

# Reducing subspaces for rank-one perturbations of normal operators

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We study the existence of reducing subspaces for rank-one perturbations of diagonal operators and, in general, of normal operators of uniform multiplicity one. As we will show, the spectral picture will play a significant role in order to prove the existence of reducing subspaces for rank-one perturbations of diagonal operators whenever they are not normal. In this regard, the most extreme case is provided when the spectrum of the rank-one perturbation of a diagonal operator  $T = D + u \otimes v$  (uniquely determined by such expression) is contained in a line, since in such a case  $T$  has a reducing subspace if and only if  $T$  is normal. Nevertheless, we will show that it is possible to exhibit non-normal operators  $T = D + u \otimes v$  with spectrum contained in a circle either having or lacking non-trivial reducing subspaces. Moreover, as far as the spectrum of  $T$  is contained in any compact subset of the complex plane, we provide a characterization of the reducing subspaces  $M$  of  $T$  such that the restriction  $T|_M$  is normal. In particular, such characterization allows us to exhibit rank-one perturbations of *completely normal* diagonal operators (in the sense of Wermer) lacking reducing subspaces. Furthermore, it determines completely the decomposition of the underlying Hilbert space in an orthogonal sum of reducing subspaces in the context of a classical theorem due to Behncke on *essentially normal operators*.

**Keywords:** Reducing subspaces; rank-one perturbation of diagonal operators; rank-one perturbation of normal operators

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## 1. Introduction and preliminaries

Let  $H$  denote an infinite-dimensional separable complex Hilbert space and  $\mathcal{L}(H)$  the Banach algebra of all bounded linear operators on  $H$ . An operator  $T \in \mathcal{L}(H)$  is *reductive* if every closed invariant subspace  $M$  of  $T$  reduces  $T$  or is a so-called *reducing subspace*; namely  $M$  is invariant under both  $T$  and its adjoint  $T^*$  (equivalently, both subspaces  $M$  and its orthogonal complement  $M^\perp$  are invariant under  $T$ ). A well-known unsolved problem in the context of bounded linear operators acting on  $H$  is if every reductive operator must be a normal operator. Indeed,

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the answer is affirmative if we restrict ourselves to the class of compact operators [1] or polynomially compact operators [36, 37]. In the general context  $\mathcal{L}(H)$ , such a problem is equivalent to the existence of non-trivial closed invariant subspaces and hence equivalent to provide a positive answer to the *Invariant Subspace Problem* in the frame of infinite-dimensional separable complex Hilbert spaces [15].

The Invariant Subspace Problem is, by now, a long-standing open question which have called the attention of many operator theorists since 1940s, producing approaching strategies leading to deep theorems and intricate examples. Within the most remarkable theorems, we mention *Lomonosov theorem* [30], while among the most relevant examples, one may find the constructions by Enflo [16] and Read [34, 35] of bounded linear operators acting on infinite-dimensional complex Banach spaces lacking non-trivial closed invariant subspaces (or even non-trivial closed invariant subsets). Recently, in [21] the authors exhibit a bounded linear operator  $T$  acting on  $\ell^1$  such that  $f(T)$  has no non-trivial closed invariant subspaces for every non-constant analytic germ  $f$ . We refer to the classical monograph by Radjavi and Rosenthal [32] and the recent one by Chalendar and Partington [9] for more on the subject.

Regarding the different aforementioned approaching strategies, there is one coming from the analysis of the behaviour of operators acting on finite-dimensional subspaces which led to the concept of *quasitriangular operators*. Recall that a bounded linear operator  $T$  acting on  $H$  is said to be quasitriangular if there exists an increasing sequence  $(P_n)_{n=1}^\infty$  of finite rank projections converging to the identity  $I$  strongly as  $n \rightarrow \infty$  such that

$$\|TP_n - P_nTP_n\| \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Clearly, given any triangular operator in  $H$ , that is, a bounded linear operator which admits a representation as an upper triangular matrix with respect to a suitable orthonormal basis, there exists an increasing sequence  $(P_n)_{n=1}^\infty$  of finite rank projections converging to the identity  $I$  strongly as  $n \rightarrow \infty$  such that

$$TP_n - P_nTP_n = (I - P_n)TP_n = 0, \quad \text{for all } n = 0, 1, 2, \dots$$

Based on the proof of Aronszajn and Smith's theorem [3], Halmos [26] introduced the concept of quasitriangular operators which, somehow, states that  $T$  has a sequence of 'approximately invariant' finite-dimensional subspaces. Compact operators, operators with finite spectrum, decomposable operators in the sense of Colojoară and Foiaş [10] or compact perturbations of normal operators are examples of quasitriangular operators. Remarkably, results due to Douglas and Pearcy [12] and Apostol, Foias and Voiculescu [2] state that the Invariant Subspace Problem is reduced to be proved for quasitriangular operators (see Herrero's book [27] for more on the subject).

Among the most simple quasitriangular operators for which the existence of non-trivial closed invariant subspaces is still open are rank-one perturbations of diagonal operators. If  $D \in \mathcal{L}(H)$  is a diagonal operator, that is, there exists an orthonormal basis  $(e_n)_{n \geq 1}$  of  $H$  and a sequence of complex numbers  $(\lambda_n)_{n \geq 1} \subset \mathbb{C}$  such that

$De_n = \lambda_n e_n$ , a rank-one perturbations of  $D$  can be written as

$$T = D + u \otimes v, \tag{1.1}$$

where  $u$  and  $v$  are non-zero vectors in  $H$  and  $u \otimes v(x) = \langle x, v \rangle u$  for every  $x \in H$ . While expression (1.1) is not unique as far as rank-one perturbations of diagonal operators concern, considering the expansions of  $u, v$  with respect to the (ordered) orthonormal basis  $(e_n)_{n \geq 1}$

$$u = \sum_{n=1}^{\infty} \alpha_n e_n, \quad v = \sum_{n=1}^{\infty} \beta_n e_n, \tag{1.2}$$

Ionascu showed that whenever both  $u$  and  $v$  have non-zero components  $\alpha_n$  and  $\beta_n$  for every  $n \geq 1$  uniqueness follows (see [28, proposition 1.1]). Moreover, he studied rank-one perturbations of diagonal operators from the standpoint of invariant subspaces identifying normal operators as well as contractions within this class. Note that, in particular, rank-one perturbations of normal operators whose eigenvectors span  $H$  belongs to such a class, since they are unitarily equivalent to those expressed by (1.1).

Later on, Foias, Jung, Ko and Percy showed that there is a large class of such operators each of which has a nontrivial hyperinvariant subspace [18]; indeed they are decomposable operators [20] (see also the papers by Fang and Xia [17] and Klaja [29] for an extension of the results in [18] to finite rank and compact perturbations).

In a more general setting, rank-one perturbations of normal operators have been extensively studied for decades (see the recent papers [4–7] and the references therein). Recently, in [31], the authors have provided conditions for a possible *dissection* of the spectrum of  $T$  along a curve implying a decomposition of  $T$  as a direct sum of two operators with localized spectrum and providing sufficient conditions to ensure the existence of invariant subspaces for  $T$ .

The aim of this work in this context is studying the existence of reducing subspaces for operators  $T$  which are rank-one perturbations of diagonal operators and, in general, of normal operators. Recently, there have been an exhaustive study on reducing subspaces for multiplication operators whenever they act on spaces of analytic functions like the Bergman space (see the works by Douglas and coauthors [13, 14] or those by Guo and Huang [22–24], for instance).

Our starting point will be a theorem of Ionascu where normal operators are characterized within the class of rank-one perturbations of normal operators. Clearly, the existence of reducing subspaces is trivial for normal operators, since the spectral measure provides plenty of projections commuting with the operator. Nevertheless, as we will show, the spectral picture will play a significant role in order to prove the existence of reducing subspaces for rank-one perturbations of diagonal operators whenever they are not normal. In this regard, the most extreme case is provided when the spectrum of the operator  $T = D + u \otimes v$  (uniquely determined by such expression) is contained in a line, since in such a case  $T$  has a reducing subspace if and only if  $T$  is normal (see theorem 2.1, § 2). As a consequence, we will exhibit operators within this class being decomposable (even *strongly decomposable*) with no reducing subspaces.

When the spectrum of  $T = D + u \otimes v$  is contained in a circle, which turns out to be the other possible case according to Ionascu's result to ensure that  $T$  is a normal operator (see [28, corollary 3.2]), the situation differs drastically from the previous case aforementioned. More precisely, it is possible to exhibit non-normal operators with spectrum contained in a circle either having or lacking non-trivial reducing subspaces (see theorem 3.1, § 3).

Indeed, theorem 3.1 is extended in a more general setting in § 4 allowing us to exhibit rank-one perturbations of diagonal operators with arbitrary spectrum lacking non-trivial reducing subspaces. The main result in this context, theorem 4.3, characterizes the reducing subspaces  $M$  of  $T$  such that the restriction of  $T$  to  $M$ , denoted by  $T|_M$ , is normal. In particular, as a consequence of theorem 4.7 it is possible to exhibit rank-one perturbation of *completely normal* diagonal operators lacking reducing subspaces. Recall that a normal operator is *completely normal* if all its invariant subspaces are reducing.

Besides, we discuss these results in the context of a classical theorem due to Behncke [8] which provides a decomposition of the underlying Hilbert space in an orthogonal sum of reducing subspaces for *essentially normal operators* (see also [25, chapter 8]). We conclude § 4 addressing some of the previous results in the more general context of rank-one perturbations of normal operators.

Finally, in § 5, we present some examples of rank-one perturbations of diagonal operators with multiplicity strictly larger than one in order to illustrate how the picture of the reducing subspaces changes whenever the assumption on the uniform multiplicity one is not assumed. Such assumption plays a key role in the proofs of the results aforementioned.

For the sake of completeness, we close this first section with some preliminaries regarding results about existence of invariant subspaces of rank-one perturbations of diagonal operators as well as of normal operators, which will be of interest throughout the paper.

### 1.1. Preliminaries

Let  $D$  be a diagonal operator in  $\mathcal{L}(H)$  and denote by  $\Lambda(D) = (\lambda_n)_{n \geq 1} \subset \mathbb{C}$  its set of eigenvalues with respect to an orthonormal basis  $(e_n)_{n \geq 1}$  of  $H$ . It is well-known that the spectrum of  $D$  is given by the closure of  $\Lambda(D)$ , that is,  $\sigma(D) = \overline{\Lambda(D)}$ .

Let  $u$ , and  $v$  be non-zero vectors in  $H$  and consider their expansions with respect to the (ordered) orthonormal basis  $(e_n)_{n \geq 1}$

$$u = \sum_{n=1}^{\infty} \alpha_n e_n, \quad v = \sum_{n=1}^{\infty} \beta_n e_n.$$

Let  $T \in \mathcal{L}(H)$  the rank-one perturbation of  $D$  given by expression (1.1), namely

$$T = D + u \otimes v$$

where  $u \otimes v(x) = \langle x, v \rangle u$  for every  $x \in H$ . As mentioned previously, Ionascu proved in [28, proposition 1.1] that if  $\alpha_n \beta_n \neq 0$  for every  $n$ , then (1.1) is unique in the sense that if  $T = D + u \otimes v = D' + u' \otimes v'$  with  $D, D'$  diagonal operators and  $u, v, u', v'$  non-zero vectors in  $H$ , then  $D = D'$  and  $u \otimes v = u' \otimes v'$ .

Moreover, he also proved that if there exists  $n_0 \in \mathbb{N}$  such that  $\alpha_{n_0}\beta_{n_0} = 0$ , then either  $\lambda_{n_0}$  is an eigenvalue of  $T$  or  $\overline{\lambda_{n_0}}$  is an eigenvalue of  $T^*$ ; in both cases associated with the same eigenvector  $e_{n_0}$  (see [28, proposition 2.1]). As a straightforward consequence  $T$  has a reducing subspace whenever there exists  $n_0 \in \mathbb{N}$  such that  $\alpha_{n_0} = \beta_{n_0} = 0$ . As we will show in theorem 2.1, this case will exhaust all the possibilities when the spectrum of the given operator  $T = D + u \otimes v$  is contained in a line.

Indeed, there are only two possible spectral pictures for a rank-one perturbation of a diagonal operator  $T = D + u \otimes v$  with uniqueness expression ( $\alpha_n\beta_n \neq 0$  for every  $n$ ) in order to be a normal operator as stated in proposition 3.1 and corollary 3.2 in [28]:

**THEOREM 1.1** (Ionascu, 2001). *Let  $T = N + u \otimes v$  be in  $\mathcal{L}(H)$  where  $N$  is a normal operator and  $u, v$  are nonzero vectors in  $H$ . Then  $T$  is a normal operator if and only if either*

- (i)  *$u$  and  $v$  are linearly dependent and  $u$  is an eigenvector for  $\text{Im}(\alpha N^*)$  where  $\alpha = \langle u, v \rangle / \|v\|^2$ , or*
- (ii)  *$u, v$  are linearly independent vectors and there exist  $\alpha, \beta \in \mathbb{C}$  such that*

$$(N^* - \overline{\alpha}I)u = \|u\|^2\beta v \quad \text{and} \quad (N - \alpha I)v = \|v\|^2\overline{\beta}u,$$

where  $\text{Re}(\beta) = -1/2$ .

*In particular, with the introduced notation, if  $D$  is a diagonal operator and  $\alpha_n\beta_n \neq 0$  for every  $n$ , then  $T = D + u \otimes v \in \mathcal{L}(H)$  is normal if and only if*

- (i') *there exist  $\alpha \in \mathbb{C}$  and  $t \in \mathbb{R}$  such that  $\Lambda(D)$  lies on the line  $\{z \in \mathbb{C} : \text{Im}(\alpha\overline{z}) = t\}$  and  $u = \alpha v$ , or*
- (ii') *there exist  $\alpha \in \mathbb{C}$  and  $t \in \mathbb{R}$  such that  $\Lambda(D)$  lies on the circle  $\{z \in \mathbb{C} : |z - \alpha| = t\}$  and*

$$\frac{tu}{\|u\|} = e^{i\theta}(D - \alpha I) \left( \frac{v}{\|v\|} \right),$$

where  $\theta \in [0, \pi)$  is determined by the equation  $\text{Re}(te^{i\theta}/(\|u\|\|v\|)) = -1/2$ .

As we will show, this dichotomy will allow us to establish the existence of reducing subspaces for rank-one perturbation of diagonal operators as far as their spectrum is contained in a line.

In order to finish this preliminary section, we turn our attention to the multiplicity of the eigenvalues of the diagonal operator. In [28, proposition 2.2], it was shown that if  $\lambda$  is an eigenvalue of  $D$  of multiplicity strictly larger than one, then  $\lambda$  is an eigenvalue of the rank-one perturbation operator  $T = D + u \otimes v$ . Indeed, a closer look at the proof shows the following in our context:

**PROPOSITION 1.2.** *Assume  $\lambda$  is an eigenvalue of  $D$  of multiplicity strictly larger than one and let  $\lambda = \lambda_{n_0} = \lambda_{n_1}$  for  $n_0, n_1 \in \mathbb{N}$ . Suppose, in addition, that  $\alpha_{n_0} = \beta_{n_0}$  and  $\alpha_{n_1} = \beta_{n_1}$ . Then  $T$  has a non-trivial reducing subspace.*

Hence, according to proposition 1.2, it turns out to be easy to construct examples of operators having non-trivial reducing subspaces if we do not assume uniform multiplicity one for the diagonal operator. In the last section, § 5, we will address examples in this context showing, in particular, that the assumption of uniform multiplicity one is essential in our approach.

**2. Reducing subspaces for rank-one perturbations of diagonal operators: when the spectrum is contained in a line**

In this section we will show, in particular, that if  $T$  is a uniquely determined rank-one perturbation of a diagonal operator with spectrum contained in a line,  $T$  has a reducing subspace if and only if it is a normal operator. The precise statement is the following:

**THEOREM 2.1.** *Let  $T = D + u \otimes v \in \mathcal{L}(H)$  where  $D$  is a diagonal operator with respect to an orthonormal basis  $(e_n)_{n \geq 1}$  and  $u = \sum_n \alpha_n e_n$ ,  $v = \sum_n \beta_n e_n$  are nonzero vectors in  $H$ . Assume  $D$  has uniform multiplicity one and its spectrum  $\sigma(D)$  is contained in a line. Then,  $T$  has a non-trivial reducing subspace if and only if  $T$  is normal or there exists  $n \in \mathbb{N}$  such that  $\alpha_n = \beta_n = 0$ .*

Note that since every normal operator  $N$  whose eigenvectors span  $H$  with uniform multiplicity one is unitarily equivalent to a diagonal operator (also with uniform multiplicity one). Theorem 2.1 could be rephrased for rank-one perturbation of such normal operators showing, in particular, the existence of a large class of rank-one perturbation of normal operators lacking non-trivial reducing subspaces. Indeed, by means of Ionascu’s theorem, it is enough to consider vectors  $u$  and  $v$  linearly dependent where  $u$  is an eigenvector for  $\text{Im}(\alpha N^*)$  with  $\alpha = \langle u, v \rangle / \|v\|^2$ .

In order to prove theorem 2.1 some previous lemmas will be necessary. Recall that a closed subspace  $M \subseteq H$  is reducing for an operator  $T \in \mathcal{L}(H)$  if and only if the orthogonal projection  $P_M : H \rightarrow M$  onto  $M$  commutes with  $T$ , that is,  $TP_M = P_M T$ . The notation  $\{T\}'$  will stand for the commutant of  $T$ .

**LEMMA 2.2.** *Let  $T = D + u \otimes v \in \mathcal{L}(H)$  where  $D$  is a diagonal operator and  $u, v$  are nonzero vectors in  $H$ . Let  $M \subseteq H$  be a non-trivial reducing subspace for  $T$ . If  $P_M$  is the orthogonal projection onto  $M$  and  $Q_M = I - P_M$ , then*

$$\begin{aligned} Q_M D P_M &= -Q_M u \otimes P_M v, \\ P_M D Q_M &= -P_M u \otimes Q_M v, \\ Q_M D^* P_M &= -Q_M v \otimes P_M u. \\ P_M D^* Q_M &= -P_M v \otimes Q_M u. \end{aligned}$$

*Proof.* We show the first equality, since the other ones follow analogously.

Since  $M$  is reducing for  $T$ ,  $Q_M T P_M = 0$  and therefore,

$$Q_M D P_M = Q_M (T - u \otimes v) P_M = Q_M (u \otimes v) P_M.$$

In addition, for every  $x \in H$

$$(Q_M u \otimes v) P_M x = \langle P_M x, v \rangle Q_M u = \langle x, P_M v \rangle Q_M u = (Q_M u \otimes P_M v) x,$$

which finally leads to the desired expression. □

Next lemma refers, roughly speaking, to the location of the vectors  $u$  and  $v$  with respect to reducing subspaces of  $T = D + u \otimes v$ .

LEMMA 2.3. *Let  $T = D + u \otimes v \in \mathcal{L}(H)$  where  $D$  is a diagonal operator with respect to an orthonormal basis  $(e_n)_{n \geq 1}$  and  $u = \sum_n \alpha_n e_n, v = \sum_n \beta_n e_n$  nonzero vectors in  $H$ . Assume  $D$  has uniform multiplicity one and  $\alpha_n \beta_n \neq 0$  for every  $n \in \mathbb{N}$ . Let  $P_M : H \rightarrow M$  be a non-trivial orthogonal projection commuting with  $T$ . Then,  $u, v, e_n \notin \ker P_M \cup \ker(I - P_M)$  for all  $n \in \mathbb{N}$ .*

In order to prove lemma 2.3, recall that given a linear space  $\mathcal{A}$  of  $\mathcal{L}(H)$ , a vector  $x$  is separating for  $\mathcal{A}$  if  $Ax = 0$  and  $A \in \mathcal{A}$ , then  $A = 0$ .

*Proof.* First, observe that  $Q_M = I - P_M$  and  $P_M$  both commute with  $T$  and  $T^*$ . Now, since  $u$  is a separating vector for  $\{T\}'$  and  $v$  for  $\{T^*\}'$  (see [19, theorem 1.4]) it follows that the vectors  $P_M u, P_M v, Q_M u, Q_M v$  are non-zero. In particular, one deduces that  $u, v \notin M \cup M^\perp$ .

On the other hand, since  $P_M T = T P_M$  then

$$D P_M - P_M D = u \otimes P_M v - P_M u \otimes v.$$

Let  $n \in \mathbb{N}$  and assume, for the moment, that  $e_n \in M$ . Then,

$$0 = D P_M e_n - P_M D e_n = \langle e_n, v \rangle u - \langle e_n, u \rangle P_M u. \tag{2.1}$$

Note that  $\langle e_n, v \rangle \neq 0$  since  $\alpha_n \beta_n \neq 0$ , so  $u = P_M u$  and therefore  $u \in M$ , which is a contradiction. Hence  $e_n \notin M$ .

The case  $e_n \in M^\perp$  is analogous. □

As lemma 2.3, the following result deals with the location of  $u$  and  $v$  but when the assumption  $\alpha_n \beta_n \neq 0$  for every  $n \in \mathbb{N}$  is replaced by one regarding the spectrum of the diagonal operator  $D$ .

LEMMA 2.4. *Let  $T = D + u \otimes v \in \mathcal{L}(H)$  where  $D$  is a diagonal operator with respect to an orthonormal basis  $(e_n)_{n \geq 1}$  and  $u = \sum_n \alpha_n e_n, v = \sum_n \beta_n e_n$  nonzero vectors in  $H$ . Assume  $D$  has uniform multiplicity one, its spectrum  $\sigma(D)$  lies in a Jordan curve and  $\alpha_n$  and  $\beta_n$  are not simultaneously zero. If  $P_M : H \rightarrow M$  is a non-trivial orthogonal projection commuting with  $T$ , then  $u, v, e_n \notin \ker P_M \cup \ker(I - P_M)$  for all  $n \in \mathbb{N}$ .*

*Proof.* Let us argue by contradiction assuming  $P_M v = 0$ . Then, by lemma 2.2, we have

$$Q_M D P_M = -Q_M u \otimes P_M v = 0,$$

where  $Q_M = I - P_M$ . Then  $0 = (I - P_M) D P_M = D P_M - P_M D P_M$  or, equivalently,  $D P_M = P_M D P_M$ . Hence the closed subspace  $M := P_M(H)$  is a non-trivial closed invariant subspace for  $D$  (see [32, theorem 0.1], for instance).

Moreover,  $D$  is a completely normal operator (see [39, theorem 3] or [32, chapter 1]). Therefore, every invariant subspace of  $D$  is reducing and it is spanned by

a subset of eigenvectors. Accordingly,  $DP_M = P_MD$  and  $M = \overline{\text{span} \{e_n : n \in \Lambda\}}$ , where  $\Lambda$  is a proper subset of the natural numbers  $\mathbb{N}$ . Observe that, in particular,  $DP_M = P_MD$  implies that  $P_M u = 0$ .

On the other hand, since  $P_M$  is the orthogonal projection onto  $M = \overline{\text{span} \{e_n : n \in \Lambda\}}$ , trivially  $P_M$  is a diagonal operator with respect to  $(e_n)_{n \geq 1}$  (since  $P_M e_n = e_n$  for every  $n \in \Lambda$  and  $P_M e_n = 0$  for every  $n \notin \Lambda$ ). Having into account that  $P_M u = P_M v = 0$ , we deduce

$$0 = \sum_{n \in \Lambda} \alpha_n e_n = \sum_{n \in \Lambda} \beta_n e_n.$$

Hence, for every  $n \in \Lambda$  we have  $\alpha_n = \beta_n = 0$ , which is a contradiction unless  $\Lambda = \emptyset$ . But, in this latter case, it would follow that  $P_M = 0$  which is also absurd since  $P_M$  is a non-trivial projection. Therefore,  $P_M v \neq 0$  as the statement asserts.

The proof of the statement  $u \notin \ker P_M \cup \ker(I - P_M)$  is analogous, just considering  $T^*$ .

Finally, in order to show that  $e_n \notin \ker P_M \cup \ker(I - P_M)$  for all  $n \in \mathbb{N}$ , we may argue as in lemma 2.3 considering (2.1) if  $e_n$  would be in  $M$  (note that, by assumption,  $\alpha_n$  and  $\beta_n$  are not simultaneously zero). □

With the previous results at hands, we are in position to prove theorem 2.1. The proof will be accomplished by studying firstly the case that the diagonal operator  $D$  is self-adjoint and therefore, its spectrum is contained in the real numbers.

*Proof of theorem 2.1.* Assume that  $T = D + u \otimes v \in \mathcal{L}(H)$  where  $D$  is a diagonal operator with respect to an orthonormal basis  $(e_n)_{n \geq 1}$  and  $u = \sum_n \alpha_n e_n$ ,  $v = \sum_n \beta_n e_n$  two nonzero vectors in  $H$ .

Clearly, if  $T$  is a normal operator, it has non-trivial reducing subspaces. In addition, if there exists  $n_0 \in \mathbb{N}$  such that  $\alpha_{n_0} = \beta_{n_0} = 0$ , then the subspace generated by the basis vector  $e_{n_0}$  is reducing for  $T$  as pointed out in the preliminary section. In both cases,  $T$  has a non-trivial reducing subspace. For the converse, assume that  $T$  has a non-trivial reducing subspace and let us show that  $T$  is normal or there exists  $n_0 \in \mathbb{N}$  such that  $\alpha_{n_0} = \beta_{n_0} = 0$ .

**Case 1:**  $D$  is a self-adjoint operator.

Assume that  $T = D + u \otimes v$  has a non-trivial reducing subspace  $M \subset H$  and that  $\alpha_n$  and  $\beta_n$  are not simultaneously zero. Let  $P_M$  be the non-trivial orthogonal projection onto  $M$  and  $Q_M = I - P_M$ . By lemma 2.2, both relations

$$Q_M u \otimes P_M v = -Q_M D P_M = Q_M v \otimes P_M u$$

and

$$P_M u \otimes Q_M v = -P_M D Q_M = P_M v \otimes Q_M u$$

hold. Moreover, lemma 2.4 ensures that the vectors  $P_M u$ ,  $Q_M u$ ,  $P_M v$  and  $Q_M v$  are non-zero.



A little computation shows that

$$\langle P_M u, P_M v \rangle Q_M u = \|P_M u\|^2 Q_M v,$$

and since  $\|P_M u\| > 0$ , we deduce

$$Q_M v = \frac{\langle P_M u, P_M v \rangle}{\|P_M u\|^2} Q_M u.$$

In a similar way, we have

$$P_M v = \frac{\langle Q_M u, Q_M v \rangle}{\|Q_M u\|^2} P_M u = \frac{\langle P_M v, P_M u \rangle}{\|P_M u\|^2} P_M u.$$

Now, let us write  $\alpha = \langle P_M v, P_M u \rangle / \|P_M u\|^2$ , so  $P_M v = \alpha P_M u$  and  $Q_M v = \bar{\alpha} Q_M u$ . Hence,

$$v = P_M v + Q_M v = \alpha P_M u + \bar{\alpha} Q_M u.$$

Notice that it is enough to show that  $\alpha \in \mathbb{R}$ , so  $v = \alpha u$  and therefore  $T$  would be a self-adjoint operator, since it would be the sum of two self-adjoint operators.

Observe that

$$\|v\|^2 = \|P_M v\|^2 + \|Q_M v\|^2 = |\alpha|^2 \|P_M u\|^2 + |\alpha|^2 \|Q_M u\|^2 = |\alpha|^2 \|u\|^2.$$

Moreover,

$$\begin{aligned} \langle u, v \rangle &= \langle P_M u, P_M v \rangle + \langle Q_M u, Q_M v \rangle \\ &= \langle P_M u, \alpha P_M u \rangle + \langle Q_M u, \bar{\alpha} Q_M u \rangle \\ &= \bar{\alpha} \|P_M u\|^2 + \alpha \|Q_M u\|^2 \\ &= (\operatorname{Re}(\alpha) - i \operatorname{Im}(\alpha)) \|P_M u\|^2 + (\operatorname{Re}(\alpha) + i \operatorname{Im}(\alpha)) \|Q_M u\|^2 \\ &= \operatorname{Re}(\alpha) (\|P_M u\|^2 + \|Q_M u\|^2) + i \operatorname{Im}(\alpha) (\|Q_M u\|^2 - \|P_M u\|^2) \\ &= \operatorname{Re}(\alpha) \|u\|^2 + i \operatorname{Im}(\alpha) (\|Q_M u\|^2 - \|P_M u\|^2). \end{aligned}$$

Note that  $\alpha u - v \in M^\perp$ , since  $P_M(\alpha u - v) = 0$ . Thus,  $Q_M(\alpha u - v) = \alpha u - v$ .

Similarly,  $\bar{\alpha} u - v \in M$ . Furthermore,  $(\alpha P_M + \bar{\alpha} Q_M)u = v$ . Since  $P_M$  and  $Q_M$  commute with  $T$ , it follows that

$$(\alpha P_M + \bar{\alpha} Q_M)T = T(\alpha P_M + \bar{\alpha} Q_M).$$

Now,

$$T(\alpha P_M + \bar{\alpha} Q_M)u = Tv = Dv + \|v\|^2 u = Dv + |\alpha|^2 \|u\|^2 u$$

and

$$\begin{aligned} (\alpha P_M + \bar{\alpha} Q_M)Tu &= (\alpha P_M + \bar{\alpha} Q_M)(Du + \langle u, v \rangle u) \\ &= \alpha P_M Du + \bar{\alpha} Q_M Du + \langle u, v \rangle (\alpha P_M u + \bar{\alpha} Q_M u) \\ &= \alpha P_M Du + \bar{\alpha} Q_M Du + \langle u, v \rangle v \end{aligned}$$

Thus,

$$Dv + |\alpha|^2 \|u\|^2 u = \alpha P_M Du + \bar{\alpha} Q_M Du + \langle u, v \rangle v.$$

So,

$$P_M Dv + Q_M Dv + |\alpha|^2 \|u\|^2 u = \alpha P_M Du + \bar{\alpha} Q_M Du + \langle u, v \rangle v.$$

Upon applying lemma 2.2 it follows that

$$\begin{aligned} |\alpha|^2 \|u\|^2 u - \langle u, v \rangle v &= P_M D(\alpha u - v) + Q_M D(\bar{\alpha} - v) \\ &= P_M DQ_M(\alpha u - v) + Q_M DP_M(\bar{\alpha} u - v) \\ &= \alpha(P_M u \otimes Q_M u)(v - \alpha u) + \alpha(P_M u \otimes Q_M u)(v - \bar{\alpha} u) \\ &= \alpha \langle v - \alpha u, u \rangle P_M u + \alpha \langle v - \bar{\alpha} u, u \rangle Q_M u \\ &= (\alpha \langle v, u \rangle - \alpha^2 \|u\|^2) P_M u + (\alpha \langle v, u \rangle - |\alpha|^2 \|u\|^2) Q_M u. \end{aligned}$$

Moreover,  $v = \alpha P_M u + \bar{\alpha} Q_M u$ , so

$$|\alpha|^2 \|u\|^2 u - \langle u, v \rangle v = (|\alpha|^2 \|u\|^2 - \alpha \langle u, v \rangle) P_M u + (|\alpha|^2 \|u\|^2 - \bar{\alpha} \langle u, v \rangle) Q_M u.$$

Consequently,

$$|\alpha|^2 \|u\|^2 - \alpha \langle u, v \rangle = \alpha \langle v, u \rangle - \alpha^2 \|u\|^2$$

and

$$|\alpha|^2 \|u\|^2 - \bar{\alpha} \langle u, v \rangle = \alpha \langle v, u \rangle - |\alpha|^2 \|u\|^2.$$

From the second equality we have

$$\begin{aligned} 2|\alpha|^2 \|u\|^2 &= \alpha \langle v, u \rangle + \bar{\alpha} \langle u, v \rangle \\ &= 2\text{Re}(\alpha \langle v, u \rangle). \end{aligned} \tag{2.2}$$

Let us write  $\alpha = a + bi$ , with  $a, b \in \mathbb{R}$  real numbers. The aim is to show that  $b = 0$ .

Recall that

$$\langle v, u \rangle = \overline{\langle u, v \rangle} = a \|u\|^2 - ib(|Q_M u|^2 - |P_M u|^2).$$

Then,

$$\begin{aligned} \alpha \langle v, u \rangle &= (a + bi)(a \|u\|^2 - ib(|Q_M u|^2 - |P_M u|^2)) \\ &= a^2 \|u\|^2 - aib(|Q_M u|^2 - |P_M u|^2) \\ &\quad + aib \|u\|^2 + b^2(|Q_M u|^2 - |P_M u|^2). \end{aligned}$$

So,  $\text{Re}(\alpha \langle v, u \rangle) = a^2 \|u\|^2 + b^2(|Q_M u|^2 - |P_M u|^2)$ . From equation (2.2) we have

$$(a^2 + b^2) \|u\|^2 = a^2 \|u\|^2 + b^2(|Q_M u|^2 - |P_M u|^2),$$

and therefore,

$$b^2 \|u\|^2 = b^2(|Q_M u|^2 - |P_M u|^2).$$

If  $b \neq 0$ , this identity yields that  $\|P_M u\| = 0$ , what contradicts lemma 2.4. Accordingly,  $b = 0$  and then  $\alpha$  is a real number as we wish to prove in order to show that  $T$  is a normal operator.

Finally, if  $T$  has a reducing subspace and it is not a normal operator, we deduce that there exists  $n \in \mathbb{N}$  such that  $\alpha_n = \beta_n = 0$ , so the proof in case 1 is complete.

**Case 2:**  $D$  is not a self-adjoint operator.

Assume first that the set of eigenvalues  $\Lambda(D) = (\lambda_n)_{n \geq 1}$  is contained in a line  $\Gamma$  in the complex plane parallel to the real axis. Let  $h \in \mathbb{R}$  such that  $\Gamma = \{x + ih : x \in \mathbb{R}\}$ .

Observe that the linear bounded operator  $D - hiI$  is a diagonal, self-adjoint operator of uniform multiplicity one. Now, observe that  $T$  has the same reducing subspaces that  $T - hiI$  has. Then, by case 1, it follows that  $T - hiI$  has a non-trivial reducing subspace if and only if either  $T - hiI$  is a normal operator or there exists  $n \in \mathbb{N}$  such that  $\alpha_n = \beta_n = 0$ . Accordingly, the same conclusion holds for  $T$ .

Finally, if  $\Lambda(D)$  is contained in a line  $\Gamma$  that intersects the real axis, let  $\theta \in [0, \pi)$  denote the angle formed by  $\Gamma$  and the real axis measured in the positive direction. Then,  $\tilde{D} = e^{-i\theta}D$  satisfies that its set of eigenvalues  $\Lambda(\tilde{D})$  is contained in a line parallel to the real axis. The final statement follows upon applying case 1 to the operator  $e^{-i\theta}T$ . This concludes the proof of theorem 2.1.  $\square$

REMARK 2.5. We point out that the assumption on  $D$  having uniform multiplicity one in theorem 2.1 is necessary and cannot be dropped off. Indeed, it is not difficult to provide examples of non-normal operators  $T = D + u \otimes v$  having non-trivial reducing subspaces such that the spectrum  $\sigma(D)$  is contained in a line and  $u$  and  $v$  are non-zero vectors with non-zero components. For instance, if  $\ell^2$  denote the classical Hilbert space consisting of complex sequences whose modulus are squared summable and  $\{e_n\}_{n \geq 1}$  the canonical basis in  $\ell^2$ , let us consider  $D$  be the diagonal operator defined by

$$De_n = \begin{cases} e_n & n = 1, 2; \\ \frac{1}{n}e_n & n \geq 3. \end{cases}$$

Clearly,  $D$  is a self-adjoint (even a compact) operator in  $\ell^2$ . Now, if  $u = \sum_{n \geq 1} (1/n)e_n$  and  $v = e_1 + (1/2)e_2 + \sum_{n \geq 3} (1/n^2)e_n$ , for instance, the operator  $T = D + u \otimes v$  has non-trivial reducing subspace (indeed the one generated by  $e_1$ , see proposition 1.2), the spectrum  $\sigma(D)$  is  $\{1/n\}_{n \geq 1} \cup \{0\} \subset [0, 1]$  and clearly  $u$  and  $v$  are non-zero vectors with non-zero components. Nevertheless,  $T$  is not a normal operator since  $u$  and  $v$  are not proportional vectors, which is required by condition (i') in Ionascu's theorem (see the preliminary section).

To close this section, we observe that if  $D$  is a diagonal operator with a set of eigenvalues  $\Lambda(D) = (\lambda_n)_{n \geq 1}$  contained in a line or a circle then  $T = D + u \otimes v$  is a decomposable operator by a result due to Radjabalipour and Radjavi (see [33, corollary 2]). Recall that an operator  $T \in \mathcal{L}(H)$  is *decomposable* if for every open cover  $U, V \subset \mathbb{C}$  of  $\sigma(T)$  there exist invariant subspaces  $M, N \subset H$  for  $T$  such that  $M + N = H$  and  $\sigma(T|_M) \subset \bar{U}$  and  $\sigma(T|_N) \subset \bar{V}$ .

Hence, as a consequence of theorem 2.1, it is possible to exhibit one-rank perturbations of diagonal operators  $D$  which are decomposable but lack non-trivial reducing subspaces. Indeed, a bit more can be achieved in this context.

Recall that a closed subspace  $M \subset H$  is called a *spectral maximal subspace* of an operator  $T \in \mathcal{L}(H)$  if

- (a)  $M$  is an invariant subspace of  $T$ , and
- (b)  $N \subset M$  for all closed invariant subspaces  $N$  of  $T$  such that the spectrum of the restriction  $T|_N$  is contained in the spectrum of the restriction  $T|_M$ , that is,  $\sigma(T|_N) \subseteq \sigma(T|_M)$ .

In addition,  $T \in \mathcal{L}(H)$  is called *strongly decomposable* if its restriction to an arbitrary spectral maximal subspace is again decomposable. Indeed, the authors in [33, corollary 2] state that if  $T^* - T$  belongs to the Schatten class  $S_p(H)$  for  $1 \leq p < \infty$ , then  $T$  is strongly decomposable.

**COROLLARY 2.6.** *There exist rank-one perturbations of self-adjoint diagonal operators  $T = D + u \otimes v \in \mathcal{L}(H)$  that are strongly decomposable operators and have no non-trivial reducing subspaces.*

*Proof.* Let  $D$  be a self-adjoint diagonal operator of uniform multiplicity one. Let us consider  $u$  and  $v$  two non-zero vectors in  $H$  with non-zero components and being not real-proportional. Then the operator  $T = D + u \otimes v$  is strongly decomposable since  $T^* - T$  is a rank two operator (and hence belongs to the Schatten class  $S_p(H)$  for every  $1 \leq p < \infty$ ). Nevertheless,  $T$  is not normal and by theorem 2.1, it has no non-trivial reducing subspaces. □

### 3. Reducing subspaces for rank-one perturbations of diagonal operators: when the spectrum is contained in a circle

In this section, we will focus on rank-one perturbations of a diagonal operator with spectrum contained in a circle. Note that this is the other possible case according to Ionascu’s result to ensure that such operators are normal (condition (ii’) in the preliminary section).

As we will show, the spectral picture does not determine the existence of non-trivial reducing subspaces for non-normal operators within this class. In other words, it is possible to exhibit non-normal operators within this class with spectrum contained in a circle either having or lacking non-trivial reducing subspaces. Our main result in this section reads as follows:

**THEOREM 3.1.** *Let  $D \in \mathcal{L}(H)$  be a diagonal operator with respect to an orthonormal basis  $(e_n)_{n \geq 1}$  with uniform multiplicity one. Assume its spectrum  $\sigma(D)$  is contained in a circle with center  $\alpha$ , and let  $u \in H$  such that  $\langle u, e_n \rangle \neq 0$  for every  $n \in \mathbb{N}$ . Then*

- (a) *The operator  $T_1 = D + u \otimes u \in \mathcal{L}(H)$  has no non-trivial reducing subspaces.*
- (b) *If  $\langle u, Du \rangle \neq \bar{\alpha} \|u\|^2$ , there exists  $v \in H$  with  $\langle v, e_n \rangle \neq 0$  for every  $n \in \mathbb{N}$  and the operator  $T_2 = D + u \otimes v \in \mathcal{L}(H)$  is not normal but has non-trivial reducing subspaces.*

*Proof.* First, we consider the case that the diagonal operator  $D$  is a unitary operator and therefore, its spectrum is contained in the unit circle  $\mathbb{T}$ .

**Case 1:**  $D$  is a unitary operator.

Assume, on the contrary, that  $M$  is a non-trivial reducing subspace for  $T = D + u \otimes u$  and denote, as usual,  $P_M$  the orthogonal projection onto  $M$ . We may assume, without loss of generality, that  $M$  is infinite dimensional (otherwise we would argue with  $M^\perp$  since it would be an infinite-dimensional reducing subspace).

Let  $x$  be a non-zero vector in  $M$ . Clearly,  $Tx = Dx + \langle x, u \rangle u \in M$ , and since  $M$  is reducing,  $T^*x = D^*x + \langle x, u \rangle u$  is also in  $M$ . Accordingly,  $(D - D^*)x \in M$  for every  $x \in M$ , or equivalently,  $M$  is an infinite-dimensional non-trivial closed invariant subspace for  $D - D^*$ .

Note that  $D - D^*$  is a non-trivial diagonal operator with the spectrum contained in the imaginary axis of the complex plane. Indeed, if  $\Lambda(D) = (\lambda_n)_{n \geq 1}$  denotes the set of eigenvalues of  $D$ , then  $\Lambda(D - D^*) = (2i \operatorname{Im}(\lambda_n))_{n \geq 1}$ . In particular, by [32, theorems 1.23 and 1.25]  $D - D^*$  is a completely normal operator and therefore,  $M$  is spanned by eigenvectors of  $D - D^*$ .

On the other hand, since  $D$  has uniform multiplicity one then the eigenvalues of  $D - D^*$  has multiplicity at most two. Actually, if  $2i \operatorname{Im}(\lambda_n)$  has multiplicity two, then there exists  $m \in \mathbb{N}$  such that  $\operatorname{Im}(\lambda_n) = \operatorname{Im}(\lambda_m)$ . For each pair of indexes of this form, let us denote by  $n$  the smaller index and by  $n'$  the greater one. That is, let us denote by  $\Omega$  the set of pairs of indexes

$$\Omega = \{(n, n') \in \mathbb{N} \times \mathbb{N} : n < n' \text{ and } \operatorname{Im}(\lambda_n) = \operatorname{Im}(\lambda_{n'}) \text{ where } \lambda_n, \lambda_{n'} \in \Lambda(D)\}.$$

Doing so, we may consider a disjoint partition of the natural numbers

$$\mathbb{N} = N_1 \cup N_2 \cup N_3,$$

where  $N_1 \subset \mathbb{N}$  consists of the indexes of the eigenvalues with multiplicity two that are the smaller index of its corresponding pair,  $N_2$  consists of the indexes of the eigenvalues with multiplicity two that are the larger index of its corresponding pair and  $N_3$  the indexes of the eigenvalues with multiplicity one. In other words, the set  $\Omega = N_1 \times N_2$ .

Then, the eigenvectors of  $D - D^*$  are

$$\{e_n : n \in N_3\} \text{ and } \{\lambda e_n + \tau e_{n'} : (n, n') \in \Omega, \lambda, \tau \in \mathbb{C}\}.$$

With the characterization of the eigenvectors of  $D - D^*$ , our goal now will be identify those eigenvectors spanning  $M$ , that is, determine the proper subset  $\Lambda$  of  $\mathbb{N}$  such that

$$M = \overline{\operatorname{span} \{e_n : n \in N_3 \cap \Lambda\} + \{\lambda e_n + \tau e_{n'} : n \in \Lambda \cap N_1, (n, n') \in \Omega, \lambda, \tau \in \mathbb{C}\}}.$$

By lemma 2.4 for every  $n \in \mathbb{N}$  we have  $e_n \notin M \cup M^\perp$ , so  $e_n \notin M$  for every  $n \in N_3$ . Assume for the moment that  $N_3 \cap \Lambda$  is not empty and let  $n_0 \in N_3 \cap \Lambda$ . Denote by

$$e_{n_0} = e_{n_0}^M \oplus e_{n_0}^{M^\perp}$$

the orthogonal decomposition of  $e_{n_0}$  with respect to  $H = M \oplus M^\perp$ . Note that, in particular,  $e_{n_0}^M$  is non-zero because otherwise  $e_{n_0} \in M^\perp$ .

Having in mind that  $(D - D^*)P_M = P_M(D - D^*)$  because  $M$  is reducing for  $D - D^*$ , we deduce upon applying it to  $e_{n_0}$  that

$$(D - D^*)P_M e_{n_0} = (D - D^*)e_{n_0}^M = (\lambda_{n_0} - \overline{\lambda_{n_0}})e_{n_0}^M.$$

That is,  $2i\text{Im}(\lambda_{n_0})$  is an eigenvalue of  $D - D^*$  with eigenspace  $\text{span}\{e_{n_0}, e_{n_0}^M\}$ . Hence,  $2i\text{Im}(\lambda_{n_0})$  is of multiplicity 2, but this contradicts the fact that  $n_0 \in N_3$ . Therefore,  $N_3 \cap \Lambda$  is empty. Accordingly, we deduce that  $\Lambda$  must be a non-void subset  $\Lambda$  of  $N_1$  (being possible  $\Lambda = N_1$ ), and

$$M = \overline{\text{span}\{\lambda e_n + \tau e_{n'} : n \in \Lambda \subseteq N_1, (n, n') \in \Omega \text{ and } \lambda, \tau \in \mathbb{C}\}}. \tag{3.1}$$

Moreover, we observe that for every eigenvector  $\lambda e_n + \tau e_{n'} \in M$ , the coefficients  $\lambda$  and  $\tau$  are non-zero (otherwise we would have  $e_n$  or  $e_{n'} \in M$ ). Accordingly, every eigenvector  $\lambda e_n + \tau e_{n'} \in M$  is a multiple of  $e_n + (\tau/\lambda)e_{n'}$ . Since  $e_{n'} \notin M$ , we deduce that there exists coefficients  $\tau_n \in \mathbb{C}$  such that

$$M = \overline{\text{span}\{e_n + \tau_n e_{n'} : n \in \Lambda \subseteq N_1, (n, n') \in \Omega\}}. \tag{3.2}$$

On the other hand, since  $M$  is a non-trivial closed invariant subspace under  $T$ , we have that  $T(e_n + \tau_n e_{n'}) \in M$  for every  $n \in \Lambda \subseteq N_1$  with  $(n, n') \in \Omega$ . If we consider  $u = \sum_n \alpha_n e_n$  where  $\alpha_n = \langle u, e_n \rangle \neq 0$  for every  $n \in \mathbb{N}$  by hypotheses, then

$$\begin{aligned} T(e_n + \tau_n e_{n'}) &= \lambda_n e_n - \overline{\lambda_n} \tau_n e_{n'} + \langle e_n + \tau_n e_{n'}, u \rangle u \\ &= \lambda_n e_n - \overline{\lambda_n} \tau_n e_{n'} + (\overline{\alpha_n} + \tau_n \overline{\alpha_{n'}})u \in M \end{aligned} \tag{3.3}$$

for every  $n \in \Lambda \subseteq N_1$  with  $(n, n') \in \Omega$ .

Now, let us prove that  $\text{Re}(\lambda_{n_0}) = 0$  for at least one positive integer  $n_0 \in N_1$ .

First, by means of the orthogonality relations of the basis elements  $\{e_n\}_{n \in \mathbb{N}}$ , it follows that for every vector  $x \in M$

$$\tau_m \langle x, e_m \rangle = \langle x, e_{m'} \rangle \quad \text{for every } m \in \Lambda \subseteq N_1 \text{ with } (m, m') \in \Omega. \tag{3.4}$$

Let  $n_0$  be any positive integer in  $\Lambda$  (recall that  $\Lambda$  is not empty). If

$$\overline{\alpha_{n_0}} + \tau_{n_0} \overline{\alpha_{n'_0}} = 0, \tag{3.5}$$

then by (3.3) the vector

$$\mathbf{a} = \lambda_{n_0} e_{n_0} - \overline{\lambda_{n_0}} \tau_{n_0} e_{n'_0}$$

is in  $M$  which, by means of (3.4) particularized in the index  $n_0$ , satisfies

$$\tau_{n_0} \langle \mathbf{a}, e_{n_0} \rangle = \langle \mathbf{a}, e_{n'_0} \rangle.$$

That is,

$$\tau_{n_0} \lambda_{n_0} = -\overline{\lambda_{n_0}} \tau_{n_0}.$$

Having in mind that  $\tau_n \neq 0$  for every  $n \in \mathbb{N}$ , we deduce that  $\lambda_{n_0} = -\overline{\lambda_{n_0}}$  as far as (3.5) holds, or equivalently  $\text{Re}(\lambda_{n_0}) = 0$  whenever (3.5) is satisfied.

Assume, now, that (3.5) is not satisfied, that is

$$\overline{\alpha_{n_0}} + \tau_{n_0} \overline{\alpha_{n'_0}} \neq 0. \tag{3.6}$$

Let us consider  $m_0 \in \Lambda$  with  $m_0 \neq n_0$ . Observe that this is possible since, in particular,  $M$  is given by (3.2) and the dimension of  $M$  is infinite.

Now, by (3.3), the vector

$$\mathbf{b} = \lambda_{n_0} e_{n_0} - \overline{\lambda_{n_0}} \tau_{n_0} e_{n'_0} + (\overline{\alpha_{n_0}} + \tau_{n_0} \overline{\alpha_{n'_0}}) u$$

is in  $M$ . We argue as before upon considering (3.4) at  $\mathbf{b}$  and  $e_{m_0}$ , that is,  $\tau_{m_0} \langle \mathbf{b}, e_{m_0} \rangle = \langle \mathbf{b}, e_{m'_0} \rangle$ . Then

$$\tau_{m_0} (\overline{\alpha_{n_0}} + \tau_{n_0} \overline{\alpha_{n'_0}}) \alpha_{m_0} = (\overline{\alpha_{n_0}} + \tau_{n_0} \overline{\alpha_{n'_0}}) \alpha_{m'_0}.$$

Since  $\overline{\alpha_{n_0}} + \tau_{n_0} \overline{\alpha_{n'_0}} \neq 0$  (i.e. our assumption (3.6)), we deduce that

$$\tau_{m_0} = \frac{\alpha_{m'_0}}{\alpha_{m_0}}.$$

Therefore, having in mind that  $e_{m_0} + \tau_{m_0} e_{m'_0} \in M$ , we deduce that

$$\mathbf{c} = T(e_{m_0} + \tau_{m_0} e_{m'_0}) = \lambda_{m_0} e_{m_0} - \overline{\lambda_{m_0}} \frac{\alpha_{m'_0}}{\alpha_{m_0}} e_{m'_0} + \left( \overline{\alpha_{m_0}} + \frac{|\alpha_{m'_0}|^2}{\alpha_{m_0}} \right) u$$

is also in  $M$ . Once again applying (3.4), we have  $\tau_{m_0} \langle \mathbf{c}, e_{m_0} \rangle = \langle \mathbf{c}, e_{m'_0} \rangle$  and therefore

$$\frac{\alpha_{m'_0}}{\alpha_{m_0}} (\lambda_{m_0} + |\alpha_{m_0}|^2 + |\alpha_{m'_0}|^2) = -\frac{\alpha_{m'_0}}{\alpha_{m_0}} \overline{\lambda_{m_0}} + \overline{\alpha_{m_0}} \alpha_{m'_0} + \frac{|\alpha_{m'_0}|^2}{\alpha_{m_0}} \alpha_{m'_0}.$$

Multiplying by  $\alpha_{m_0}/\alpha_{m'_0}$  we obtain

$$\lambda_{m_0} + |\alpha_{m_0}|^2 + |\alpha_{m'_0}|^2 = -\overline{\lambda_{m_0}} + |\alpha_{m_0}|^2 + |\alpha_{m'_0}|^2,$$

from where, clearly,  $\text{Re}(\lambda_{m_0}) = 0$ .

Consequently, independently if (3.5) is or not satisfied, there exists at least a positive integer  $n_0 \in N_1$  such that  $\text{Re}(\lambda_{n_0}) = 0$ , as we wished to show.

In order to finish the proof of case 1, let us show that the existence of such  $n_0 \in N_1$  yields the desired contradiction. Since  $D$  is unitary, its spectrum is contained in the unit circle and therefore, such  $\lambda_{n_0}$  is either  $i$  or  $-i$ . Assume  $\lambda_{n_0} = i$  (the other case is analogous). If  $n_0 \in N_1$ , by definition, there exist  $n'_0 \in \mathbb{N}$  with  $n_0 < n'_0$  and  $\lambda_{n'_0}$  an eigenvalue of  $D$ ,  $\lambda_{n'_0} \in \Lambda(D)$ , such that  $\text{Im}(\lambda_{n_0}) = \text{Im}(\lambda_{n'_0})$ . Hence,  $\text{Im}(\lambda_{n'_0}) = 1$ , and therefore  $\lambda_{n'_0}$  must be also  $i$ . Accordingly,  $i$  is an eigenvalue of  $D$  of multiplicity 2, which contradicts the hypotheses that the uniform multiplicity of  $D$  is one.

**Case 2:**  $D$  is not a unitary operator.

Now, assume  $D$  is a diagonal operator such that  $\sigma(D)$  is contained in a circle with center  $\alpha$  and radius  $r > 0$ . As in case 1, we argue by contradiction assuming  $T = D + u \otimes u$  has a non-trivial reducing subspace  $M$ . Since  $M$  is reducing for  $T$ ,

it is also reducing for  $(1/r)(T - \alpha I) = (1/r)(D - \alpha I) + (1/r)u \otimes u$ , which is the contradiction.

This concludes the proof that statement (a) of theorem 3.1 holds.

Now, we will show statement (b) of theorem 3.1. As before, the key argument will be to prove the result when  $D$  is a unitary diagonal operator.

Hence, suppose  $D$  is a unitary and  $u \in H$  such that  $\langle u, e_n \rangle \neq 0$  for every  $n \in \mathbb{N}$  and  $\langle u, Du \rangle \neq 0$ . In order to show that there exists  $v \in H$  such that  $T = D + u \otimes v \in \mathcal{L}(H)$  is not normal but has non-trivial reducing subspaces, we take

$$v = -\frac{1}{\langle u, Du \rangle} D^{*2}u.$$

The goal will be showing that the subspace  $M := \text{span} \{D^*u\}$  reduces the operator  $T = D + u \otimes v$ .

First, note that

$$\langle D^{*2}u, e_n \rangle = \langle u, D^2e_n \rangle = \overline{\lambda_n^2} \langle u, e_n \rangle \neq 0$$

since  $|\lambda_n| = 1$ . Then,  $\langle v, e_n \rangle \neq 0$  for every  $n \in \mathbb{N}$ .

Now, observe that

$$\langle Dv, u \rangle = \left\langle -\frac{1}{\langle u, Du \rangle} D^*u, u \right\rangle = -\frac{1}{\langle u, Du \rangle} \langle D^*u, u \rangle = -1.$$

So, we have

$$TD^*u = (D + u \otimes v)D^*u = u + \langle D^*u, v \rangle u = u + \langle u, Dv \rangle u = u - u = 0.$$

Moreover,

$$T^*D^*u = (D^* + v \otimes u)D^*u = D^{*2}u + \langle u, Du \rangle v = D^{*2}u - D^{*2}u = 0.$$

So,  $D^*u$  is a non-zero vector which turns out to be an eigenvector associated with the eigenvalue 0 for  $T$  and  $T^*$ .

Finally, let us show that  $T$  is not a normal operator. Assume, on the contrary, that  $T$  is normal. Having into account condition (ii') in Ionascu theorem 1.1, we deduce the existence of  $\gamma \in \mathbb{C}$  such that  $Dv = \gamma u$ . Then, since  $Dv = -(1/\langle u, Du \rangle)D^*u$ , we have

$$D^*u = -\gamma \langle u, Du \rangle u.$$

That is,  $u$  is an eigenvector for  $D^*$  associated with the eigenvalue  $-\gamma \langle u, Du \rangle$ . Observe that  $\gamma \neq 0$  since  $Dv \neq 0$  and  $\langle u, Du \rangle \neq 0$  by hypothesis. Therefore, there exists  $n_0 \in \mathbb{N}$  such that  $u \in \text{span} \{e_{n_0}\}$ , and this contradicts the fact that  $\langle u, e_n \rangle \neq 0$  for every  $n \in \mathbb{N}$ . Hence  $T$  is not a normal operator.

Now, let us prove the general case. Assume  $D$  is not unitary but  $\sigma(D)$  is contained in a circle with center  $\alpha$  and radius  $r > 0$ . Then the operator  $\tilde{D} = (1/r)(D - \alpha I)$  is a diagonal unitary operator satisfying  $\langle u, \tilde{D}u \rangle \neq 0$  (since  $\langle u, \tilde{D}u \rangle = 0$ , would imply  $\langle u, (D - \alpha I)u \rangle = 0$ , and therefore  $\langle u, Du \rangle = \bar{\alpha} \|u\|^2$ , which contradicts our



hypotheses). Hence, from the unitary case, we ensure the existence of  $\tilde{v} \in H$  such that

$$\tilde{T} = \tilde{D} + u \otimes \tilde{v}$$

is not normal but has a non-trivial reducing subspace  $M$ . Thus,

$$r\tilde{T} + \alpha I = D + r(u \otimes \tilde{v}) = D + u \otimes (r\tilde{v})$$

is a non-normal operator with  $M$  as a reducing subspace. Accordingly,  $v = r\tilde{v}$  completes the argument and the proof of theorem 3.1.  $\square$

As a consequence, the following corollary holds.

**COROLLARY 3.2.** *Let  $D \in \mathcal{L}(H)$  be a diagonal operator with respect to an orthonormal basis  $(e_n)_{n \geq 1}$  with uniform multiplicity one. Assume its spectrum  $\sigma(D)$  is contained in a circle with center  $\alpha$ , and let  $u$  and  $v$  vectors in  $H$  such that both  $\langle u, e_n \rangle$  and  $\langle v, e_n \rangle$  are not zero for every  $n \in \mathbb{N}$ . If  $T = D + u \otimes v \in \mathcal{L}(H)$  has non-trivial reducing subspaces, then  $u$  and  $v$  are linearly independent vectors.*

#### 4. Reducing subspaces for rank-one perturbations of diagonal operators: general case

In this section, we consider the existence of reducing subspaces for rank-one perturbations of diagonal operators  $D$  with uniform multiplicity one but not imposing restrictions to the spectrum of  $D$  as in the previous two sections. In particular, we focus on the existence of reducing subspaces  $M$  such that  $T|_M$  is normal as well as the existence of reducing subspaces for self-adjoint perturbations. We start by setting a result in the spirit of lemmas 2.3 and 2.4.

**PROPOSITION 4.1.** *Let  $T = D + u \otimes v \in \mathcal{L}(H)$  where  $D$  is a diagonal operator with respect to an orthonormal basis  $(e_n)_{n \geq 1}$  and  $u = \sum_n \alpha_n e_n$ ,  $v = \sum_n \beta_n e_n$  nonzero vectors in  $H$ . Assume  $D$  has uniform multiplicity one and for each  $n \in \mathbb{N}$  the coordinates  $\alpha_n$  and  $\beta_n$  are not simultaneously zero. Let  $M$  be a reducing subspace for  $T$ . If  $u, v \in M$  or  $u, v \in M^\perp$  then  $M$  is trivial.*

The key of the proof of proposition 4.1 relies on an argument of a theorem of Wermer [39], which we isolate and include for the sake of completeness (see also [32, theorem 1.25]). Recall that a normal operator  $T \in \mathcal{L}(H)$  is diagonalizable if the set of eigenvectors of  $T$  spans  $H$ .

**THEOREM 4.2 (Wermer, 1952).** *Let  $T \in \mathcal{L}(H)$  be a diagonalizable normal operator. Then, every non-zero reducing subspace of  $T$  is spanned by eigenvectors of  $T$ .*

*Proof.* First, we claim that every non-zero reducing subspace of  $T$  contains at least one eigenvector. In order to show the claim, let  $M$  be a non-zero reducing subspace. Since the eigenvectors of  $T$  span  $H$ , there exists an eigenvector  $x$  not orthogonal to  $M$ . Let  $Tx = \lambda x$  and write  $x = x_1 + x_2$  with respect to the orthogonal

decomposition  $H = M \oplus M^\perp$ . Hence, from

$$Tx = T(x_1 + x_2) = \lambda(x_1 + x_2),$$

it follows

$$Tx_1 - \lambda x_1 = x_2 - Tx_2$$

and since  $M$  is reducing,  $Tx_1 - \lambda x_1 \in M$  and  $x_2 - Tx_2 \in M^\perp$ . Thus,  $Tx_1 = \lambda x_1$  and  $M$  contains at least one eigenvector of  $T$  as claimed.

Now, let us prove the statement of the theorem. Take  $M$  a non-zero reducing subspace for  $T$  and denote by  $N$  the closed subspace of  $M$  generated by all the eigenvectors of  $T$  in  $M$ . Note that  $N$  is a non-zero reducing subspace because  $T$  is normal. We claim that  $N = M$ .

Indeed, arguing by contradiction, suppose that  $N \neq M$ . Then  $N' = M \cap N^\perp$  is a non-zero invariant subspace for  $T$  which is reducing (since it is the intersection of two reducing subspaces). Accordingly,  $N'$  contains at least one eigenvector  $z \neq 0$  of  $T$ . Thus  $z \in N^\perp$ . But, since  $z \in M$  and is an eigenvector of  $T$ ,  $z \in N$  by definition. But this implies that  $z = 0$ , which is a contradiction. Accordingly,  $N = M$  which yields the statement. □

We are now in position to prove proposition 4.1.

*Proof of proposition 4.1.* Let  $P_M : H \rightarrow M$  be the orthogonal projection onto  $M$ , which clearly commutes with  $T$  since  $M$  is reducing. The goal is to show that either  $P_M$  is the identity or the zero operator. Assume  $P_M$  is not the zero operator,  $Q_M = I - P_M$  and suppose first that  $u, v \in M$ . Hence,  $Q_M u = Q_M v = 0$ .

By lemma 2.2 we have  $Q_M D P_M = Q_M D^* P_M = 0$ . Then,  $D P_M = P_M D P_M$  and  $D^* P_M = P_M D^* P_M$ , so it follows that  $M$  is an invariant subspace for  $D$  and  $D^*$ , that is,  $M$  is a reducing subspace for  $D$ . Thus, according to Wermer’s theorem,  $M$  is spanned by a set of eigenvectors of  $D$  and, therefore, there exists  $\Lambda \subset \mathbb{N}$  such that

$$M = \overline{\text{span} \{e_n : n \in \Lambda\}}.$$

Note that, if  $n \in \Lambda$  we have  $Q_M e_n = 0$  and if  $n \in \mathbb{N} \setminus \Lambda$  then  $Q_M e_n = e_n$ .

Now, since  $u = \sum_n \alpha_n e_n$  and  $v = \sum_n \beta_n e_n$  and both  $Q_M u = Q_M v = 0$ , we deduce

$$\sum_{n \notin \Lambda} \alpha_n e_n = \sum_{n \notin \Lambda} \beta_n e_n = 0.$$

Hence,  $\alpha_n = \beta_n = 0$  for every  $n \in \mathbb{N} \setminus \Lambda$  which contradicts the hypothesis that  $\alpha_n$  and  $\beta_n$  are not simultaneously zero unless  $\Lambda = \mathbb{N}$  and,  $P_M$  the identity operator as we wished to prove.

The case  $u, v \in M^\perp$  is analogous. □

With proposition 4.1 at hands, it is possible to characterize the reducing subspaces  $M$  of  $T$  such that  $T|_M$  is normal.

**THEOREM 4.3.** *Let  $T = D + u \otimes v \in \mathcal{L}(H)$  where  $D$  is a diagonal operator with respect to an orthonormal basis  $(e_n)_{n \geq 1}$  and  $u = \sum_n \alpha_n e_n$ ,  $v = \sum_n \beta_n e_n$  nonzero vectors in  $H$ . Assume  $D$  has uniform multiplicity one and for each  $n \in \mathbb{N}$  the coordinates  $\alpha_n$  and  $\beta_n$  are not simultaneously zero. Then  $T$  has a non-trivial reducing subspace such that  $T|_M$  is normal if and only if one of the following conditions holds:*

- (a)  $T$  is normal.
- (b) There exists  $\alpha, \beta \in \mathbb{C}$  and  $x \in H$  such that

$$(D - \alpha I)x = \frac{\langle (D - \bar{\beta}I)x, x \rangle}{\langle u, x \rangle} u, \quad (D^* - \beta I)x = -\langle x, u \rangle v,$$

and  $M = \text{span} \{x\}$ .

Before proceeding with the proof, let us point out that it is possible to exhibit normal operators such that their restrictions to invariant subspaces are not normal (see [39], for instance). On the other hand, there exist normal operators such that their restrictions to every invariant subspace is normal. This property is equivalent to be a completely normal operator. For instance, every diagonal operator whose spectrum is contained in a Jordan curve is completely normal. As a consequence, every Hermitian operator is completely normal.

The proof of theorem 4.3 depends on the linear independence of  $\{u, v, D^*u, Dv\}$ , which we examine in the next lemma.

**LEMMA 4.4.** *Let  $T = D + u \otimes v \in \mathcal{L}(H)$  where  $D$  is a diagonal operator with respect to an orthonormal basis  $(e_n)_{n \geq 1}$  and  $u = \sum_n \alpha_n e_n$ ,  $v = \sum_n \beta_n e_n$  nonzero vectors in  $H$ . Assume  $D$  has uniform multiplicity one and for each  $n \in \mathbb{N}$  the coordinates  $\alpha_n$  and  $\beta_n$  are not simultaneously zero. Let  $M$  be a reducing subspace for  $T$  such that  $T|_M$  is normal. Then:*

- (i) If  $\{u, v, D^*u, Dv\}$  are linearly independent, then  $M = \{0\}$ .
- (ii) If  $u$  is an eigenvector for  $D^*$  or  $v$  is for  $D$ , then  $M = \{0\}$ .
- (iii) If  $M \neq \{0\}$  and  $u = \alpha v$  for some  $\alpha \in \mathbb{C}$ , then the spectrum of  $D$  is contained in a line and  $T$  is normal.
- (iv) If  $M \neq \{0\}$  and  $(D - \alpha I)v = \lambda u$  for some  $\alpha, \lambda \in \mathbb{C}$  then the spectrum of  $D$  is contained in a circle centred in  $\alpha$  and  $T$  is normal. The same conclusion follows if it is assumed  $M \neq \{0\}$  and  $(D^* - \beta I)u = \lambda v$  for some  $\beta, \lambda \in \mathbb{C}$ .
- (v) If  $Dv = \alpha u + \beta v + \mu D^*u$  for some scalars  $\alpha, \beta$ , and  $\mu \in \mathbb{C}$  and  $T$  is not normal, then  $\dim M \leq 1$ .

Note that the statements (i)–(v) in lemma 4.4 cover all the possible situations regarding the linear dependence of  $\{u, v, D^*u, Dv\}$ . Clearly, the linear independence case is covered by (i). So, suppose  $\{u, v, D^*u, Dv\}$  is a set of linearly dependent vectors. In such a case, (ii) covers the cases where  $u$  and  $D^*u$  or  $v$

and  $Dv$  are proportional. Statement (iii) deals with the situation in which  $u$  and  $v$  are proportional, while (iv) considers the cases when  $\{u, v, Dv\}$  or  $\{u, v, D^*u\}$  are sets of linearly dependent vectors. Finally, statement (v) deals with the general linear dependence case assuming, in addition, that  $T$  is not normal. Note that this assumption can be made since, otherwise,  $T$  would be normal and Ionascu's theorem would lead us to either statement (iii) or (iv) aforementioned.

In what follows, given  $T$  and  $S$  bounded linear operators in  $H$ , the commutator of  $T$  and  $S$  is the operator in  $\mathcal{L}(H)$  defined by

$$[T, S] := TS - ST.$$

Clearly,  $[T, S] = 0$  if and only if  $T$  and  $S$  commutes.

In the proof of lemma 4.4, the following straightforward result will be repeatedly invoked.

LEMMA 4.5. *Let  $A = \sum_{k=1}^n x_k \otimes y_k \in \mathcal{L}(H)$ , where  $x_k, y_k \in H$  for each  $1 \leq k \leq n$ . Suppose that  $x_1, \dots, x_n$  are linearly independent vectors. If  $Ax = 0$  for  $x \in H$  then  $\langle x, y_k \rangle = 0$  for every  $1 \leq k \leq n$ .*

*Proof of lemma 4.4.* Let us begin by proving statement (i). Since  $\{u, v, D^*u, Dv\}$  are linearly independent, Ionascu theorem implies that  $T$  is not normal (observe that statement (i) in Ionascu theorem asks  $u$  and  $v$  be linearly dependent vectors while (ii) asks  $Dv$  and  $u$  be linearly dependent). Since  $T|_M$  is normal, we deduce that  $M \subsetneq H$ .

On the other hand, a straightforward computation shows

$$[T, T^*] = Dv \otimes u + u \otimes (Dv + \|v\|^2 u) - D^*u \otimes v - v \otimes (D^*u + \|u\|^2 v). \tag{4.1}$$

By hypotheses,  $T|_M$  is normal so  $[T, T^*]x = 0$  for every  $x \in M$ . Since  $u, v, D^*u, Dv$  are linearly independent, lemma 4.5 yields that

$$\langle x, u \rangle = \langle x, v \rangle = 0$$

for every  $x \in M$ . Hence,  $u, v \in M^\perp$  and therefore, by proposition 4.1,  $M$  is trivial. Now,  $T$  is not normal and accordingly  $M = \{0\}$ .

To prove (ii), assume  $u$  is an eigenvector for  $D^*$  (a similar reasoning applies if  $v$  is an eigenvector for  $D$ ). Since  $D$  has uniform multiplicity one, there exists  $n_0 \in \mathbb{N}$  such that  $u \in \text{span} \{e_{n_0}\}$ . Without loss of generality, we may assume  $u = e_{n_0}$ . Then equation (4.1) becomes

$$\begin{aligned} [T, T^*] &= Dv \otimes e_{n_0} + e_{n_0} \otimes (Dv + \|v\|^2 e_{n_0}) - \overline{\lambda_{n_0}} e_{n_0} \otimes v - v \otimes (\overline{\lambda_{n_0}} e_{n_0} + v) \\ &= (D - \lambda_{n_0} I)v \otimes e_{n_0} + e_{n_0} \otimes ((D - \lambda_{n_0} I)v + \|v\|^2 e_{n_0}) - v \otimes v. \end{aligned}$$

Now,  $v = \sum_k \beta_k e_k$  and  $\beta_k \neq 0$  for every  $k \neq n_0$  and each eigenvalue  $\lambda_k$  of  $D$  is of multiplicity one. Hence,  $e_{n_0}$  is linearly independent of  $(D - \lambda_{n_0} I)v$  and  $v$ . Moreover, assume there exists  $\beta \in \mathbb{C} \setminus \{0\}$  such that

$$(D - \lambda_{n_0} I)v = \beta v.$$

Then  $(\lambda_k - \lambda_{n_0}) = \beta$  for every  $k \neq n_0$  which is a contradiction. Accordingly,  $(D - \lambda_{n_0} I)v, e_{n_0}$  and  $v$  are linearly independent vectors, again lemma 4.5 yields that

$v, e_{n_0} \in M^\perp$  and  $M \neq H$ . Thus, proposition 4.1 yields that  $M = \{0\}$ , as stated in (ii).

Let us prove now statement (iii). Assume  $u = \alpha v$  for some  $\alpha \in \mathbb{C} \setminus \{0\}$  (observe that  $\alpha \neq 0$  since  $u$  is a nonzero vector).

Assume  $M$  is a non-zero reducing subspace. We may assume  $M \neq H$  since on the contrary, the result is just a consequence of Ionascu theorem condition (i'). We are required to show that  $\Lambda(D)$  is contained in a line and that  $T$  is a normal operator.

First, observe that if  $v \in M^\perp$  then  $u \in M^\perp$  and  $M = \{0\}$  by proposition 4.1. Assume, therefore, that  $v \notin M^\perp$ .

Equation (4.1) turns out to be in this case

$$[T, T^*] = (\bar{\alpha}D - \alpha D^*)v \otimes v + v \otimes (\bar{\alpha}D - \alpha D^*)v. \tag{4.2}$$

Now, we claim that the fact that  $v \notin M^\perp$  implies  $(\bar{\alpha}D - \alpha D^*)v = \beta v$  for some  $\beta \in \mathbb{C}$ . Indeed, if  $x \in M$  such that  $\langle x, v \rangle \neq 0$ , (4.2) yields

$$0 = \langle x, v \rangle (\bar{\alpha}D - \alpha D^*)v + \langle x, (\bar{\alpha}D - \alpha D^*)v \rangle v,$$

from where the claims follows having  $\beta = -\langle x, (\bar{\alpha}D - \alpha D^*)v \rangle / \langle x, v \rangle$ .

So, equation (4.2) becomes

$$[T, T^*] = 2\text{Re}(\beta)v \otimes v.$$

But, once again, since  $T|_M$  is normal and  $v \notin M^\perp$ , it follows that  $\text{Re}(\beta) = 0$ . Hence,  $\beta = it$  with  $t \in \mathbb{R}$ .

Finally, observe that  $(\bar{\alpha}D - \alpha D^*)v = \beta v = itv$  implies

$$\text{Im}(\alpha \bar{\lambda}_n) = t,$$

so  $\Lambda(D)$  lies in the line  $\{z \in \mathbb{C} : \text{Im}(\alpha \bar{z}) = t\}$  and by Ionascu theorem condition (i') it follows that  $T$  is normal.

This proves statement (iii).

In order to show condition (iv), assume  $Dv - \alpha v = \lambda u$  for some  $\alpha, \lambda \in \mathbb{C}$ . If  $\lambda = 0$ , case (ii) yields that  $M = \{0\}$ , which is a contradiction. So, we may assume  $\lambda \neq 0$ . Clearly, without loss of generality, we can also assume  $\alpha = 0$  since  $D - \alpha I$  is a diagonal operator of uniform multiplicity one. Thus,  $Dv = \lambda u$  for  $\lambda \neq 0$ .

In addition, observe that if  $u$  and  $v$  are linearly dependent then  $Dv$  and  $v$  are linearly dependent, and  $M = \{0\}$  by (ii), which is a contradiction. So we may also suppose that  $u$  and  $v$  are linearly independent vectors.

So, assume  $M$  is a non-zero reducing subspace,  $Dv = \lambda u$  for  $\lambda \neq 0$  and  $\{u, v\}$  are linearly independent. As in the proof of condition (iii), we may assume  $M \neq H$  and the goal is to show that  $\Lambda(D)$  is contained in a circle and that  $T$  is normal.

Now, equation (4.1) becomes

$$\begin{aligned} [T, T^*] &= u \otimes (2\text{Re}(\lambda) + \|v\|^2)u - (D^* - \bar{\alpha}I)u \otimes v - v \otimes ((D^* - \bar{\alpha}I)u + \|u\|^2 v) \\ &= u \otimes (2\text{Re}(\lambda) + \|v\|^2)u - D^*u \otimes v - v \otimes (D^*u + \|u\|^2 v) \end{aligned} \tag{4.3}$$

First, let us assume that  $v \in M^\perp$ . Upon applying (4.3) to any vector  $x \in M$ , we deduce

$$0 = (2\text{Re}(\lambda) + \|v\|^2)\langle x, u \rangle u - \langle x, D^*u \rangle v,$$

which along with the linear independence of  $\{u, v\}$  yields that  $(2\operatorname{Re}(\lambda) + \|v\|^2) = 0$  and  $D^*u \in M^\perp$ . In addition, since  $M$  is reducing for  $T$  it follows that

$$Tv = Dv + \|v\|^2 u = \lambda u + \|v\|^2 u \in M^\perp.$$

Having in mind that  $\operatorname{Re}(\lambda) = -\|v\|^2/2$ , we deduce that  $Tv \neq 0$ . Thus,  $u \in M^\perp$ . At this point, we may argue either using proposition 4.1 which would yield that  $M = \{0\}$  since  $v \in M^\perp$  or as follows. First,  $M$  is a reducing subspace for  $D$  since  $M$  is reducing for  $T$  and  $u, v \in M^\perp$ . Then, by means of Wermer theorem, there exists  $n_0 \in \mathbb{N}$  such that  $e_{n_0} \in M$ . In particular, this implies that  $\langle u, e_{n_0} \rangle = \langle v, e_{n_0} \rangle = 0$ , or equivalent,  $\alpha_{n_0} = \beta_{n_0} = 0$ , what contradicts the hypotheses of the lemma. This latter argument will be of use in a remark after finishing the proof of lemma 4.4.

Now, let us assume  $v \notin M^\perp$ . Once again, upon applying (4.3) to any vector  $x \in M$  with  $\langle x, v \rangle \neq 0$ , it follows that

$$D^*u = \mu u + \beta v, \tag{4.4}$$

where  $\mu, \beta \in \mathbb{C}$ . It can be assumed that  $\beta \neq 0$ . We claim that  $\alpha_n \neq 0$  for every  $n \in \mathbb{N}$ . Indeed, from  $D^*u = \mu u + \beta v$ , it follows that for every  $n \in \mathbb{N}$

$$\overline{\lambda_n} \alpha_n = \mu \alpha_n + \beta \cdot \beta_n.$$

So, if  $\alpha_{n_0} = 0$  for some  $n_0 \in \mathbb{N}$ , it would imply that  $\beta_{n_0} = 0$  which contradicts the hypotheses.

We will show that  $\mu = 0$  in (4.4) since otherwise we are led to a contradiction.

Assume, on the contrary, that  $\mu \neq 0$  in (4.4). Applying  $D$  to (4.4) we obtain

$$DD^*u = \mu Du + \beta Dv = \mu Du + \beta \lambda u.$$

Then, the eigenvalues  $\lambda_n$  of  $D$  satisfy

$$|\lambda_n|^2 = \mu \lambda_n + \beta \lambda,$$

for every  $n \in \mathbb{N}$ , since  $\alpha_n \neq 0$  for every  $n \in \mathbb{N}$ . Dividing out by  $-\beta \lambda$  (which is different from 0), we deduce that there exist complex numbers  $a, b \in \mathbb{C} \setminus \{0\}$  such that every  $\lambda_n$  with  $n \neq n_0$  lies in

$$A = \{z \in \mathbb{C} : a|z|^2 + bz = -1\}.$$

The equation  $a|z|^2 + bz = -1, z \in \mathbb{C}$  is equivalent to the system

$$A = \begin{cases} \operatorname{Re}(a)(x^2 + y^2) + \operatorname{Re}(b)x - \operatorname{Im}(b)y = -1, \\ \operatorname{Im}(a)(x^2 + y^2) + \operatorname{Re}(b)y + \operatorname{Im}(b)x = 0, \end{cases}$$

where  $x = \operatorname{Re}(z), y = \operatorname{Im}(z) \in \mathbb{R}$ . Observe that  $A$  is therefore the intersection of two different conics, and hence  $A$  is a finite set of points, what contradicts the uniform multiplicity one of  $D$ .

Hence,  $\mu = 0$  as claimed and (4.4) becomes

$$D^*u = \beta v.$$

Note that  $u$  is linearly independent of  $D^*u$  because  $\{u, v\}$  are linearly independent. Having in mind this and applying (4.3) to any vector  $x \in M$  once more time we deduce that

$$(2\operatorname{Re}(\lambda) + \|v\|^2) = 0.$$

That is,  $\operatorname{Re}(\lambda) = -\|v\|^2/2$ . Then, equation (4.3) turns out to be

$$[T, T^*] = \beta v \otimes v + v \otimes (\beta + \|u\|^2)v = (2\operatorname{Re}(\beta) + \|u\|^2)v \otimes v.$$

On the other hand, having in mind that  $Dv = \lambda u$ , we deduce

$$DD^*u = D(\beta v) = \beta \lambda u.$$

Now, the fact that  $\alpha_n \neq 0$  for every  $n \in \mathbb{N}$  along with the previous equality implies that

$$DD^* = \beta \lambda I.$$

Note that  $\beta \lambda$  is a positive number. Accordingly, the spectrum of  $D$  is contained in a circle of center 0 and radius  $r = \sqrt{\beta \lambda} > 0$ , which is half of the statement we wished to prove. Let us see also that  $T$  is normal.

Now, an easy computation involving coordinates in the expression  $Dv = \lambda u$  leads to

$$r \|v\| = |\lambda| \|u\|,$$

and therefore

$$|\lambda| = \frac{r \|v\|}{\|u\|}. \tag{4.5}$$

Since  $Dv = \lambda u = |\lambda|e^{i\theta}u$  for  $\theta \in [0, 2\pi)$ , it follows

$$|\lambda|u = e^{-i\theta}Dv,$$

and by (4.5)

$$\frac{ru}{\|u\|} = e^{-i\theta} \frac{Dv}{\|v\|}.$$

Moreover, from  $\operatorname{Re}(\lambda) = -\|v\|^2/2$  one deduces  $\operatorname{Re}(re^{-i\theta}/(\|u\|\|v\|)) = -(1/2)$ , which implies that  $T$  is normal by Ionascu theorem condition (ii'). This completes the proof of the statement (iv).

Finally, let us prove the statement (v). Assume  $Dv = \alpha u + \beta v + \mu D^*u$  for some  $\alpha, \beta, \mu \in \mathbb{C}$ , and  $T$  is not normal. Since we can express

$$(D - \beta I)v = (\mathbf{a} - \mu\bar{\beta})u + \mu(D^* - \bar{\beta}I)u$$

where  $\mathbf{a} = \alpha + 2\mu\bar{\beta}$ , we can assume with no loss of generality that  $\beta = 0$ .

Observe that, in this case, equation (4.1) becomes

$$\begin{aligned}
 [T, T^*] &= u \otimes ((2\operatorname{Re}(\alpha) + \|v\|^2)u + \mu D^*u) \\
 &\quad - D^*u \otimes (\bar{\mu}u - v) - v \otimes (D^*u + \|u\|^2 v).
 \end{aligned}$$

Moreover, we can assume that  $D^*u$ ,  $u$  and  $v$  are linearly independent and  $\mu \neq 0$ , otherwise cases (iii) and (iv) would yield that  $T$  is normal, a contradiction.

By hypotheses,  $[T, T^*]x = 0$  for every  $x \in M$ , so lemma 4.5 yields that  $(2\operatorname{Re}(\alpha) + \|v\|^2)u + \mu D^*u$ ,  $\bar{\mu}u - v$  and  $D^*u + \|u\|^2 v$  belongs to  $M^\perp$ .

We will argue by contradiction. Assume, on the contrary, that the reducing subspace  $M$  has dimension strictly larger than 1. Then there exists  $x \in M \setminus \{0\}$  such that  $\langle x, u \rangle = 0$ . Since  $\langle x, \bar{\mu}u - v \rangle = 0$ , we have  $\langle x, v \rangle = 0$ . Moreover, since  $x$  is also orthogonal to  $D^*u + \|u\|^2 v$ , we have  $\langle x, D^*u \rangle = 0$ . The fact that  $M$  is invariant under  $T$  implies, in particular,

$$Tx = Dx + \langle x, v \rangle u = Dx \in M.$$

So, for every  $x \in M \cap (\operatorname{span} \{u\})^\perp$ ,  $Dx \in M$  and  $\langle Dx, u \rangle = \langle x, D^*u \rangle = 0$ . Hence, the closed subspace  $M \cap (\operatorname{span} \{u\})^\perp$  is invariant under  $D$ .

On the other hand,  $M$  is also invariant under  $T^*$  because it is reducing and, therefore,

$$T^*x = D^*x + \langle x, u \rangle v = D^*x \in M.$$

Moreover, since  $Dv = \alpha u + \mu D^*u$  we deduce  $\langle D^*x, v \rangle = \langle x, Dv \rangle = 0$ . In addition, from  $D^*x \in M$ , it follows

$$\langle D^*x, \bar{\mu}u - v \rangle = 0,$$

and therefore,  $\langle D^*x, u \rangle = 0$ . Accordingly,  $M \cap (\operatorname{span} \{u\})^\perp$  is a reducing subspace for  $D^*$ .

Now, since every reducing subspace of  $D$  contains an eigenvector because of Werners theorem, it follows that there exists  $n_0 \in \mathbb{N}$  such that  $e_{n_0} \in M \cap (\operatorname{span} \{u\})^\perp$ .

Now, recalling that  $\bar{\mu}u - v \in M^\perp$ , we deduce  $\langle e_{n_0}, v \rangle = 0$ . Thereby,  $e_{n_0}$  is orthogonal to  $u$  and  $v$ , that is,  $\alpha_{n_0} = \beta_{n_0} = 0$ , which contradicts the hypotheses of the lemma. Hence, the dimension of  $M$  is equal or less than one, as we wished to prove. This concludes the proof of the statement (v) and hence that of lemma 4.4.  $\square$

REMARK 4.6. A closer look at the proof of statement (iv), specifically at the argument regarding the intersection of the two different conics, allows us to remark that the hypothesis of the uniform multiplicity one for  $D$  might be relaxed. Indeed, observe that as far as  $u$  and  $v$  are linearly independent, a contradiction would follow if we would just assume that the spectrum of  $D$  contains five or more different eigenvalues, since an intersection of two different conics in the plane consists, at most, of four different points. Moreover, the assumption regarding the multiplicity of  $D$  is only used in this argument in the proof of (iv), so our approach allows restating statement (iv) just assuming that  $D$  has five or more different eigenvalues.

Now, we can prove theorem 4.3.



*Proof of theorem 4.3.* If  $T$  is normal, the spectral theorem for normal operators provides plenty of reducing subspaces for  $T$ . In addition, if condition (b) is satisfied, a simple computation shows that  $M$  is reducing for  $T$ , and since  $M$  has dimension 1,  $T|_M$  is normal.

Assume now there exists a non-trivial reducing subspace  $M$  for  $T$  such that  $T|_M$  is normal. By lemma 4.4 we deduce that either  $T$  is normal or  $\dim M = 1$ . Assume  $T$  is not normal. Hence,  $\dim M = 1$  and therefore, there exists a non-zero vector  $x \in M$  and  $\alpha, \beta \in \mathbb{C}$  such that  $Tx = \alpha x$  and  $T^*x = \beta x$ . From  $Tx = \alpha x$  we have  $(D - \alpha I)x + \langle x, v \rangle u = 0$ , so

$$(D - \alpha I)x = -\langle x, v \rangle u.$$

From  $(T^* - \beta I)x = 0$  we have  $(D^* - \beta I)x + \langle x, u \rangle v = 0$ . Having into account that  $u \notin M^\perp$ , we have  $\langle x, u \rangle \neq 0$ , and therefore

$$v = -\frac{1}{\langle x, u \rangle} (D^* - \beta I)x.$$

Hence,

$$(D - \alpha I)x = \frac{1}{\langle u, x \rangle} \langle x, (D - \beta I)x \rangle u,$$

which yields the result. □

Next result generalizes condition (a) in theorem 3.1 whenever rank-one perturbations of diagonal operators with arbitrary spectrum are considered.

**THEOREM 4.7.** *Let  $D \in \mathcal{L}(H)$  be a diagonal operator with respect to an orthonormal basis  $(e_n)_{n \geq 1}$  of uniform multiplicity one. Let  $u$  be a nonzero vector in  $H$  such that  $\langle u, e_n \rangle \neq 0$  for every  $n \in \mathbb{N}$  and  $T = D + u \otimes u$ . Then,  $T$  has a non-trivial reducing subspace if and only if  $T$  is normal.*

Clearly, as a consequence of theorem 4.7, it is possible to exhibit rank-one perturbations of completely normal diagonal operators lacking reducing subspaces.

*Proof.* It suffices to prove that if  $T$  has a non-trivial reducing subspace, then  $T$  is normal since the converse is straightforward. In addition, we may assume that  $D$  is not self-adjoint, since otherwise  $T$  would be self-adjoint and the result holds trivially.

Thus, let  $M$  be a non-trivial reducing subspace and suppose that  $D$  is not self-adjoint.

First, let us assume  $T|_M$  is normal. By theorem 4.3 we have that either  $T$  is normal or  $\dim M = 1$ . Let us suppose that  $\dim M = 1$  since otherwise we would be done.

Then there exist  $\alpha, \beta \in \mathbb{C}$  such that  $Dx + \langle x, u \rangle u = \alpha x$  and  $D^*x + \langle x, u \rangle u = \beta x$ . Hence

$$x_n := \langle x, e_n \rangle \neq 0$$

for every  $n \in \mathbb{N}$ . Now,

$$(D - D^*)x = (\alpha - \beta)x,$$

from where  $\text{Im}(\lambda_n)$  is constant for every  $n \in \mathbb{N}$ . Thus, the spectrum of  $D$  is contained in a parallel line to the real axis and by Ionascu’s theorem (i’), it follows  $T$  is normal.

If  $T|_{M^\perp}$  is normal, we can argue equivalently to show that  $T$  is normal.

Now, assume that neither  $T|_M$  nor  $T|_{M^\perp}$  are normal operators. If  $u$  and  $(D - D^*)u$  are linearly dependent, then  $u$  is an eigenvector for  $(D - D^*)$ , associated with an eigenvalue  $\lambda \in \mathbb{C}$ . Since  $\langle u, e_n \rangle \neq 0$  for every  $n \in \mathbb{N}$ ,  $\lambda_n - \overline{\lambda_n} = \lambda$  for every  $n \in \mathbb{N}$ , and hence  $D - D^* = \lambda I$  and  $T$  is normal. So, we may assume that  $u$  and  $(D - D^*)u$  are linearly independent.

By assumption  $T|_M$  and  $T|_{M^\perp}$  are not normal, so there exist  $x_0 \in M$  and  $y_0 \in M^\perp$  such that

$$0 \neq [T, T^*]x_0 \in M \text{ and } 0 \neq [T, T^*]y_0 \in M^\perp. \tag{4.6}$$

Note that in this case

$$[T, T^*] = (D - D^*)u \otimes u + u \otimes (D - D^*)u \tag{4.7}$$

and  $u \notin M \cup M^\perp$  because of proposition 4.1.

If  $\langle x_0, (D - D^*)u \rangle = \langle y_0, (D - D^*)u \rangle = 0$ , it follows that  $(D - D^*)u \in M \cap M^\perp$ , so  $(D - D^*)u = 0$  and this occurs if and only if  $D = D^*$  since  $\langle u, e_n \rangle \neq 0$  for every  $n \in \mathbb{N}$ . Hence, we can assume  $\langle x_0, (D - D^*)u \rangle \neq 0$ .

In addition,  $u \notin M$  so, by means of (4.6) and (4.7),  $\langle x_0, u \rangle \neq 0$ . By considering  $P_M$  the orthogonal projection onto  $M$  and  $Q_M = I - P_M$ , it follows that

$$\langle x_0, (D - D^*)u \rangle Q_M u = -\langle x_0, u \rangle Q_M (D - D^*)u.$$

Moreover, since  $Q_M$  commutes with  $T$  and  $T^*$  we have

$$DQ_M = Q_M D + Q_M u \otimes u - u \otimes Q_M u$$

and

$$D^*Q_M = Q_M D^* + Q_M u \otimes u - u \otimes Q_M u,$$

so  $Q_M(D - D^*) = (D - D^*)Q_M$ . Hence,  $P_M$  and  $Q_M$  belongs to the commutant  $\{D - D^*\}'$ . Moreover,

$$Q_M u = -\frac{\langle x_0, u \rangle}{\langle x_0, (D - D^*)u \rangle} (D - D^*)Q_M u,$$

so  $Q_M u$  is an eigenvector for  $(D - D^*)$ .

Now, let  $y_0 \in M^\perp$  considered in (4.6), that is,  $0 \neq [T, T^*]y_0 \in M^\perp$ . If  $\langle y_0, (D - D^*)u \rangle = 0$  then  $(D - D^*)u \in M^\perp$ , so

$$(D - D^*)P_M u = P_M(D - D^*)u = 0,$$

and therefore  $P_M u$  is an eigenvector associated with the eigenvalue 0. If  $\langle y_0, (D - D^*)u \rangle \neq 0$ , we can argue similarly to deduce that  $P_M u$  is also a eigenvector for  $(D - D^*)$ .

Now, recall that the eigenvalues of  $(D - D^*)$  are  $(2i\text{Im}(\lambda_n))_n$  and for each  $n \in \mathbb{N}$ , the space of eigenvectors associated with  $2i\text{Im}(\lambda_n)$  is given by

$$M_n = \overline{\text{span} \{e_k : \text{Im}(\lambda_k) = \text{Im}(\lambda_n)\}}.$$

Note that the eigenspaces  $M_n$  and  $M_m$  are orthogonal if  $n \neq m$ . Moreover, since  $P_M u + Q_M u = u$  and  $M_n \neq H$  for every  $n \in \mathbb{N}$  ( $D$  is not self-adjoint), it follows that both  $P_M u$  and  $Q_M u$  are eigenvectors associated with different eigenvalues.

Moreover, there exists a partition of the positive integers  $\mathbb{N}$ , that is, non-empty sets  $N_1, N_2 \subset \mathbb{N}$  such that  $N_1 \cup N_2 = \mathbb{N}$  and  $N_1 \cap N_2 = \emptyset$ , such that

$$P_M u = \sum_{n \in N_1} \alpha_n e_n \quad Q_M u = \sum_{n \in N_2} \alpha_n e_n.$$

Now,

$$TP_M u = DP_M u + \|P_M u\|^2 u \in M$$

since  $M$  is invariant under  $T$ . Since  $Q_M u \in M^\perp$ , we have  $\langle TP_M u, Q_M u \rangle = 0$ . Now, since  $DP_M u = \sum_{n \in N_1} \lambda_n \alpha_n e_n$  it follows that  $\langle DP_M u, Q_M u \rangle = 0$ , so we deduce that  $\langle \|P_M u\|^2 u, Q_M u \rangle = 0$ , but

$$\langle \|P_M u\|^2 u, Q_M u \rangle = \|P_M u\|^2 \langle u, Q_M u \rangle = \|P_M u\|^2 \|Q_M u\|^2,$$

so  $P_M u = 0$  or  $Q_M u = 0$ , what contradicts lemma 2.3. □

As a byproduct of the previous results, we state the following corollary.

**COROLLARY 4.8.** *Let  $D$  a diagonal operator with respect to an orthonormal basis  $(e_n)_{n \geq 1}$  of uniform multiplicity one and  $u \in H$  a non-zero vector such that  $\langle u, e_n \rangle \neq 0$  for every  $n \in \mathbb{N}$ . Then*

- (i)  $T = D + u \otimes u$  has a non-trivial reducing subspace if and only if  $T$  is normal. In particular, if the spectrum of  $D$  is not contained in a line parallel to the real axis,  $T$  has no non-trivial reducing subspaces.
- (ii) Moreover, if there exists  $\alpha, \beta \in \mathbb{C}$  and  $x \in H$  such that

$$(D - \alpha I)x = \frac{\langle (D - \bar{\beta} I)x, x \rangle}{\langle u, x \rangle} u$$

and  $\langle x, u \rangle \neq 0$ , then  $T = D + u \otimes v$  has a non-trivial reducing subspace, where  $v = (D^* - \beta I)x$ .

**4.1. A remark on essentially normal operators: Behncke theorem**

In this subsection, we recall Behncke’s theorem concerning the algebraic structure of essentially normal operators and remark it in the context of the results proved.

Recall that an operator  $T$  is called *essentially normal* if  $[T, T^*]$  is compact. Behncke’s theorem [8] generalizes a previous result of [38], where the case of  $T - T^*$  being compact is addressed (we refer to [11, p. 159, theorem 5.4] and [25, chapter 8] for more on the subject).

**THEOREM 4.9** (Behncke). *Let  $T \in \mathcal{L}(H)$  be an essential normal operator. Then  $H$  admits an orthogonal decomposition*

$$H = H_0 \oplus H_1 \oplus H_2 \oplus \dots \tag{4.8}$$

where

- (1) each  $H_n$  is a reducing subspace for  $T$ ;
- (2)  $T_0 = T|_{H_0}$  is a maximal normal operator, that is, there is no closed subspace  $K_0 \supsetneq H_0$  such that  $K_0$  is reducing for  $T$  and  $T|_{K_0}$  is normal.
- (3) For  $n \geq 1$  each  $T_n = T|_{H_n}$  has no non-trivial reducing subspaces and it is essentially normal.

Moreover, the decomposition is unique in the sense that if  $T_i$  and  $H_i$  with  $i \geq 0$  are replaced with  $\tilde{T}_i$  and  $\tilde{H}_i$  satisfying (1)–(3), and both  $T_0$  and  $\tilde{T}_0$  are maximal normal operators, then after reordering  $\tilde{H}_i$  for  $i \geq 1$ , there is a unitary operator  $U$  commuting with  $T$  such that

$$U^*P_{H_i}U = P_{\tilde{H}_i} \text{ and } U^*TU|_{H_i} = \tilde{T}_i \quad (i \geq 0).$$

Observe that, if  $T = D + u \otimes v \in \mathcal{L}(H)$  is a non-normal rank-one perturbation of a diagonal operator where  $D$ , as usual, is a diagonal operator with respect to an orthonormal basis  $(e_n)_{n \geq 1}$  with uniform multiplicity one and  $u = \sum_n \alpha_n e_n$ ,  $v = \sum_n \beta_n e_n$  are nonzero vectors in  $H$  with non-simultaneously zero coordinates for each  $n \in \mathbb{N}$ , we deduce on one hand as a byproduct of lemma 4.4 and Behncke theorem that

- (a) If  $\{u, v, D^*u, Dv\}$  are linearly independent, or  $u$  is an eigenvector for  $D^*$  or  $v$  is an eigenvector for  $D$ , the Hilbert space  $H$  admits an orthogonal decomposition

$$H = H_1 \oplus H_2 \oplus \dots$$

where for every  $n \geq 1$ ,  $H_n$  is a reducing subspace for  $T$  and  $T_n = T|_{H_n}$  is a non-normal essentially normal operator with no non-trivial reducing subspaces.

- (b) The same conclusion as in (a) follows if  $u = \alpha v$ , or  $(D - \alpha I)v = \lambda u$  or  $(D^* - \beta I)u = \lambda v$  for some scalars  $\alpha, \lambda, \beta \in \mathbb{C}$  since  $T$  is not normal.

(c) If  $Dv = \alpha u + \beta v + \mu D^*u$  for some  $\alpha, \beta, \mu \in \mathbb{C}$ , then the Hilbert space  $H$  admits an orthogonal decomposition

$$H = H_0 \oplus H_1 \oplus H_2 \oplus \dots$$

where for every  $n \geq 0$ ,  $H_n$  is a reducing subspace for  $T$ ,  $H_0$  is at most a one-dimensional Hilbert space and for each  $n \geq 1$ , the operators  $T_n = T|_{H_n}$  are non-normal essentially normal operator with no non-trivial reducing subspaces.

In addition, by means of corollary 4.3 and Behncke theorem, the following consequence holds:

**COROLLARY 4.10.** *Let  $T = D + u \otimes v \in \mathcal{L}(H)$  be a non-normal rank-one perturbation of a diagonal operator  $D$  with respect to an orthonormal basis  $(e_n)_{n \geq 1}$ , where  $u = \sum_n \alpha_n e_n, v = \sum_n \beta_n e_n$  are nonzero vectors in  $H$  with non-zero simultaneously coordinates  $\alpha_n$  and  $\beta_n$  for each  $n \in \mathbb{N}$ . Then  $H$  admits an orthogonal decomposition*

$$H = H_0 \oplus H_1 \oplus H_2 \oplus \dots$$

*satisfying Behncke’s theorem conditions (1)–(3) with  $H_0$  non-trivial if and only if  $H_0$  is one-dimensional. Moreover, if  $H_0 = \text{span}\{x\}$  for  $x \in H \setminus \{0\}$ , there exists  $\alpha, \beta \in \mathbb{C}$  such that*

$$(D - \alpha I)x = \frac{\langle (D - \bar{\beta}I)x, x \rangle}{\langle u, x \rangle} u, \quad \text{and} \quad (D^* - \beta I)x = -\langle x, u \rangle v.$$

On the other hand, in the context of Behncke theorem, one may exhibit easy examples of rank-one perturbation of diagonal operators of multiplicity one such that the orthogonal decomposition of  $H$  in (4.8) is trivial, namely,  $H_0 = \{0\}, H_1 = H$  and  $H_n = \{0\}$  for every  $n \geq 2$ . Actually, theorem 2.1 or theorem 4.7 in this context provide many of such examples. We state them as corollaries:

**COROLLARY 4.11.** *Let  $T = D + u \otimes v \in \mathcal{L}(H)$  where  $D$  is a diagonal operator with respect to an orthonormal basis  $(e_n)_{n \geq 1}$  and  $u, v$  are nonzero vectors in  $H$  satisfying  $\langle u, e_n \rangle \neq 0$  and  $\langle v, e_n \rangle \neq 0$ . Assume  $D$  has uniform multiplicity one and its spectrum  $\sigma(D)$  is contained in a line. Then, the orthogonal decomposition of  $H$  in (4.8) satisfying Behncke’s theorem conditions (1)–(3) is trivial if and only if  $T$  is not normal.*

**COROLLARY 4.12.** *Let  $D \in \mathcal{L}(H)$  be a diagonal operator with respect to an orthonormal basis  $(e_n)_{n \geq 1}$  of uniform multiplicity one. Let  $u$  be a nonzero vector in  $H$  such that  $\langle u, e_n \rangle \neq 0$  for every  $n \in \mathbb{N}$  and  $T = D + u \otimes u$ . Then, the orthogonal decomposition of  $H$  in (4.8) satisfying Behncke’s theorem conditions (1)–(3) is trivial if and only if  $T$  is not normal.*

## 4.2. Rank-one perturbations of normal operators

Some of the previous results can be addressed in the context of rank-one perturbations of normal operators of multiplicity one and some consequences can be derived. The first result deals with the class of unitary operators.

**THEOREM 4.13.** *Suppose  $U \in \mathcal{L}(H)$  is a unitary operator and  $u \in H$  is a vector such that  $\langle u, Uu \rangle \neq 0$  and  $\{u, U^*u, (U^*)^2u\}$  are linearly independent. Then there exists  $v \in H$  such that  $T = U + u \otimes v$  has a one-dimensional reducing subspace and  $T$  is not a normal operator.*

*Proof.* Let us consider  $v = (-1/\langle u, Uu \rangle)(U^*)^2u$ . Note that  $v$  is well defined and

$$\langle Uv, u \rangle = \frac{-1}{\langle u, Uu \rangle} \langle U^*u, u \rangle = \frac{-1}{\langle u, Uu \rangle} \langle u, Uu \rangle = -1.$$

Then,

$$TU^*u = (U + u \otimes v)U^*u = u\langle u, Uv \rangle u = u - u = 0.$$

Moreover,

$$T^*U^*u = (U^*)^2u + \langle u, Nu \rangle v = (U^*)^2u - (U^*)^2u = 0.$$

Then,  $M := \text{span} \{U^*u\}$  reduces  $T$ . It remains to show that  $T$  is not normal.

Assume, on the contrary, that  $T$  is a normal operator. By [28, proposition 3.1] we have two possibilities:

- (i)  $u$  and  $v$  are linearly dependent, which is absurd because  $v = (-1/\langle u, Uu \rangle)(U^*)^2u$  and  $u$  and  $(U^*)^2u$  are linearly independent by hypotheses.
- (ii)  $u$  and  $v$  are linearly independent and there exists  $\alpha, \beta \in \mathbb{C}$  such that

$$(U^* - \alpha I)u = \|u\|^2 \beta v.$$

That is,

$$U^*u - \alpha u = \frac{-\|u\|^2 \beta}{\langle u, Uu \rangle} (U^*)^2u.$$

Now  $u, U^*u$  and  $(U^*)^2u$  are linearly independent, so  $U^*u = 0$  and therefore  $\langle u, Uu \rangle = \langle U^*u, u \rangle = 0$ , which yields a contradiction.

Hence,  $T$  is not a normal operator and the proof is finished. □

In the same circle of ideas, the following result holds.

**PROPOSITION 4.14.** *Let  $N \in \mathcal{L}(H)$  be a normal operator and  $u \in H$  a non-zero vector. Assume that there exist  $\alpha, \beta \in \mathbb{C}$  and  $x \in H$  such that  $\langle x, u \rangle \neq 0$  and  $(N - \alpha I)x = (\langle (N - \bar{\beta}I)x, x \rangle / \langle u, x \rangle)u$ . Then, the operator  $T := N + u \otimes v$  has a non-trivial reducing subspace, where  $v = -(1/\langle x, u \rangle)(N^* - \beta I)x$ .*

The proof is a just a computation showing that  $T$  reduces  $M := \text{span} \{x\}$ .

**5. Rank-one perturbations of diagonal operators with multiplicity strictly larger than one: examples**

In this final section we present some examples of rank-one perturbations of diagonal operators with multiplicity strictly larger than one in order to illustrate how the picture of the reducing subspaces changes if we drop off the uniform multiplicity one assumption.

EXAMPLE 5.1. There exists  $T = D + u \otimes v$  a rank-one perturbation of a diagonal operator  $D$  with multiplicity strictly larger than one and spectrum contained in a line such that  $\langle u, e_n \rangle \neq 0$  and  $\langle v, e_n \rangle \neq 0$ ,  $T$  has a non-trivial reducing subspace and it is not a normal operator. It is enough to consider the operator described in remark 2.5.

As we mentioned in § 2, this example shows that the assumption of uniform multiplicity one cannot be dropped off the hypothesis of theorem 2.1, so the result is sharp in that sense.

Next example is somehow an extreme case regarding rank-one perturbations of diagonal operators with multiplicity strictly larger than one with reducing subspaces such that lemmas 2.3 and 2.4 and proposition 4.1 do not hold.

EXAMPLE 5.2. Let  $u, v \in H$  and consider  $T := I + u \otimes v$ , where  $I$  denote the identity operator. Observe that  $I$  is a self-adjoint and unitary diagonal operator, but every closed subspace  $M$  such that  $u, v \in M$  is reducing for  $T$ . Clearly, the behaviour of this operator differs completely from those satisfying theorems 2.1 and 4.3 since the aforementioned lemmas and proposition play an essential role in their corresponding proofs.

Finally, we use proposition 1.2 to show that the assumption of uniform multiplicity cannot be dropped off theorem 4.7.

EXAMPLE 5.3. Let  $(\lambda_n)_n$  be any sequence in the complex plane such that  $\lambda_1 = \lambda_2$  and let  $D$  be the diagonal operator such that  $De_n = \lambda_n e_n$  for every  $n \in \mathbb{N}$ . Consider  $u \in H$  such that  $\langle u, e_n \rangle \neq 0$  for every  $n \in \mathbb{N}$ . Then, the operator  $T := D + u \otimes u$  has a non-trivial reducing subspace.

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