' \\
{\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\
X & \xrightarrow{p} & Y
\end{array}
$$

where p is an S-fibration (and hence a fibration in $\mathcal{C}$) and v is an S-equivalence. Our goal is to show that u is also an S-equivalence.

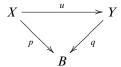The fibre of a map $p : X \longrightarrow Y$ is defined as the pullback of the diagram

$$
X \xrightarrow{p} Y \longleftarrow *.
$$

Since $\mathcal{C}$ is right proper, Hirschhorn [10, Proposition 13.4.6] tells us that the fibre of p is also the homotopy fibre of p, Fp. Similarly, the fibre of p' is also its homotopy fibre Fp'. The fibres are isomorphic since we started with a pullback square; hence, the homotopy fibres are weakly equivalent. Now consider the comparison of exact triangles in $\mathrm{Ho}(\mathcal{C})$

$$
\begin{array}{ccccccc}
\Omega Y' & \longrightarrow & Fp' & \longrightarrow & X' & \longrightarrow & Y' \\
{\scriptstyle \Omega v}\downarrow & & {\scriptstyle \cong}\downarrow & & {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\
\Omega Y & \longrightarrow & Fp & \longrightarrow & X & \longrightarrow & Y.
\end{array}
$$

Since S is stable, this is also a morphism of exact triangles in $\mathrm{Ho}(L_S\mathcal{C})$. Furthermore, Ωv is an S-equivalence. Hence, the five lemma for triangulated categories implies that u is also an S-equivalence, which is what we wanted to show. $\square$

We now need a pair of technical lemmas, the second of which gives a useful characterisation of S-fibrations.

LEMMA 4.8. *Let $\mathcal{C}$ be a stable model category and S a stable class of maps. Assume that $L_S\mathcal{C}$ exists and that we have a commutative triangle in $\mathcal{C}$*

$$
\begin{array}{ccc}
X & \xrightarrow{\ \ u\ \ } & Y \\
 & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\
 & B &
\end{array}
$$

such that the homotopy fibres of p and q are S-local. Then, u is an S-equivalence if and only if it is a weak equivalence in $\mathcal{C}$.

Proof. The above gives a distinguished triangle in Ho($\mathcal{C}$) and hence in Ho($L_S\mathcal{C}$),

$$
\begin{array}{ccccccc}
\Omega B & \longrightarrow & Fp & \longrightarrow & X & \xrightarrow{p} & B \\
\Big\| & & \Big\downarrow{\scriptstyle v} & & \Big\downarrow{\scriptstyle u} & & \Big\| \\
\Omega B & \longrightarrow & Fq & \longrightarrow & Y & \xrightarrow{q} & B
\end{array}
$$

Since S-equivalences between S-local objects are weak equivalences, the result follows. $\qquad\square$

LEMMA 4.9. *Let $\mathcal{C}$ be a stable right proper model category such that $L_S\mathcal{C}$ exists. Consider a fibration $p : X \longrightarrow Y$ in $\mathcal{C}$. Then, p is an S-fibration if and only if the fibre of p is S-fibrant.*

Proof. Since pullbacks of fibrations are fibrations, the fibre of an S-fibration is S-fibrant. Conversely, assume that the fibre Fp is S-fibrant. Since $\mathcal{C}$ is assumed to be right proper, Fp is also the homotopy fibre of p. We factor p in $L_S\mathcal{C}$ as follows:

$$
\begin{array}{ccc}
X & \xrightarrow[\sim]{\;\;j\;\;} & B \\
 & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\
 & Y &
\end{array}
$$

Since the homotopy fibres of p and q are both S-fibrant and hence S-local, j is a weak equivalence in $\mathcal{C}$ by Lemma 4.8. As p is a fibration in $\mathcal{C}$, it has the right lifting property with respect to j :

$$
\begin{array}{ccc}
X & =\!=\!= & X \\
{\scriptstyle\sim}\Big\downarrow{\scriptstyle j} & \nearrow & \Big\downarrow{\scriptstyle p} \\
B & \xrightarrow[q]{\;\;f\;\;} & Y.
\end{array}
$$

The commutative diagram

$$
\begin{array}{ccccc}
X & \xrightarrow{\;j\;} & B & \xrightarrow{\;f\;} & X \\
{\scriptstyle p}\Big\downarrow & & {\scriptstyle q}\Big\downarrow & & {\scriptstyle p}\Big\downarrow \\
Y & =\!=\!= & Y & =\!=\!= & Y
\end{array}
$$

shows that p is a retract of the S-fibration q and hence an S-fibration itself, which is what we wanted to show. $\qquad\square$

We are now almost ready to prove our main theorem for this section which gives a very convenient description of the generating cofibrations and acyclic cofibrations of $L_S\mathcal{C}$ when S is assumed to be stable. For technical reasons, we want S to consist of cofibrations between cofibrant objects. Any map is weakly equivalent to such a map and changing the maps in S up to weak equivalence does not alter the weak equivalences of $L_S\mathcal{C}$, so this is no restriction.

Before we give the theorem, we need an extra piece of terminology (see Hirschhorn [**10**, Definition 3.3.8]. Recall that in Section 3 we defined the action of simplicial sets

on $\mathcal{C}$ via framings, which gives a bifunctor

$$- \otimes - : \mathcal{C} \times \mathrm{sSet}_* \longrightarrow \mathcal{C}.$$

In particular, if the model category $\mathcal{C}$ is simplicial, then this agrees with the given simplicial action on $\mathcal{C}$.

DEFINITION 4.10. Let $f : A \to B$ be a map of $\mathcal{C}$ and let $i_n : \partial\Delta[n]_+ \to \Delta[n]_+$ be the standard inclusion of pointed simplicial sets. Then we define a set of horns on a set of maps S in $\mathcal{C}$ to be the set of maps of $\mathcal{C}$ below.

$$\Lambda S = \left\{ f \square i_n : A \otimes \Delta[n]_+ \underset{A \otimes \partial\Delta[n]_+}{\amalg} B \otimes \partial\Delta[n]_+ \to B \otimes \Delta[n]_+ \mid (f : A \to B) \in S, \ n \geqslant 0 \right\}.$$
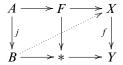
In the above definition, one has to choose cosimplicial resolutions of A and B such that f induces a Reedy cofibration between the resolutions. However, the theorem below is independent of these choices. Note that if S consists of cofibrations between cofibrant objects, so does ΛS.

THEOREM 4.11. *Let $\mathcal{C}$ be a stable, proper, cellular model category with generating cofibrations I and generating acyclic cofibrations J. Let S be a stable set of cofibrations between cofibrant objects. Then, $L_S \mathcal{C}$ is cellular with respect to the sets I and $J \cup \Lambda S$. Hence, in particular, $J \cup \Lambda S$ is a set of generating acyclic cofibrations for the S-local model structure on $\mathcal{C}$.*

Proof. Note that our assumptions imply that $L_S \mathcal{C}$ exists and is cellular with respect to the set I and some set of generating acyclic cofibrations that is constructed in Hirschhorn [**10**, Proposition 4.5.1]. Our task is to show that the triple $(\mathcal{C}, I, J \cup \Lambda S)$ satisfies the conditions of the definition of a cellular model category (see [**10**, Definition 12.1.1]). Hence, we must show that I and $J \cup \Lambda S$ are generating sets for the S-local model structure on $\mathcal{C}$ and that these sets satisfy the additional smallness conditions of Hirschhorn's definition.

First of all, let us prove the following claim. Assume that T is a set of cofibrations that are also S-equivalences. Furthermore, assume that $Z \in \mathcal{C}$ is S-fibrant if and only if the map $Z \longrightarrow *$ has the right lifting property with respect to $J \cup T$. Then a map f is an S-fibration if and only if it has the right lifting property with respect to $J \cup T$.

If f is an S-fibration, then of course it has the right lifting property with respect to both J and T. So let us assume conversely that $f : X \longrightarrow Y$ is a map that has the right lifting property with respect to $J \cup T$. We want to use Lemma 4.9 and show that F, the fibre of f, is S-fibrant. Take some $j : A \to B$ in $J \cup T$ and consider a lifting square between j and $F \to *$. We may extend that square to include f, as below.

$$
\begin{array}{ccccc}
A & \longrightarrow & F & \longrightarrow & X \\
\downarrow{\scriptstyle j} & & \downarrow & & \downarrow{\scriptstyle f} \\
B & \longrightarrow & * & \longrightarrow & Y
\end{array}
$$

Since f is assumed to have the right lifting property with respect to j, the lift in the diagram exists. By the universal property of the pullback, there is also a map $B \longrightarrow F$ making the left square commute. Thus, F also has the right lifting property with respect to $J \cup T$. Hence, by our assumptions and Lemma 4.9, f is an S-fibration.

Now that we have proved our claim, we are ready to prove that $J \cup \Lambda S$ is indeed a set of generating acyclic cofibrations of $L_S \mathcal{C}$. Thus, we have to show that the assumptions of the above claim hold for $J \cup \Lambda S$. This means we have to show that an object Z is S-fibrant if and only if the map $Z \longrightarrow *$ has the right lifting property with respect to $J \cup \Lambda S$.

The maps of $J \cup \Lambda S$ are cofibrations that are S-local equivalences by Hirschhorn [**10**, Proposition 4.2.3]. Hence, if Z is S-fibrant, then $Z \longrightarrow *$ has the right lifting property with respect to $J \cup \Lambda S$. For the converse, we use [**10**, Proposition 4.2.4], noting that our naming conventions are slightly different from the reference. Thus, we have shown that $(\mathcal{C}, I, J \cup \Lambda S)$ is a cofibrantly generated model category.

Finally, we must show that I and $J \cup \Lambda S$ satisfy the additional smallness conditions of [**10**, Definition 12.1.1]. Since $\mathcal{C}$ is cellular, this amounts to proving that the domains of ΛS are small relative to I. The domains of ΛS are cofibrant; hence, they are small with respect to I by [**10**, Lemma 12.4.2]. $\square$

REMARK 4.12. The work of Hirschhorn [**10**] uses in an essential manner the assumption that $\mathcal{C}$ is cellular to obtain a set of generating set of acyclic cofibrations for $L_S \mathcal{C}$. The reference then uses this set to show that $L_S \mathcal{C}$ exists. We have used stability to find such a set and then used the assumption that $\mathcal{C}$ is cellular to see that this set satisfies the conditions of the small object argument.

Hence, we have a partial refinement of the above theorem to the case when $\mathcal{C}$ is not cellular. Assume that $\mathcal{C}$ is a stable, proper cofibrantly generated model category and S is a stable set of cofibrations. If the domains of $J \cup \Lambda S$ are small relative to the class of transfinite compositions of pushouts of $J \cup \Lambda S$, then $L_S \mathcal{C}$ exists and is cofibrantly generated by the sets I and $J \cup \Lambda S$. Furthermore, it is stable and proper.

Theorem 4.11 is a considerable improvement on the general situation where $\mathcal{C}$ has not been assumed to be stable. Without stability, the results of Hirschhorn [**10**] only prove the existence of some set of generating acyclic cofibrations. Indeed, the set $J \cup \Lambda S$ is not always a generating set of acyclic cofibrations for $L_S \mathcal{C}$, as shown by [**10**, Example 2.1.6], which we will spell out below. The proof that $L_S \mathcal{C}$ exists and is cofibrantly generated in the unstable case uses the Bousfield-Smith cardinality argument. So in general it is all but impossible to obtain a useful description of the generating acyclic cofibrations from the proof.

EXAMPLE 4.13. Consider the model category of topological spaces with weak equivalences the weak homotopy equivalences. Let $n > 0$ and let $f : S^n \to D^{n+1}$ be the inclusion. We now look at localisation with respect to $S = \{f\}$.

The path space fibration

$$p : PK(\mathbb{Z}, n) \to K(\mathbb{Z}, n)$$

has the right lifting property with respect to $J \cup \Lambda \{f\}$. Hence, every $J \cup \Lambda \{f\}$-cofibration has the left lifting property with respect to p. But the cofibration $* \to S^n$ does not have this left lifting property. The composite map $* \to S^n \to D^{n+1}$ is clearly an $\{f\}$-local equivalence as is f itself. Hence, $* \to S^n$ is a cofibration and an $\{f\}$-local-equivalence that is not a $J \cup \Lambda \{f\}$-cofibration.

We can also use Theorem 4.11 to consider smashing localisations of spectra. Recall that Bousfield localisation of a model category of spectra $\mathcal{S}$, such as symmetric spectra

or EKMM $\mathbb{S}$-modules is called *smashing* if for every spectrum X the map

$$\lambda \wedge^L \text{Id}_X : X \longrightarrow X \wedge^L L_E \mathbb{S}$$

is an E-localisation.

LEMMA 4.14. *If localisation with respect to E is smashing, then $L_E \mathbb{S} = L_\Gamma \mathbb{S}$ for*

$$\Gamma = \{\Sigma^n \lambda : \mathbb{S}^n \longrightarrow L_E \mathbb{S}^n \mid n \in \mathbb{Z}\}.$$

Proof. Every element in Γ is an E-equivalence; hence, every Γ-equivalence is an E-equivalence. Let us now consider the following commutative diagram, where $f : X \longrightarrow Y$ is a map of spectra and Z is Γ-local:

$$
\begin{array}{ccc}
[Y, Z]_* & \xrightarrow{\quad f^* \quad} & [X, Z]_* \\
{\scriptstyle \cong} \big\uparrow & & \big\uparrow {\scriptstyle \cong} \\
[Y \wedge^L L_E \mathbb{S}, Z]_* & \xrightarrow{(f \wedge^L L_E \mathbb{S})^*} & [X \wedge^L L_E \mathbb{S}, Z]_*
\end{array}
$$

The vertical arrows are isomorphisms because the map $X \longrightarrow X \wedge^L L_E \mathbb{S}$ is a Γ-equivalence and Z is Γ-local. To see this, note that the class of objects X for which this is a Γ-equivalence is closed under coproducts and exact triangles, and contains the sphere.

Now let f be an E-equivalence. By assumption, this is equivalent to $f \wedge^L L_E \mathbb{S}$ being a weak equivalence. This implies that the bottom row of the commutative square is an isomorphism. Hence, the top row is an isomorphism and thus f is a Γ-equivalence. □

COROLLARY 4.15. *Let $\mathbb{S}$ be the model category of symmetric spectra or EKMM $\mathbb{S}$-modules with generating cofibrations I and acyclic cofibrations J. Let L_E be a smashing Bousfield localisation with respect to a homology theory E. Then, $L_E \mathbb{S}$ is proper, stable and cellular with generating cofibrations I and generating acyclic cofibrations $J \cup \Lambda \Gamma$.* □

A further refinement on the generating sets appears as Corollary 6.7.

5. Stable right localisations. In this section, we introduce the notion of right Bousfield localisation with respect to a *stable* class of objects. We then proceed by showing that in this framework, the right Bousfield localisation of a stable model category remains stable. We will see that if $\mathcal{C}$ is a stable model category and K is a stable class of maps, then $R_K \mathcal{C}$ (provided it exists) is left proper whenever $\mathcal{C}$ is. Furthermore, if $\mathcal{C}$ is cellular and right proper, we can specify a very convenient set of generating cofibrations and acyclic cofibrations for $R_K \mathcal{C}$.

Right Bousfield localisation is the dual notion to left Bousfield localisation, as we have mentioned above. We defined K-coequivalences and K-colocal objects in Section 2. Note that our definitions imply that any object of K is K-colocal, but the converse is not necessarily true. Also, any weak equivalence of $\mathcal{C}$ is a K-coequivalence.

In well-behaved cases, it is possible to construct a right localisation of $\mathcal{C}$ with respect to K. We state the general result [**10**, Theorem 5.1.1] below.

THEOREM 5.1 (*Hirschhorn*). *Let* $\mathcal{C}$ *be a right proper cellular model category and* K *a set of objects in* $\mathcal{C}$. *Then there exists a model structure* $R_K\mathcal{C}$ *on the underlying category* $\mathcal{C}$ *such that*

- *the weak equivalences in* $R_K\mathcal{C}$ *are the* K-*coequivalences*
- *the fibrations in* $R_K\mathcal{C}$ *are the fibrations of* $\mathcal{C}$.

One has to distinguish between K-cofibrant, cofibrant in $\mathcal{C}$ and K-colocal. Note that an object is K-cofibrant if and only if it is K-colocal and cofibrant in $\mathcal{C}$. The cofibrant replacement functor Q_K of $R_K\mathcal{C}$ provides a colocalisation for an object X, that is, a K-coequivalence

$$Q_K(X) \longrightarrow X$$

with $Q_K(X)$ a K-colocal object of $\mathcal{C}$.

EXAMPLE 5.2. Let us again return to the example where $\mathcal{C} = \mathrm{Ch}(R)$ and A is a perfect R-module. In this special case, the cofibrant replacement Q_A provides the A-cellular approximation

$$\mathrm{Cell}_A(M) \longrightarrow M.$$

This means that $\mathrm{Cell}_A(M)$ is 'built' from A using exact triangles and coproducts (see Dwyer and Greenless [**6**, Section 4]). In this setting, cellular approximation satisfies

$$\mathrm{Cell}_A(M) \cong \mathrm{Cell}_A(R) \otimes_R^L M,$$

giving rise to the cofibrant replacement map

$$\mathrm{Cell}_A(R) \otimes_R^L M \xrightarrow{\sim} M.$$

Analogous to the definition of a smashing left localisation, we can call this right localisation right smashing: a right localisation of a monoidal model category $\mathcal{C}$ with unit S is right smashing if

$$Q_K S \otimes^L X \longrightarrow X$$

is a K-cofibrant approximation for all X.

Dually to the local case, we see that the class of K-coequivalences is closed under Ω. Also, the class of K-colocal objects is closed under Σ.

DEFINITION 5.3. Let K be a class of objects in $\mathcal{C}$. We say that K is stable if the class of K-colocal objects is also closed under Ω.

We also have the dual result to Lemma 4.4: if $\mathcal{C}$ is a stable model category, then a class of objects K is stable if and only if the collection of K-coequivalences objects is closed under Σ. In particular, if K is closed under Ω, then it is stable.

REMARK 5.4. As with Remark 4.5, we see that if K is a stable set of objects, then a map $f : X \to Y$ is a K-coequivalence if and only if

$$[k, f]_*^{\mathcal{C}} : [k, X]_*^{\mathcal{C}} \longrightarrow [k, Y]_*^{\mathcal{C}}$$

is an isomorphism of graded abelian groups for all $k \in K$. Similarly, A is K-colocal if and only if for all K-coequivalences $f : X \to Y$, the map

$$[A, f]_*^{\mathcal{C}} : [A, X]_*^{\mathcal{C}} \longrightarrow [A, Y]_*^{\mathcal{C}}$$

is an isomorphism of graded abelian groups.

EXAMPLE 5.5. The case of A-torsion modules for a perfect R-module A provides an example of a class of stable colocal objects.

PROPOSITION 5.6. *Let $\mathcal{C}$ be a stable model category and K a stable class of objects. Assume that $R_K \mathcal{C}$ exists. Then, $R_K \mathcal{C}$ is also stable.* $\square$

We omit the proof since it is very similar to the proof of Proposition 4.6.

We can always make a set of objects stable, but this usually changes the resulting model structure and homotopy category drastically.

LEMMA 5.7. *Let K be a class of cofibrant objects in a stable model category $\mathcal{C}$. Define $\Omega^\infty K$ to be the collection of objects $Q\Omega^n X$ for $X \in K$ and $n \geqslant 0$. Then, provided it exists, $L_{\Omega^\infty K} \mathcal{C}$ is a stable model category. Furthermore, K is stable if and only if $L_{\Omega^\infty K} \mathcal{C}$ is equal to $L_K \mathcal{C}$.* $\square$

We know that the right localisation of a right proper model category is again right proper by Hirschhorn [**10**, Theorem 5.1.5]. If K is stable, then we also see that $R_K \mathcal{C}$ is left proper whenever $\mathcal{C}$ is.

PROPOSITION 5.8. *Let $\mathcal{C}$ be a stable left proper model category. Let K be a stable class of objects. If $R_K \mathcal{C}$ exists, then it is left proper.*

Proof. Consider a pushout

$$\begin{array}{ccc} A & \xrightarrow{p} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow{q} & P \end{array}$$

where p is a K-cofibration and f is a K-coequivalence. We see immediately that q is a K-cofibration. We would like to show that g is a K-coequivalence.

Since $\mathcal{C}$ is left proper, the cofibre of p (the pushout of p along $A \to *$) is also the homotopy cofibre Cp of p. Similarly, the cofibre of q agrees with the homotopy cofibre Cq of q. Since we have a pushout, the two cofibres are isomorphic; hence, the map c below is a weak equivalence in $R_K \mathcal{C}$. By Proposition 5.6, $R_K \mathcal{C}$ is stable, so the following is a morphism of exact triangles:

$$\begin{array}{ccccccc} A & \longrightarrow & C & \longrightarrow & Cp & \longrightarrow & \Sigma A \\ {\scriptstyle\sim}\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & {\scriptstyle\sim}\downarrow{\scriptstyle c} & & {\scriptstyle\sim}\downarrow{\scriptstyle \Sigma f} \\ B & \longrightarrow & P & \longrightarrow & Cq & \longrightarrow & \Sigma B. \end{array}$$

By the five lemma for triangulated categories g is a K-coequivalence. $\square$

We know that $R_K\mathcal{C}$ has the same fibrations (and hence acyclic cofibrations) as $\mathcal{C}$ but fewer cofibrations. Generally, it is very hard to specify a set of generating cofibrations for $R_K\mathcal{C}$. However, if $\mathcal{C}$ and K are stable, we obtain a convenient description.

Following the previous section, a set of *horns* on K is defined as

$$\Lambda K = \{X \otimes \partial\Delta[n]_+ \longrightarrow X \otimes \Delta[n]_+ \mid n \geqslant 0, X \in K\}.$$

Remember that the operation $\otimes$ is defined via framings in $\mathcal{C}$ as in Section 3. We have assumed that the set K consists of cofibrant objects, so ΛK consists of cofibrations of $\mathcal{C}$.

THEOREM 5.9. *Let $\mathcal{C}$ be a stable, right proper, cellular model category with a set of generating cofibrations I and generating acyclic cofibrations J. Let K be a stable set of cofibrant objects. Then, $R_K\mathcal{C}$ is cellular with generating cofibrations $J \cup \Lambda K$ and acyclic cofibrations J.*

Proof. We know that the model structure exists, is stable and is right proper. We prove that $R_K\mathcal{C}$ is a cellular model category, via Hirschhorn [10, Theorem 12.1.9]. The various smallness and compactness arguments follow from the corresponding statements for $\mathcal{C}$ and the fact that K consists of cofibrant objects.

All that remains is to show that a map f is a trivial K-fibration if and only if it has the right lifting property with respect to $J \cup \Lambda K$. By [10, Proposition 5.2.5] the maps of $J \cup \Lambda K$ are cofibrations of $R_K\mathcal{C}$. Hence, if f is a fibration and K-coequivalence, then f has the right lifting property with respect to $J \cup \Lambda K$. Now assume that f has the right lifting property with respect to $J \cup \Lambda K$. Since f has the right lifting property with respect to J, it is a fibration in $\mathcal{C}$ and hence it is a fibration in $R_K\mathcal{C}$. Now we want to show that f is a K-coequivalence.

By [10, Proposition 5.2.4], a map $g : A \longrightarrow B$ with B *fibrant* has the right lifting property with respect to $J \cup \Lambda K$ if and only if g is a fibration and a K-coequivalence. But this is not true for general B and we cannot simply assume B to be fibrant.

However, we are working in a stable setting. Since $R_K\mathcal{C}$ is stable, f being a K-coequivalence is equivalent to asking for its fibre (which in our setting is also its homotopy fibre) to be K-coacyclic. The fibre F is the pullback of the diagram

$$* \longrightarrow B \xleftarrow{\ f\ } A.$$

As f has the right lifting property with respect to $J \cup \Lambda K$ and F is a pullback, $F \longrightarrow *$ also has this right lifting property. The terminal object $*$ is fibrant, so by [10, Proposition 5.2.4] F is K-coacyclic, which is what we needed to prove. $\qquad\square$

REMARK 5.10. Just as with Remark 4.12 we can replace the assumption that $\mathcal{C}$ is cellular with the assumption that $\mathcal{C}$ is cofibrantly generated and the domains of $J \cup \Lambda K$ are small with respect to the class of transfinite compositions of pushouts of $J \cup \Lambda K$. Thus, the theorem also provides a refinement of the general existence theorem of right localisations for the stable case.

The theorem is again an improvement on the general setting where $\mathcal{C}$ has not been assumed to be stable. Without stability, the results of Hirschhorn [10] only prove the existence of some set of generating cofibrations. Indeed, the set $J \cup \Lambda K$ is not always a generating set of cofibrations for $L_S\mathcal{C}$, as shown by [10, Example 5.2.7], which we will spell out now.

EXAMPLE 5.11. Consider the model category of pointed simplicial sets sSet$_*$. Let A be the quotient of $\Delta[1]$ obtained by identifying the the vertices of $\Delta[1]$. The geometric realisation of this simplicial set is homeomorphic to the circle. We consider the right localisation of sSet$_*$ with respect to $K = \{A\}$, having one 0-simplex and one 1-simplex.

Let Y be $\partial\Delta[2]$, whose geometric realisation is also homeomorphic to the circle. Let X be the simplicial set built from six 1-simplices with vertices identified so that the geometric realisation of X is a circle. There is a fibration $p: X \to Y$, whose geometric realisation is the double covering of the circle.

Now let $F(A, X)$ denote the simplicial set of maps from A to X. We observe that $F(A, X)$ has only one simplex in each degree. The reason for this is the fact that the only pointed map from A to X is the constant map to the basepoint. By induction, this also holds for maps from $A \wedge \Delta[n]_+$ to X. The same is true for $F(A, Y)$, so

$$F(A, p) : F(A, X) \longrightarrow F(A, Y)$$

is an isomorphism.

The map p is a fibration, so it has the right lifting property with respect to J. The above argument shows that p also has the right lifting property with respect to $\Lambda(A)$; hence, it has the right lifting property with respect to $J \cup \Lambda(A)$.

But p is not a K-coequivalence as we shall show now. Consider the map below, which is induced by p :

$$\operatorname{sing}\operatorname{hom}(|A|, |p|) : \operatorname{sing}\operatorname{hom}(|A|, |X|) \longrightarrow \operatorname{sing}\operatorname{hom}(|A|, |Y|),$$

where $\operatorname{hom}(|A|, |X|)$ denotes the space of maps between the topological spaces $|A|$ and $|X|$ and sing the singular complex functor. However, since $|p|$ is a double cover of the circle, the map

$$\pi_0(\operatorname{sing}\operatorname{hom}(|A|, |p|)) : \pi_0(\operatorname{sing}\operatorname{hom}(|A|, |X|)) \longrightarrow \pi_0 \operatorname{sing}(\operatorname{hom}(|A|, |Y|))$$

is multiplication by 2 on the integers. Thus, $\operatorname{sing}\operatorname{hom}(|A|, |p|)$ is not a weak equivalence. Now we note that for any simplicial sets P and Q, $\operatorname{Map}(P, Q)$ is naturally weakly equivalent to $\operatorname{sing}\operatorname{hom}(|P|, |Q|)$. Thus, $\operatorname{Map}(A, p)$ is not a weak equivalence as claimed.

6. Monoidal left localisations.

Let $\mathcal{C}$ be a cellular and left proper model category and let S be a set of maps in $\mathcal{C}$. Then we can ask the following: if $\mathcal{C}$ is monoidal, when is $L_S\mathcal{C}$ also monoidal? When $\mathcal{C}$ is stable, we can use our preceding results to examine monoidality in a convenient way.

For this we need to know that $L_S\mathcal{C}$ satisfies the pushout product axiom. Recall that the *pushout product* of two maps $f : A \to B$ and $g : C \to D$ is defined as

$$f \square g : A \otimes D \coprod_{A \otimes C} B \otimes C \longrightarrow B \otimes D.$$

A model category with monoidal product and unit $(\mathcal{C}, \otimes, \mathbb{S})$ is a *monoidal model category* if the pushout product of two cofibrations is again a cofibration which is trivial if either f or g is. Furthermore, the unit $\mathbb{S}$ of $\mathcal{C}$ has to satisfy a cofibrancy condition (see Hovey [**11**, Definition 4.2.6]).

Thus, the usual method to examine monoidality of a model category is to examine its sets of generating cofibrations and generating acyclic cofibrations. By Theorem 4.11

we know that if $\mathcal{C}$ is stable and proper and that S is a stable set of cofibrations between cofibrant objects, then the generating acyclic cofibrations of $L_S\mathcal{C}$ have the form $J \cup \Lambda S$. Since $\mathcal{C}$ is assumed to be monoidal, we know that $J\square I$ consists of weak equivalences in $\mathcal{C}$. Thus, if $\Lambda S\square I$ is a set of S-equivalences, then $L_S\mathcal{C}$ is monoidal. Conversely, if $L_S\mathcal{C}$ is monoidal, then $\Lambda S\square I$ consists of S-equivalences.

Now we apply Hovey [**11**, Theorem 5.6.5], which essentially states that framings and monoidal products interact well, to see that the image of the set $\Lambda S\square^L I$ in $\text{Ho}(\mathcal{C})$ is isomorphic to the image of the set $\Lambda(S\square^L I)$ in $\text{Ho}(\mathcal{C})$. If we assume that the domains of I are cofibrant, then the derived pushout product $S\square^L I$ is equal to the actual pushout product $S\square I$. Similarly, $\Lambda S\square^L I = \Lambda S\square I$. Thus, $\Lambda S\square I$ consists of S-equivalences if and only if $\Lambda(S\square I)$ consists of S-equivalences. Furthermore, $S\square I$ consists of S-equivalences if and only if $\Lambda(S\square I)$ consists of S-equivalences. Hence, we have the following result and definition.

LEMMA 6.1. *Let $\mathcal{C}$ be a proper, cellular and monoidal stable model category. Let S be a stable set of cofibrations between cofibrant objects. Assume that the domains of the generating cofibrations I are cofibrant. Then the set $S\square I$ is contained in the class of S-equivalences if and only if $L_S\mathcal{C}$ is a monoidal model category.* $\square$

DEFINITION 6.2. A stable set of cofibrations S in a monoidal model category $\mathcal{C}$ is said to be monoidal if $S\square I$ is contained in the class of S-equivalences.

We can use this to restate a well-known fact.

EXAMPLE 6.3. The generating set $\mathcal{J}$ of E_*-equivalences in $\mathcal{MS}$, the model category of EKMM $\mathbb{S}$-modules, is monoidal. This follows from the fact that if f is an E_*-equivalence and A is a cofibrant spectrum, then $f \otimes A$ is also an E_*-equivalence. Hence, by Lemma 6.1, $L_E(\mathcal{MS})$ is a monoidal model category.

LEMMA 6.4. *Let $\mathcal{C}$ be a proper, stable, cellular, monoidal model category. Assume that S is a stable set of cofibrations between cofibrant objects and that the domains of the generating cofibrations I are cofibrant. Then $S\square I$ is a monoidal stable set of maps. Hence, $L_{S\square I}\mathcal{C}$ is a stable monoidal model category in which the maps S are weak equivalences.*

Proof. Take any $s \in S$ and any cofibration a. Then the map a is a retract of pushouts of transfinite compositions of maps in I. Hence, $s\square a$ a retract of pushouts of transfinite compositions of maps in the set $S\square I$. Thus, $s\square a$ is an $S\square I$-equivalence.

We need to check that $S\square I$ is still stable, so consider some $s\square i$. Let $\mathbb{S}^{-1}$ be some cofibrant desuspension of the unit $\mathbb{S}$ of $\mathcal{C}$. We know that $(s\square i) \otimes \mathbb{S}^{-1}$ is isomorphic to $s\square(i \otimes \mathbb{S}^{-1})$, which, by the above, is an $S\square I$-equivalence. It follows immediately that the $S\square I$-equivalences are closed under desuspension, so our set is stable.

Now, we must check that $(\Lambda(S\square I))\square I$ consists of $S\square I$-equivalences. But every element in $(\Lambda(S\square I)\square I)$ is weakly equivalent to an element in $\Lambda(S\square(I\square I))$ by Hovey [**11**, Theorem 5.6.5] and our assumption on the domains of I. We know that any map in $S\square(I\square I)$ is an $S\square I$-equivalence and a cofibration. Furthermore, a horn on such a map is still an $S\square I$-equivalence.

Finally, to see that S consists of $S\square I$-equivalences, consider the cofibration $\eta\colon * \to Q\mathbb{S}$. For any $s \in S$, $s\square\eta$ is isomorphic to $s \otimes Q\mathbb{S}$, which is weakly equivalent to s since the domains and codomains of S are cofibrant. $\square$
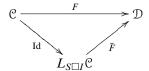
We may also conclude that if S is monoidal, then $L_{S\square I}\mathcal{C}$ is equal to $L_S\mathcal{C}$. Usually, however, localising at $S\square I$ and S gives different model categories. While the above

result makes more maps into weak equivalences than we might want, it actually does so in quite a minimal way, as the result below shows. We can think of this as saying that $L_{S\square I}\mathcal{C}$ is the *monoidal* left Bousfield localisation of $\mathcal{C}$ at the stable set S.

THEOREM 6.5. *Let*

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

be a lax monoidal Quillen pair between monoidal model categories $\mathcal{C}$ and $\mathcal{D}$. Assume that $\mathcal{C}$ is proper, stable and cellular and that the domains of its generating cofibrations I are cofibrant. Let S be a stable set of cofibrations between cofibrant objects in $\mathcal{C}$. If $F(s)$ is a weak equivalence in $\mathcal{D}$ for all $s \in S$, then this adjoint pair factors uniquely over the change of model structures adjunction between $\mathcal{C}$ and $L_{S\square I}\mathcal{C}$. That is, we have a commutative diagram of left adjoints of weak monoidal Quillen pairs

Proof. We must show that the image under F of every element $S\square I$ is an isomorphism in $\mathrm{Ho}(\mathcal{D})$. We must show that for any $s \in S$, $F(s\square i)$ is a weak equivalence in $\mathcal{D}$. We have a weak monoidal Quillen pair and the domain and codomain of $s\square i$ are cofibrant. Thus, we see that $F(s\square i)$ is weakly equivalent to $Fs\square Fi$. Since $\mathcal{D}$ is monoidal, this is an acyclic cofibration of $\mathcal{D}$.

Hence, we have the desired factorisation of Quillen functors via the universal property of left Bousfield localisations (see Hirschhorn [**10**, Definition 3.1.1]). Furthermore, the thus obtained $\bar{F}$ and its right adjoint $\bar{G}$ form a lax monoidal Quillen pair between the monoidal model categories $L_{S\square I}\mathcal{C}$ and $\mathcal{D}$. □

If we restrict ourselves to spectra, then we can use the above to obtain a very concise description of the generating sets of a monoidal stable localisation. For the result below we can use EKMM $\mathbb{S}$-modules, symmetric spectra, orthogonal spectra or their equivariant versions for a compact Lie group.

PROPOSITION 6.6. *Let $\mathbb{S}$ be a monoidal model category of spectra from the list above. Let S be some set of cofibrations between cofibrant objects in $\mathcal{C}$. Then $L_{S\square I}\mathbb{S}$ exists, is cellular, proper, stable and monoidal. It has generating sets given by I and $J \cup (S\square I)$.*

Proof. The model category of spectra $\mathbb{S}$ comes equipped with a collection of evaluation functors

$$U_V : \mathcal{C} \longrightarrow \mathrm{sSet}_*$$

for each V of the indexing category (such as the non-negative integers or finite dimensional real inner product spaces). Let F_V be the left adjoint to U_V.

We see that the set of generating cofibrations I of $\mathbb{S}$ can be chosen to consist of maps of the form $F_V l$, where l is some generating cofibration for simplicial sets. It follows immediately that they have cofibrant domains and $S\square I$ is stable. Hence, by Lemma 6.4, we know that $L_{S\square I}\mathbb{S}$ is monoidal. By the results of Section 4 we know that it is also stable, proper and cellular and cofibrantly generated by I and $J \cup \Lambda(S\square I)$.

We now need to show that a map f has the right lifting property with respect to $\Lambda(S\square I)$ if and only if it has the right lifting property with respect to $S\square I$.

Because $\mathcal{S}$ is a simplicial model category, we can assume that an element in $\Lambda(S\square I)$ is of the form $(s\square F_V l)\square k$, where l and k are generating cofibrations for simplicial sets and $s\in S$. But this is isomorphic to $S\square F_V(l\square k)$. It follows that the sets $\Lambda(S\square I)$ and $S\square(I\square I)$ agree.

We have $I\subseteq I\square I$ because $\iota\square F_V l = F_V l$, where

$$\iota = (F_0\partial\Delta[0]_+ \longrightarrow F_0\Delta[0]_+).$$

Thus,

$$S\square I \subseteq S\square(I\square I) = \Lambda(S\square I)$$

and hence if we define A–cof to be the class of maps with the left lifting property with respect to all maps with the right lifting property with respect to A, then we see that

$$(S\square I)\text{–cof} \subseteq \Lambda(S\square I)\text{–cof}.$$

For the other inclusion, we know that model category $\mathcal{S}$ is monoidal, so $I\square I \subseteq I$–cof. Thus,

$$\Lambda(S\square I) = S\square(I\square I) \subseteq S\square(I\text{–cof}) \subseteq (S\square I)\text{–cof}.$$

$\square$

Recall from Section 6 that in the case of a smashing localisation we have $L_E\mathcal{S} = L_\Gamma\mathcal{S}$ for

$$\Gamma = \{\Sigma^n\lambda : \Sigma^n\mathbb{S} \longrightarrow \Sigma^n L_E\mathbb{S} \mid n\in\mathbb{Z}\}.$$

Together with Corollary 4.15, we achieve the following.

COROLLARY 6.7. *Let $\mathcal{S}$ be a monoidal model category of spectra with generating cofibrations I and acyclic cofibrations J. Let L_E be a smashing Bousfield localisation. Then $L_E\mathcal{S}$ is proper, cellular, stable and monoidal with generating cofibrations I and generating acyclic cofibrations $J\cup(\Gamma\square I)$.* $\square$

7. Monoidal right localisations. Let $\mathcal{C}$ be a cellular and right proper model category and let K be a set of objects in $\mathcal{C}$. Then we can ask the following: if $\mathcal{C}$ is monoidal, when is $R_K\mathcal{C}$ also monoidal? We can use our preceding work on stability and generating cofibrations to give a compact and useful answer. We will then apply this to some examples.

We start with an observation. Recall that an object in $\mathcal{C}$ is K-cofibrant if and only if it is K-colocal and cofibrant in $\mathcal{C}$. The elements of K are K-cofibrant. Thus, if $R_K\mathcal{C}$ is monoidal, then any element of the form $k\otimes k'$ for $k, k'\in K$ will also be K-cofibrant. We show that this necessary condition is almost sufficient for monoidality of $R_K\mathcal{C}$.

DEFINITION 7.1. Let K be a set of cofibrant objects in a right proper, cellular, monoidal model category $\mathcal{C}$. We say that K is monoidal if the following two conditions hold:

- Any object of the form $k \otimes k'$, for $k, k' \in K$, is K-colocal.
- For $Q_K \mathbb{S}$ a K-cofibrant replacement of the unit $\mathbb{S}$ of $\mathcal{C}$ and any $k \in K$, the map $Q_K \mathbb{S} \otimes k \to k$ is a K-coequivalence.

Note that if the first condition holds, then the domain and codomain of $Q_K \mathbb{S} \otimes k \to k$ are both K-cofibrant. Hence, this map is a K-coequivalence if and only if it is a weak equivalence of $\mathcal{C}$. Obviously, if the monoidal unit is an element of K, then the second condition holds automatically.

Recall that a model category satisfies the *monoid axiom* if all transfinite compositions of pushouts of maps of the from $j \otimes Z$, for j an acyclic cofibration and Z any object of $\mathcal{C}$, are weak equivalences. This is a very useful tool for considering the category of modules over a monoid R in $\mathcal{C}$: if $\mathcal{C}$ is cofibrantly generated, monoidal and satisfies the monoid axiom (and some smallness assumptions hold), then the category of R-modules in $\mathcal{C}$ is also a cofibrantly generated model category by Schwede and Shipley [15, Theorem 4.1].

THEOREM 7.2. *Let $\mathcal{C}$ be a stable, proper, cellular and monoidal model category. Let K be a stable collection of cofibrant objects. Then $R_K \mathcal{C}$ is monoidal if and only if K is monoidal.*

Furthermore, if K is monoidal and $\mathcal{C}$ also satisfies the monoid axiom, then so does $R_K \mathcal{C}$.

Proof. If $R_K \mathcal{C}$ is monoidal, then the pushout product axiom implies that K is monoidal. For the converse, assume that K is monoidal. To show that $R_K \mathcal{C}$ is monoidal, we must verify the two conditions of Hovey [11, Definition 4.2.6]. The second of these, namely that

$$Q_K \mathbb{S} \otimes k \to k$$

is a K-coequivalence, holds by assumption.

Remember from Theorem 5.9 that $R_K \mathcal{C}$ has generating cofibrations $\Lambda K \cup J$ and acyclic cofibrations J. Hence, we must check that $(\Lambda K \cup J) \square (\Lambda K \cup J)$ consists of K-cofibrations. This amounts to proving that the following three collections $\Lambda K \square \Lambda K$, $\Lambda K \square J$ and $J \square J$ consist of K-cofibrations. For the first, consider

$$i = (\partial \Delta[n]_+ \otimes k \to \Delta[n]_+ \otimes k) \square (\partial \Delta[m]_+ \otimes k' \to \Delta[m]_+ \otimes k') \in \Lambda K \square \Lambda K,$$

which is a cofibration in $\mathcal{C}$ since $\mathcal{C}$ was assumed to be monoidal. We can rewrite i, up to weak equivalence, as the following map which is a cofibration of $\mathcal{C}$ between K-colocal objects:

$$((\partial \Delta[n]_+ \to \Delta[n]_+) \square (\partial \Delta[m]_+ \to \Delta[m]_+)) \otimes (k \otimes k').$$

Thus, the domain and codomain of i are K-colocal, so by Hirschhorn [10, Proposition 3.3.16] i is also K-cofibration.

Let us now look at the second collection, $\Lambda K \square J$. A map in this set is contained in the class of maps $I \square J$–cof, which consists of acyclic cofibrations of $\mathcal{C}$. Any such map is a K-cofibration. The same argument holds for the third collection, $J \square J$. Thus, the pushout product of two K-cofibrations is again a K-cofibration which is acyclic if either of the two maps is.

The monoid axiom holds in $R_K\mathcal{C}$ if it holds in $\mathcal{C}$, since the set of generating acyclic cofibrations has not changed. □

We can apply this to Dwyer and Greenlees' example of right Bousfield localisation, where $\mathcal{C} = \mathrm{Ch}(R)$ and $K = \{A\}$ a perfect R-module (see Section 2).

COROLLARY 7.3. *The model category $R_{\{A\}}(\mathrm{Ch}(R))$ of A-torsion R-modules is a monoidal model category.*

Proof. We consider $\mathrm{Ch}(R)$ with the projective model structure. Since A is a perfect chain complex of R-modules, it is of finite length and is degreewise projective. Hence, A is cofibrant in $\mathrm{Ch}(R)$. We are now going to check that $K = \{A\}$ satisfies the two conditions of Definition 7.1.

We remember from Example 5.2 that in this case the cofibrant replacement is the same as cellular approximation and that cellular approximation is given by the weak equivalence

$$\mathrm{Cell}_A(R) \otimes_R^L M \longrightarrow \mathrm{Cell}_A(M).$$

For the unit condition, we must prove that

$$\mathrm{Cell}_A(R) \otimes_R^L \mathrm{Cell}_A(M) \longrightarrow \mathrm{Cell}_A(M)$$

is an $\{A\}$-coequivalence for any M. But this map is simply cellular approximation of a cellular object; hence, it is a weak equivalence.

We now have to check that $A \otimes A$ is $\{A\}$-colocal. For this, we have to show that

$$\mathrm{Map}_{\mathrm{Ch}(R)}(A \otimes A, N) \simeq * \quad \text{for any } N \text{ with} \quad \mathrm{Map}_{\mathrm{Ch}(R)}(A, N) \simeq *.$$

But in this case, $\mathrm{Map}_{\mathrm{Ch}(R)}(X, Y) \simeq *$ is equivalent to $R\mathrm{Hom}_R(X, Y) = 0$ as

$$\pi_k(\mathrm{Map}_{\mathrm{Ch}(R)}(X, Y)) \cong [S^0, \mathrm{Map}_{\mathrm{Ch}(R)}(\Sigma^{-k}X, Y)] \cong R\mathrm{Hom}_R^{-k}(X, Y).$$

We also have by adjunction

$$R\mathrm{Hom}_R(A \otimes A, N) \cong R\mathrm{Hom}_R(A, R\mathrm{Hom}_R(A, N)),$$

so our claim follows. □

Just as we may make any set of objects K stable, we may also make any stable set into a monoidal stable set. Let $\bar{K}$ denote the collection of objects $k_1 \otimes k_2 \cdots \otimes k_n$ for all $n \geqslant 0$, with the zero-fold product being the cofibrant replacement of the unit. This set is clearly monoidal so $R_{\bar{K}}\mathcal{C}$ is a monoidal model category. However, $R_{\bar{K}}\mathcal{C}$ has *fewer* weak equivalences, so in general a K-coequivalence is not a $\bar{K}$-coequivalence. So this notion of replacing K by $\bar{K}$ is perhaps less useful than the version for left localisations.

Dually to Theorem 6.5 we can show that $R_{\bar{K}}\mathcal{C}$ is the best we can achieve. The following result essentially says that $R_{\bar{K}}$ is the 'closest' right localisation to $R_K\mathcal{C}$ for an arbitrary stable K that is also monoidal.

PROPOSITION 7.4. *Let $\mathcal{C}$ be a right proper, stable, cellular monoidal model category. Then the identity adjunction gives Quillen pairs as below where the right hand adjunction*

is a monoidal Quillen pair.

$$R_K \mathcal{C} \rightleftarrows R_{\bar{K}} \mathcal{C} \rightleftarrows \mathcal{C}$$

Proof. Every object of K is cofibrant in $\mathcal{C}$. Since $\mathcal{C}$ is monoidal, every object of the form $k_1 \otimes k_2 \otimes \cdots \otimes k_n$ for $k_i \in K$ and $n \geqslant 0$ is also cofibrant in $\mathcal{C}$. It follows that $R_K \mathcal{C} \rightleftarrows \mathcal{C}$ factors over $R_{\bar{K}} \mathcal{C}$ as required, giving a monoidal Quillen pair $R_{\bar{K}} \mathcal{C} \rightleftarrows \mathcal{C}$. □

8. Replacing stable model categories by spectral ones. Model categories are fundamentally linked to simplicial sets via framings. But framings are only well behaved on the homotopy category. For many tasks, it is preferable to have a simplicial model category. Hence the question: when is a model category Quillen equivalent to a simplicial one? The paper [**4**] by Dugger provides an answer to this question.

Stable model categories are fundamentally linked to spectra via stable framings (see Lenhardt [**13**]). Stable framings are even more poorly behaved on the model category level than framings. Hence, we would like an answer to the question: when is a model category Quillen equivalent to a spectral one?

DEFINITION 8.1. *A spectral model category* is a model category that is enriched, tensored and cotensored over symmetric spectra. Furthermore, it satisfies the analogue of Quillen's SM7 with simplicial sets replaced by symmetric spectra $\Sigma \mathcal{S}$. In the language of Hovey [**11**, Definition 4.2.18], it is a $\Sigma \mathcal{S}$-model category.

We can now use our work on left localisations to weaken the known assumptions that a model category has to satisfy in order to be Quillen equivalent to a spectral one. Because of Proposition 4.7 we can now combine results from Dugger and Schwede-Shipley to acquire the following result.

THEOREM 8.2. *If $\mathcal{C}$ is a model category that is stable, proper and cellular, then it is Quillen equivalent to a spectral model category that is also stable, proper and cellular.*

Proof. Because $\mathcal{C}$ is cellular and left proper, Dugger [**4**, Theorem 1.2] states that $\mathcal{C}$ is Quillen equivalent to a simplicial model category. Specifically, $\mathcal{C}$ is Quillen equivalent to a non-standard model structure on the category of simplicial objects in $\mathcal{C}$, which we write as $s\mathcal{C}_{hc}$.

In more detail, one starts by equipping the category of simplicial objects in $\mathcal{C}$ with the Reedy model structure. A Reedy weak equivalence is a map of simplicial objects $f : A \to B$ such that on each level f_n is a weak equivalence of $\mathcal{C}$. Every Reedy cofibration is a levelwise cofibration and every Reedy fibration is a levelwise fibration (see Hirschhorn [**10**, 15.3.11]). It follows immediately that $s\mathcal{C}$ is still stable. Since $\mathcal{C}$ is cellular and proper, so is $s\mathcal{C}$ by [**10**, Theorems 15.7.6 and 15.3.4].

The model category $s\mathcal{C}_{hc}$ is defined as a left Bousfield localisation of $s\mathcal{C}$ at a set S of maps defined just above Theorem 5.2 in [**4**]. Since $s\mathcal{C}_{hc}$ is Quillen equivalent to $\mathcal{C}$, it must also be stable. Hence, by Proposition 4.7, $s\mathcal{C}_{hc}$ is right proper. Thus, we now know that $s\mathcal{C}_{hc}$ is a proper, cellular, stable model category.

We now use the results of Schwede and Shipley [**16**] to replace this by a Quillen equivalent spectral model category. We rename $s\mathcal{C}_{hc}$ as $\mathcal{D}$ and denote the category of symmetric spectra in $\mathcal{D}$, by $\Sigma \mathcal{S}(\mathcal{D}, S^1)$. We can equip this category with the levelwise

(or projective) model structure, where fibrations and weak equivalences are defined levelwise. This model structure is cellular, proper and stable.

We then left localise the model structure at a set of cofibrations to obtain the 'stable' model structure on $\Sigma\mathcal{S}(\mathcal{D}, S^1)$. By [**16**, Theorem 3.8.2], this model structure is spectral and there is a Quillen equivalence between $\mathcal{D}$ and $\Sigma\mathcal{S}(\mathcal{D}, S^1)$ equipped with the stable model structure. Our previous results also show that this stable model structure on $\Sigma\mathcal{S}(\mathcal{D}, S^1)$ is proper. $\qquad\square$

Results along this line have been proved by Dugger in [**5**]. In that paper, it is shown that a stable, presentable model category is Quillen equivalent to a spectral model category. We replace the notion of presentable (which essentially means Quillen equivalent to a combinatorial model category) with the more familiar notion of cellular. While we have to add proper to our list of assumptions, our method of replacing a model category by a spectral one involves no choices and requires much less technical work to understand the resulting category and model structure.

9. Right localisation and Morita theory. In [**6**, Theorem 2.1], Dwyer and Greenlees show that the category of A-torsion R-modules (with A a perfect R-module) is equivalent to the derived category of the ring $\mathrm{End}_R(A)$. In this section, we prove a more general version of this, namely that for a set of well-behaved objects K, the model category $R_K\mathcal{C}$ is Quillen equivalent to the category of modules over the endomorphism ring spectrum with several objects mod– $\mathrm{End}(K)$.

We say that an object X in a stable model category $\mathcal{C}$ is *homotopically compact* if for any family of objects $\{Y_a\}_{a\in A}$ the canonical map below is an isomorphism:

$$\bigoplus_{a\in A}[X, Y_a]^{\mathcal{C}} \to [X, \coprod_{a\in A} Y_a]^{\mathcal{C}}$$

Homotopically compact objects have obvious technical advantages over general ones, so it is natural to ask what happens if one right localises at a set of homotopically compact objects. We show that, with some minor assumptions, such right localisations are well understood, and we identify their homotopy categories.

Let $\mathcal{C}$ be a stable, cellular, right proper, spectral model category and let K be a stable set of homotopically compact cofibrant–fibrant objects of $\mathcal{C}$. The assumption that $\mathcal{C}$ be spectral is less demanding than it appears, by Theorem 8.2.

Define $\mathrm{End}(K)$ to be the category enriched over symmetric spectra with object set given by K and morphism spectra given by $\mathrm{hom}(k, k')$ defined using the enrichment of $\mathcal{C}$ in symmetric spectra. Consider the category of contravariant enriched functors from $\mathrm{End}(K)$ to symmetric spectra, with morphisms the enriched natural transformations. We call this category mod– $\mathrm{End}(K)$. It has a model structure with weak equivalences and fibrations defined termwise (see Schwede and Shipley [**16**, Theorem A.1.1]).

There is a Quillen pair

$$\mathrm{mod\text{–}End}(K) \rightleftarrows \mathcal{C}$$

whose right adjoint takes $X \in \mathcal{C}$ to $\mathrm{hom}(-, X)$ in mod-$\mathrm{End}(K)$. We call this right adjoint $\mathrm{hom}(K, -)$ and we write $- \wedge_{\mathrm{End}(K)} K$ for its left adjoint.

We are almost ready to start relating mod– $\mathrm{End}(K)$ and $R_K\mathcal{C}$, but we first need a technical result.

LEMMA 9.1. *Let $\mathcal{C}$ a stable, cellular right proper spectral model category and let K be a stable set of cofibrant objects in $\mathcal{C}$. Then $R_K\mathcal{C}$ is a spectral model category.*

Proof. Since $\mathcal{C}$ is spectral, all we must show is the spectral analogue of (SM7), namely that if a is a cofibration of $R_K\mathcal{C}$ and i is a cofibration of $\Sigma\mathcal{S}$, then $a\square i$ is a cofibration of $R_K\mathcal{C}$. It suffices to prove this for $a \in \Lambda K$ and i a generating cofibration of $\Sigma\mathcal{S}$. We know that $a\square i$ is a cofibration of $\mathcal{C}$. We must show that it is in fact a K-cofibration.

Consider a generating cofibration i. It is of the form $F_nA \to F_nB$ for A and B simplicial sets and F_n the left adjoint to evaluation at level n. If $X \in \mathcal{C}$ is K-colocal, then $X \otimes F_nA$ is weakly equivalent to $(\Sigma^{-n}X) \otimes A$. Since K is stable, $\Sigma^{-n}X$ is K-colocal and hence so is $(\Sigma^{-n}X) \otimes A$. It follows that the domain and codomain of $a\square i$ are both K-colocal. By Hirschhorn [**10**, Proposition 3.3.16] a cofibration between K-colocal objects is a K-cofibration. Hence, $a\square i$ is a K-cofibration, which is what we wanted to prove. $\square$

We need some new terms in order to state the main result of this section.

DEFINITION 9.2. Let $\mathcal{C}$ be a stable model category. A full triangulated subcategory of Ho($\mathcal{C}$) with shift and triangles induced from Ho($\mathcal{C}$) is called localising if it is closed under coproducts in Ho($\mathcal{C}$). A set P of objects of Ho($\mathcal{C}$) is called a set of generators if the only localising subcategory which contains the objects of P is Ho($\mathcal{C}$) itself.

THEOREM 9.3. *Let $\mathcal{C}$ a stable, cellular right proper spectral model category and let K be a stable set of cofibrant–fibrant objects of $\mathcal{C}$. Then the Quillen pair*

$$- \wedge_{\mathrm{End}(K)} K : \mathrm{mod\text{--}End}(K) \rightleftarrows \mathcal{C} : \hom(K, -)$$

factors over $R_K\mathcal{C}$. Hence, one has a diagram of Quillen pairs as below:

$$\mathrm{mod\text{--}End}(K) \xrightleftharpoons[\hom(K,-)]{-\wedge_{\mathrm{End}(K)}K} R_K\mathcal{C} \xrightleftharpoons[\mathrm{Id}]{\mathrm{Id}} \mathcal{C}$$

If the set K consists of homotopically compact objects, then the left-hand Quillen pair in this diagram is a Quillen equivalence. Furthermore, the homotopy category of $R_K\mathcal{C}$ is triangulated equivalent to the localising subcategory of Ho($\mathcal{C}$) generated by K.

Proof. A generating cofibration of mod–End K takes form $\hom(-, k) \wedge i$, where i is a generating cofibration in symmetric spectra, $\wedge$ is the smash product in symmetric spectra and $\hom(-, k) \in$ mod-End K. The functor $- \wedge_{\mathrm{End}\,K} K$ sends this to $k \wedge i$, which is a cofibration of the spectral model category $R_K\mathcal{C}$. Hence, we have a factorisation of the Quillen functors as above.

It is easy to check that if k is compact in $\mathcal{C}$, then it is also compact in $R_K\mathcal{C}$. The set of cofibres of $\Lambda K \cup J$ (the generating cofibrations for $R_K\mathcal{C}$) is a generating set for the homotopy category of $R_K\mathcal{C}$. Since the cofibres of J are contractible, we may ignore these. The cofibres of the sets ΛK are simply suspensions of K up to weak equivalence; hence, it follows that K is a generating set for the homotopy category of $R_K\mathcal{C}$. We now apply Schwede and Shipley [**16**, Theorem 3.9.3] to see that we have a Quillen equivalence and that the statement on homotopy categories holds. $\square$

Thus, we have shown that in good circumstances a right localisation is Quillen equivalent to the simpler notion of modules over an endomorphism ringoid. In this

setting, we can identify $\operatorname{Ho} R_K \mathcal{C}$ as the smallest localising subcategory of $\mathcal{C}$ containing K. Hence, it is perfectly correct to think of $R_K \mathcal{C}$ as modelling the homotopy theory of objects of $\mathcal{C}$ built from K via coproducts, shifts and triangles. Thus, right localisation in these circumstances simply alters which objects we think of as generators for the homotopy category. We also obtain an explicit description of K-colocalisation. If X is fibrant in $\mathcal{C}$, then K-colocalisation is given by

$$\hom(K, X) \wedge_{\operatorname{End} K}^{L} K \longrightarrow X.$$

This leads to questions for future research: if the set K is not homotopically compact, how well does $R_K \mathcal{C}$ model mod–$\operatorname{End}(K)$? Similarly, if $\mathcal{C}$ is spectral but not cellular or right proper, and K is a stable set of homotopically compact objects, how well does mod–$\operatorname{End}(K)$ model $R_K \mathcal{C}$, which may not exist?

EXAMPLE 9.4. One half of [6, Theorem 2.1] by Dwyer and Greenlees is the statement that the category of A-torsion R-modules is equivalent to the derived category of modules over $\operatorname{End}_R(A)$, for A a perfect complex. We are now able to give a model category level version of that result: the right localisation of $\operatorname{Ch}(R)$ at the perfect complex A is Quillen equivalent to mod–$\operatorname{End}_R(A)$.

We now use a duality argument to show that in some special cases, $R_K \mathcal{C}$ is Quillen equivalent to a left localisation of $\mathcal{C}$ at a set of maps S. In particular, this applies to the case of A-torsion R-modules. For the rest of this section, assume that $\mathcal{C}$ is a stable model category whose homotopy category $\operatorname{Ho}(\mathcal{C})$ is monoidal with product $\wedge$ and unit $\mathbb{S}$. Furthermore, we require $\mathbb{S}$ to be a homotopically compact generator. We also assume that $\operatorname{Ho}(\mathcal{C})$ is closed in the sense that it possesses function objects $F(-, -)$. For example, any smashing localisation of EKMM $\mathbb{S}$-modules satisfies these assumptions.

Recall that $X \in \operatorname{Ho}(\mathcal{C})$ is said to be *strongly dualisable* if the natural map

$$F(X, \mathbb{S}) \wedge Y \longrightarrow F(X, Y)$$

is an isomorphism for all Y (see [12, Definition 1.1.2] by Hovey, Palmieri and Strickland). In our setting, the class of homotopically compact objects is equal to the class of strongly dualisable objects by [12, Theorem 2.1.3].

Let K be a set of objects in $\mathcal{C}$. By $DX := F(X, \mathbb{S})$ we denote the dual of an object X. Furthermore, we define

$$DK := \coprod_{k \in K} Dk.$$

DEFINITION 9.5. We say that a morphism $f : X \longrightarrow Y$ in $\mathcal{C}$ is a DK_*-equivalence if

$$DK \wedge f : DK \wedge X \longrightarrow DK \wedge Y$$

is an isomorphism in $\operatorname{Ho}(\mathcal{C})$.

We let $L_{DK_*} \mathcal{C}$ denote the left Bousfield localisation of $\mathcal{C}$ at the class of DK_*-equivalences, provided it exists.

It is now easy to prove the proposition below, which we combine with Theorem 9.3 to obtain the subsequent corollary.

PROPOSITION 9.6. *Let $\mathcal{C}$ be a monoidal, stable, cellular, proper model category with unit $\mathbb{S}$ a homotopically compact generator. Let K be a set of homotopically compact cofibrant objects in $\mathcal{C}$. Then the class of K-coequivalences is precisely the class of DK_*-equivalences. Furthermore, if $L_{DK_*}\mathcal{C}$ exists, then the identity functors provide a Quillen equivalence*

$$R_K\mathcal{C} \rightleftarrows L_{DK_*}\mathcal{C}.$$

$\square$

COROLLARY 9.7. *Let $\mathcal{C}$ be a monoidal, stable, cellular, proper, spectral model category with unit $\mathbb{S}$ a homotopically compact generator. Assume that K is a set of homotopically compact cofibrant–fibrant objects in $\mathcal{C}$ such that $L_{DK_*}\mathcal{C}$ exists. Then the model categories $R_K\mathcal{C}$, $L_{DK_*}\mathcal{C}$ and mod–$\mathrm{End}(K)$ are Quillen equivalent.*

This can be applied to the special case of A-torsion and A-complete R-modules for a perfect R-module A, obtaining Theorem 2.1 of Dwyer and Greenlees [6]. In this case, we consider A-torsion modules $R_A\,\mathrm{Ch}(R)$ and A-complete R-modules $L_{DA_*}\,\mathrm{Ch}(R)$. Hence, we recover Dwyer and Greenlees' result that A-torsion and A-complete R-modules are Quillen equivalent.

We can further specify to the case of $R = \mathbb{Z}$ and $A = (\mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z}) \cong \mathbb{Z}/p$. In this case, we obtain that $DA \cong A[1]$. Since DA_*-equivalences form a stable set, we recover Dwyer and Greenlees' 'paradoxical' result that left and right localisations at $\mathbb{Z}/p$ agree.

10. A correspondence between left and right localisations. We now turn to comparing left and right localisations. We show that given any left localisation, there is a corresponding right localisation and vice versa. These two localisations can be thought as 'opposite' to each other the sense of Proposition 10.3.

LEMMA 10.1. *Let $\mathcal{C}$ be a cellular, proper, stable model category and S be a stable set of maps in $\mathcal{C}$. Now let T be the set of maps $* \to Cs$, where $s \in S$ and Cs is the cofibre of s. Then, T is a stable set of maps and $L_S\mathcal{C} = L_T\mathcal{C}$.*

Proof. Consider the exact triangle in $\mathrm{Ho}(\mathcal{C})$:

$$X \xrightarrow{s} Y \longrightarrow \Sigma Cs \longrightarrow \Sigma X$$

for $s \in S$. Applying the graded homotopy classes of maps functor $[-, Z]_*^{\mathcal{C}}$ gives a long exact sequence. Remark 4.5 now proves the claim. $\square$

One advantage of replacing S by the set T is that we can see that the generating cofibrations for $L_S\mathcal{C}$ can be taken to be the set $\Lambda T \cup J$, where

$$\Lambda T = \{Cs \otimes \partial\Delta[n]_+ \longrightarrow Cs \otimes \Delta[n]_+ \mid n \geqslant 0, s \in S\}.$$

We also see that S is monoidal if and only if T is monoidal, which might be easier to check in practice. Thus, localising at S is the same as making the set of objects of the form Cs acyclic. This is why left localisations are sometimes known as acyclicisations.

Another advantage is that this description of left localisation illuminates the relation between left and right localisations. Let $\mathcal{C}$ be a cellular, proper, stable model category with generating sets I and J and let K be a stable set of cofibrant objects of $\mathcal{C}$. Then we can see that the difference between left and right localising is whether

to take $\Lambda K \cup J$ as the set of generating acyclic cofibrations or the set of generating cofibrations. This is the model category version of choosing to declare a set of objects to be trivial, or declaring a set of objects to be generators.

DEFINITION 10.2. For a set of maps S, define a set of objects $K_S = \{Cs \mid s \in S\}$. Conversely, given a set of objects K define a set of maps $S_K := \{* \to k \mid k \in K\}$.

Clearly, if S is stable, then so is K_S. Similarly, if K is stable, so is S_K. We immediately see that right localising at the set K_{S_K} is the same as right localising at the set K. Similarly, left localising at S_{K_S} gives the same model category as left localising at S.

PROPOSITION 10.3. *Choose some stable set of cofibrations S and let $K = K_S$ or choose a set of cofibrant objects K and let $S = S_K$. Assume that $\mathcal{C}$ is stable, proper and cellular. Then there is a diagram of Quillen pairs*

$$R_K\mathcal{C} \rightleftarrows \mathcal{C} \rightleftarrows L_S\mathcal{C}$$

such that the composite adjunction $\mathrm{Ho}(R_K\mathcal{C}) \rightleftarrows \mathrm{Ho}(L_S\mathcal{C})$ is trivial in the sense that both functors send every object to $*$.

Proof. Every object in $\mathrm{Ho}(R_K\mathcal{C})$ is isomorphic to a K-colocal object while every object in $\mathrm{Ho}(L_S\mathcal{C})$ is isomorphic to an S-local one. By construction, being K_S-colocal is equivalent to being S-acyclic and being K-colocal is equivalent to being S_K-acyclic. $\square$

The above adjunctions give a decomposition of the homotopy category of $\mathcal{C}$ into two pieces which are *orthogonal* in the sense that if A is K-colocal and Z is S-local, then $[A, Z]^{\mathcal{C}} = 0$. More clearly, the K-colocal objects are precisely the S-acyclic objects. Similarly, the K-acyclic objects are exactly the S-local objects.

Let us now turn to the subject of chromatic homotopy theory. A left localisation at a spectrum E is said to be *finite* if the class of E-acyclic objects is generated, in the sense of triangulated categories, by a set of finite spectra.

This is especially interesting in the case of the Johnson–Wilson theories $E(n)$. The Johnson–Wilson theories are Landweber exact modules over BP with

$$E(n)_* \cong \mathbb{Z}_{(p)}[v_1, ..., v_n, v_n^{-1}], \quad |v_i| = 2p^i - 2, \quad p \text{ prime}.$$

Localisation with respect to $E(n)$ is smashing and is usually denoted by L_n instead of $L_{E(n)}$. These localisations are of great importance to stable homotopy theory as they play a role in major structural results concerning the stable homotopy category such as the nilpotency theorem, periodicity theorem, chromatic convergence theorem and thick subcategory theorem. Furthermore, L_1 equals localisation with respect to p-local complex topological K-theory whereas L_2 is related to elliptic homology theories. One of the great open conjectures in stable homotopy theory, the *telescope conjecture*, claims that localisation with respect to $E(n)$ is finite in the above sense.

REMARK 10.4. This conjecture can be put into an even more concrete setting. Ravenel showed in [**14**] that the only finite localisations of spectra are of the form $L_{L_n^f\mathbb{S}}\mathbb{S}$, where $L_n^f\mathbb{S}$ is a finite localisation of the sphere. This is also a smashing localisation.

We can restate this in the language of right localisations. By Lemma 4.14, we have

$$L_n\mathbb{S} = L_\Gamma\mathbb{S} \text{ for } \Gamma = \{\Sigma^k\lambda : \mathbb{S}^k \longrightarrow L_n\mathbb{S}^k \mid k \in \mathbb{Z}\}.$$

By Proposition 10.3, the question of whether $L_n\mathbb{S}$ is finite is now equivalent to the question of whether R_{K_Γ} is finite. Hence, we can now use the tools of right localisation to study the telescope conjecture in future research.

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