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A CRITERION FOR AUTOMORPHISMS OF CERTAIN GROUPS TO BE INNER

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Abstract

Let R be a normal subgroup of the free group F, and set $G = F/[R, R]$. We assume that F/R is a torsion-free group which is either solvable and not cyclic, or has a non-trivial center and is not cyclic-by-periodic. Then any automorphism of G whose restriction to $R/[R, R]$ is trivial is an inner automorphism, determined by some element of $R/[R, R]$. This result extends a theorem of Šmel'kin (1967).

When S is a non-cyclic free solvable group, we may write $S = F/R'$ where F is free, $R = \delta_k(F)$ is the k-th term of the derived series of F, and $R' = [R, R] = \delta_{k+1}(F)$. If α is any automorphism of S whose restriction to $\delta_k(F)/\delta_{k+1}(F)$ is trivial, Šmel'kin (1967) proved that α is an inner automorphism determined by some element $r \in \delta_k(F)/\delta_{k+1}(F)$; that is, for all $x \in S$, $\alpha(x) = rxr^{-1}$. His proof involves an analysis of the way in which S is embedded in a certain wreath product. The purpose of this note is to extend Šmel'kin's Theorem to a larger class of groups of the form F/R', where F is free of rank at least two and R is a proper normal subgroup of F. Specifically, we prove that any automorphism of F/R' which acts trivially on R/R' is inner, provided F/R is a torsion-free group which is either solvable and not cyclic, or has a non-trivial center and is not cyclic-by-periodic. The main tool we use is an application of the free differential calculus to F/R' due to Fox (1953) and Magnus (1939).

Suppose M is a left $\mathbf{Z}G$ module, where G is a group. A *derivation* $D: G \to M$ is a function which satisfies

$$D(uv) = D(u) + uD(v)$$

for all $u, v \in G$; it is determined by its values on any generating set of G. When D is extended linearly to give $D: \mathbf{Z}G \to M$, we have for all $f, g \in \mathbf{Z}G$

$$D(fg) = D(f)\varepsilon(g) + fD(g) \tag{1}$$

where $\varepsilon: \mathbf{Z}G \to \mathbf{Z}$ is the augmentation. When F is free and $\{X_i : i \in I\}$ is a basis for F then we may choose $DX_i \in M$ arbitrarily and extend to obtain a derivation $D: F \to M$. In particular, define $D_j: F \to \mathbf{Z}F$ to be that derivation for which

$$D_j X_i = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}.$$

We note that, for all $X \in F$,

$$X - 1 = \Sigma (D_j X)(X_j - 1) \tag{2}$$

(since the left and right hand sides define derivations which agree on $\{X_i : i \in I\}$).

Now let R be a normal subgroup of the free group F, and set

$$G = F/R'; \qquad H = F/R.$$

Let $\{X_i : i \in I\}$ be a basis for F, and define $x_i \in G$ to be the image of $X_i \in F$. Let T be the free left $\mathbf{Z}H$ module with basis $\{t_i : i \in I\}$. Under the canonical homomorphisms $F \to G \to H$, T will also be viewed as a $\mathbf{Z}F$ or $\mathbf{Z}G$ module. Let $D: F \to T$ be the derivation defined by

$$D(X) = \Sigma\, D_j(X) t_j.$$

Since $R' \leqq \operatorname{Ker} D$ there are induced derivations

$$\partial_j : G \to \mathbf{Z}H, \quad \partial : G \to T$$

for which

$$\partial_j(x_i) = \delta_{ij}, \qquad \partial(x) = \Sigma\, \partial_j(x) t_j.$$

It is a theorem of Fox (1953) and Magnus (1939) that $\partial: G \to T$ is an injection. We will exploit this fact in the remainder of this paper.

The group

$$\bar{R} = R/R'$$

has a left $\mathbf{Z}H$ module structure with the action of H on $\bar{R}$ induced by

$$r \to grg^{-1} = r^g,$$

$r \in \bar{R}$, $g \in G$. The restriction of ∂ to $\bar{R}$ is an injective $\mathbf{Z}H$ homomorphism.

LEMMA. *Suppose H is torsion-free, and let $g \in G$, $r \in \bar{R}$. Then $[g, r] = 1$ implies $g \in \bar{R}$ or $r = 1$.*

PROOF. We have $\partial[g, r] = (g-1)\partial r$. If $g \notin \bar{R}$, then the image of $g - 1$ in $\mathbf{Z}H$ is not a zero divisor. Therefore $g \notin \bar{R}$ implies $\partial r = 0$, whence $r = 1$.

COROLLARY 1 (*see L. Auslander and Schenkman* (1965), *M. Auslander and Lyndon* (1955)). *If $H = F/R$ is torsion-free, then $G = F/R'$ has trivial center and $\bar{R} = R/R'$ is a characteristic subgroup of G.*

PROOF. If $[z, r] = [z, g] = 1$, where $g \in G - \bar{R}$ and $1 \neq r \in \bar{R}$, then $z = 1$ by the Lemma above. If $\alpha \in \mathrm{Aut}(G)$, then $[\bar{R}, \alpha(\bar{R})] \leqq \bar{R} \cap \alpha(\bar{R})$. If $[\bar{R}, \alpha(\bar{R})] = 1$, then $\alpha(\bar{R}) \leqq \bar{R}$; otherwise there is an $1 \neq r \in \bar{R} \cap \alpha(\bar{R})$. Since $[r, \alpha(\bar{R})] = 1$, $\alpha(\bar{R}) \leqq \bar{R}$.

COROLLARY 2. *Let $\alpha \in Aut(G)$ with H torsion-free. Assume that the restriction of α to $\bar{R}$ is trivial.*

(a) *For all $x \in G$, $\alpha(x)x^{-1} \in \bar{R}$.*

(b) *If $\alpha(x) = x^r$ for some (fixed) $r \in \bar{R}$ and all x in a normal subgroup N of G which contains $\bar{R}$ properly, then $\alpha(x) = x^r = rxr^{-1}$ for all $x \in G$.*

PROOF. For each $r \in \bar{R}$ and $x \in G$,

$$x^{-1}rx = \alpha(x^{-1}rx) = \alpha(x)^{-1}r\alpha(x),$$

so $\alpha(x)x^{-1}$ commutes with r. By the Lemma above, $\alpha(x)x^{-1} \in \bar{R}$.

To prove (b), let $x \in N - \bar{R}$ and $y \in G$. Now

$$ryxy^{-1}r^{-1} = \alpha(yxy^{-1}) = \alpha(y)rxr^{-1}\alpha(y)^{-1},$$

or $y^{-r}\alpha(y)$ commutes with x^r. Since $y^{-r}\alpha(y) \in \bar{R}$ and $x \notin \bar{R}$ the Lemma above implies that $\alpha(y) = ryr^{-1}$.

We can recognize elements of $\partial(\bar{R})$ in T by means of the criterion which follows; it appears in Bachmuth (1965) when G is free metabelian, and in Remeslennikov and Sokolov (1970) for arbitrary $G = F/R'$.

PROPOSITION (*Bachmuth* (1965), *Remeslennikov and Sokolov* (1970)). *Let $t = \Sigma p_j t_j \in T$. Then $t = \partial(r)$ for some $r \in \bar{R}$ if and only if $\Sigma p_j(x_j - 1) = 0$.*

PROOF. If $t = \partial r$, then $p_j = \partial_j r$ so $\Sigma p_j(x_j - 1) = 0$ follows from equation (2). To prove the converse, select $q_j \in \mathbf{Z}G$ which maps to $p_j \in \mathbf{Z}H$ when $p_j \neq 0$, and $q_j = 0$ otherwise. Put

$$q = \Sigma q_j(x_j - 1).$$

Since $q \in \mathrm{Ker}\{\mathbf{Z}G \to \mathbf{Z}H\}$, there exist $r_1, \cdots, r_m \in \bar{R}$, $s_1, \cdots, s_m \in \mathbf{Z}G$ such that

$$\Sigma q_j(x_j - 1) = q = \Sigma(r_k - 1)s_k.$$

By equation (1),

$$\begin{aligned} \partial q &= \partial \Sigma q_j(x_j - 1) = \Sigma p_j t_j = t \\ &= \partial \Sigma(r_k - 1)s_k = \Sigma(\partial r_k)\varepsilon(s_k). \end{aligned}$$

Define

$$r = r_1^{\varepsilon(s_1)} \cdots r_m^{\varepsilon(s_m)}.$$

Then $r \in \bar{R}$ and $\partial r = t$, as required.

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COROLLARY 3. *Suppose H is torsion-free. If $r \in \bar{R}$, $g \in G$, $t \in T$ satisfy*

$$\partial r = (1 - g)t$$

then there exists $s \in \bar{R}$ such that $r = [s, g] = sgs^{-1}g^{-1}$.

PROOF. Set $t = \Sigma p_j t_j$. If $g \in \bar{R}$ then $r = 1 = [1, g]$. Suppose therefore that $g \notin \bar{R}$. We have

$$(1 - g)\Sigma p_j(x_j - 1) = \Sigma(\partial_j r)(x_j - 1) = 0$$

by equation (2). Since $1 - g$ is not zero and not a zero divisor in $\mathbf{Z}H$, $\Sigma p_j(x_j - 1) = 0$ and $t = \partial s$ for some $s \in \bar{R}$. Using equation (1).

$$\partial[s, g] = (1 - g)\partial s = \partial r$$

and so $r = [s, g]$.

We now prove the extension of Šmel'kin's Theorem stated in the introduction.

THEOREM. *Suppose $H = F/R$ is torsion-free. If either*
(a) *H is not cyclic-by-periodic and the center of H is not trivial, or*
(b) *H is solvable and not cyclic,*
then the kernel of the restriction map $Aut(G) \to Aut(\bar{R})$ is $Inn(\bar{R})$.

PROOF. Assume that $\alpha \in \mathrm{Aut}(G)$ and $\alpha(r) = r$ for all $r \in \bar{R}$. We must show that α is inner. Define

$$D(x) = \partial(\alpha(x)x^{-1}).$$

Note that $\alpha(x)x^{-1} \in \bar{R}$ by Corollary 2(a), that $D\colon G \to \partial\bar{R} \leqq T$ is a derivation, and that $\alpha(x) = x^r$ if and only if $D(x) = (1 - x)\partial r$. By Corollary 3, it suffices to show that $D(x) = (1 - x)t$ for some $t \in T$, and by Corollary 2(b) it is enough to verify that $D(x) = (1 - x)t$ for some $t \in T$ and all x in some normal subgroup of G which contains $\bar{R}$ properly.

Since $D(r) = 0$ for all $r \in \bar{R}$, D induces a derivation $H \to \partial\bar{R}$ which we again denote by D. It suffices to prove that there exists some $t \in T$ such that

$$D(x) = (1 - x)t \tag{3}$$

for all x in some non-trivial normal subgroup of H.

CASE (a). Let $1 \neq z$ belong to the center of H. Then for all $x \in H$, $D([x, z]) = 0$ which implies that

$$(1 - z)D(x) = (1 - x)D(z).$$

Since H is not cyclic-by-periodic, there exists an $x \in H$ which is of infinite order modulo $\langle z\rangle$. Then the image of $1-x$ in $\mathbf{Z}(H/\langle z\rangle)$ is not a zero divisor, and so

$$D(z)=(1-z)t$$

for some $t\in T$. It follows from equation (1) that (3) above holds for all $x\in\langle z\rangle$.

CASE (b) Since H is solvable, there exists a positive integer k such that

$$\delta_{k+1}(H)=1, \qquad \delta_k(H)\neq 1$$

(that is, H has solvable length k, where $\delta_k(H)$ is the k-th term of the derived series of H). We will proceed by induction on k.

Suppose first that $k=1$. Then H is abelian, and by case (a) we may also assume that H is cyclic-by-finite. Since H is torsion-free, it has unique roots and is locally cyclic. Let $1\neq h\in H$ and let $\{h_i\colon i\in I\}$ be any generating set for H. There exist relatively prime integers $e(i)$, $f(i)$ such that

$$h^{e(i)}=h_i^{f(i)}, \qquad i\in I. \tag{4}$$

There are then $y_i\in H$ such that

$$h_i=y_i^{m(i)}, \qquad h=y_i^{n(i)}$$

for suitable integers $m(i)$, $n(i)$. Equation (4) implies that $e(i)$ divides $m(i)$, and we may replace h_i by a suitable power of y_i so as to assume $e(i)=1$ and $f(i)\geqq 1$ in (4). If $\{f(i)\colon i\in I\}$ is bounded, then H is finitely generated and therefore cyclic. This is excluded by hypothesis.

Since D is a derivation, (4) implies that

$$D(h)=(1+\cdots+h_i^{f(i)-1})D(h_i).$$

Therefore each $\mathbf{Z}H$ component of $D(h)\in T$ has augmentation divisible by all $f(i)$, and so has augmentation 0. Thus $D(h)\in(1-H)T$. Since H is locally cyclic, there exist $g\in H$, $s\in T$ such that

$$D(h)=(1-g)s.$$

Choose $z\in H$ such that

$$g=z^a, \qquad h=z^b$$

where b is positive and $a,b\in\mathbf{Z}$. Then

$$D(h)=D(z^b)=(1+\cdots+z^{b-1})D(z)=(1-z^a)t.$$

Since $\mathbf{Z}H$ is a free left $\mathbf{Z}\langle z\rangle$ module (with basis any complete set of right coset representatives of $\langle z\rangle$ in H), the equation above implies that $D(z)=(1-z)t$ for some $t\in T$. Thus (3) holds for all $x\in\langle z\rangle$.

For the inductive step, suppose that H has solvable length $k\geqq 2$ and that (3) is satisfied for derivations from non-cyclic torsion-free solvable groups of smaller solvable length. Then (3) holds for all $x \in H'$ if H' is not cyclic. We may therefore assume that H' is cyclic.

Let K be the centralizer of H' in H. Then K is a normal subgroup of H whose index is at most 2. Since the only torsion-free extension of a cyclic group by a group of order 2 is necessarily cyclic, it follows that K is not cyclic. If $K' = 1$, we are done by the induction hypothesis. Otherwise, $1 \neq K' \leqq H'$ so K' is cyclic and contained in the center of K. Thus K is nilpotent, so K/K' is not periodic since $1 \neq K$ is torsion-free. Now case (a) applies to K, whence (3) is valid for all $x \in K$.

We conclude this paper by noting that G has automorphisms which are not inner but which act trivially on $\bar{R}$, when H is cyclic. In this case, we can choose a basis $X \cup \{X_i: i \in I\}$ for F such that $X \notin R$, $X_i \in R$ for all $i \in I$. Then $\partial\bar{R}$ is the free left $\mathbf{Z}H = \mathbf{Z}\langle x\rangle$ module on $\{t_i: i \in I\}$. Any automorphism of G whose restriction to $\bar{R}$ is trivial fixes all x_i. It follows that the map

$$\alpha \mapsto \Sigma \varepsilon \partial_i(\alpha(x)x^{-1})t_i$$

induces an isomorphism from $\operatorname{Ker}\{\operatorname{Aut}(G) \to \operatorname{Aut}(\bar{R})\}/\operatorname{Inn}(\bar{R})$ to the free $\mathbf{Z}$ module on $\{t_i: i \in I\}$.

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