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EVERY INVERTIBLE HILBERT SPACE OPERATOR IS A PRODUCT OF SEVEN POSITIVE OPERATORS

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ABSTRACT. We prove that every invertible operator in a properly infinite von Neumann algebra, in particular in $L(H)$ for infinite dimensional H, is a product of 7 positive invertible operators. This improves a result of Wu, who proved that every invertible operator in $L(H)$ is a product of 17 positive invertible operators.

Ballantine's theorem [1] states that an invertible $n \times n$ complex matrix a is a product of finitely many positive invertible matrices if and only if $\det(a) > 0$; that in this case a is always a product of at most 5 positive invertible matrices; and that for $n \geq 2$, we cannot replace 5 by any smaller number. Much more recently, Wu proved in [10] that if H is an infinite dimensional Hilbert space, then every invertible operator $a \in L(H)$ is a product of positive invertible operators, and in fact of at most 17 of them. (Also see Theorem 2.26 of [11].) He states in [11] that it is not known if 17 is best possible. We show here that it is not; in fact, 7 factors suffice. By contrast, it follows from [4] that every invertible $a \in L(H)$ is a limit of products of 5 positive invertible elements.

Since our proof actually works in properly infinite von Neumann algebras, we give it in that generality.

For results on products of positive invertible elements in the C^*-algebras $C(X) \otimes M_n$, including a proof that in general there is no finite upper bound on the number of factors needed, see Section 2 of [5]. For recent results on limits of products of positive elements in AF algebras, algebras of measurable matrix valued functions, and other related algebras, see [6] and [7]. If instead one considers products of selfadjoint invertible operators, the following facts are known. Every invertible element of $L(H)$ is a product of at most 5 selfadjoint invertible elements ([10], Lemma 2.2), but there is an invertible $a \in L(H)$ which is not the product of fewer than 4 selfadjoint invertible elements ([8], Theorem 5).

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We use the following notation. $M_n(A)$ is the $n \times n$ matrices over A; if A is omitted, it is understood to be $\mathbb{C}$. We write $\operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ for the diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$. If $a \in M_m(A)$ and $b \in M_n(A)$, then $a \oplus b$ is the block diagonal matrix $\operatorname{diag}(a, b) \in M_{m+n}(A)$. When A is a C^*-algebra, $M_n(A)$ is given the C^* norm. We further let $C(X, A)$ be the C^*-algebra of continuous functions from the compact Hausdorff space X to a C^*-algebra A, and we let $B(X, A)$ be the C^*-algebra of bounded Borel functions

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from a metric space X to A, with the supremum norm. For any Banach algebra B, we denote its invertible group by $\mathrm{inv}(B)$. The unit circle in $\mathbb{C}$ is S^1.

Our first lemma is a measurable version of a special case of one of the equivalences in Ballantine's characterization [1] of the elements of M_n which are products of three positive invertible matrices. As we will see below, the conclusion of the lemma easily implies that u is a product of three positive invertible elements of $B(S^1, M_6)$. This lemma can probably be generalized as follows. Let X be a metric space, let $a \in B(X, M_n)$, and suppose that the conditions in [1] for $a(x)$ to be a product of three positive matrices are satisfied for each x, uniformly in x in an appropriate sense. Then a should be a product of three positive invertible elements of $B(X, M_n)$. We don't attempt to prove such a result because we don't need it.

LEMMA 1. *Let $\zeta_0 = \exp(2\pi i/3)$, which is a primitive cube root of* 1. *Let $u \in C(S^1, M_6)$ be given by*

$$u(\zeta) = \zeta\, \mathrm{diag}(1, \zeta_0, \zeta_0^2) \oplus \bar{\zeta}\, \mathrm{diag}(1, \zeta_0, \zeta_0^2).$$

Then there exists $c \in \mathrm{inv}\big(B(S^1, M_6)\big)$ such that $c(\zeta)^* u(\zeta) c(\zeta)$ is lower triangular, with diagonal entries equal to 1.

PROOF. Since we only require c to be a Borel function, we can work on the interval $[0, 2\pi)$ instead of S^1. We thus write

$$u(\theta) = \exp(i\theta)\, \mathrm{diag}(1, \zeta_0, \zeta_0^2) \oplus \exp(-i\theta)\, \mathrm{diag}(1, \zeta_0, \zeta_0^2).$$

We can further restrict to the half open intervals $\big[k\pi/3, (k+1)\pi/3\big)$ for $k = 0, \ldots, 5$, constructing c separately on each of them. So let I be one of these intervals. For θ in the interior of I, the eigenvalues of $u(\theta)$ are all distinct, and none of them are equal to -1. Therefore we can write

$$u(\theta) = \mathrm{diag}\Big(\exp\big(i\alpha_1(\theta)\big), \exp\big(i\alpha_2(\theta)\big), \ldots, \exp\big(i\alpha_6(\theta)\big)\Big), \tag{1}$$

with the α_j continuous and satisfying, for some permutation σ of $\{1, \ldots, 6\}$,

$$-\pi < \alpha_{\sigma(1)}(\theta) < \alpha_{\sigma(2)}(\theta) < \cdots < \alpha_{\sigma(6)}(\theta) < \pi.$$

By continuity, we can extend (1) over all of I, and still have

$$-\pi \leq \alpha_{\sigma(1)}(\theta) \leq \alpha_{\sigma(2)}(\theta) \leq \cdots \leq \alpha_{\sigma(6)}(\theta) \leq \pi. \tag{2}$$

Looking at the interior of I, we see that either $\sigma(1)$, $\sigma(3)$, $\sigma(5) \in \{1, 2, 3\}$ and $\sigma(2)$, $\sigma(4)$, $\sigma(6) \in \{4, 5, 6\}$, or $\sigma(1)$, $\sigma(3)$, $\sigma(5) \in \{4, 5, 6\}$ and $\sigma(2)$, $\sigma(4)$, $\sigma(6) \in \{1, 2, 3\}$. In either case, the averages $\frac{1}{2}\big(\alpha_{\sigma(j)}(\theta) + \alpha_{\sigma(j+1)}(\theta)\big)$ must be constant and take on distinct values, increasing with j, in $\{-2\pi/3, -\pi/3, 0, \pi/3, 2\pi/3\}$. Therefore

$$\frac{1}{2}\big(\alpha_{\sigma(1)}(\theta) + \alpha_{\sigma(2)}(\theta)\big) = -2\pi/3, \quad \frac{1}{2}\big(\alpha_{\sigma(3)}(\theta) + \alpha_{\sigma(4)}(\theta)\big) = 0, \quad \text{and} \tag{3}$$
$$\frac{1}{2}\big(\alpha_{\sigma(5)}(\theta) + \alpha_{\sigma(6)}(\theta)\big) = 2\pi/3$$

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on the interior of I, and thus, by continuity, on all of I. Also note that

$$(4) \qquad 0 \leq \alpha_{\sigma(j+1)}(\theta) - \alpha_{\sigma(j)}(\theta) \leq 2\pi/3$$

for all $\theta \in I$.

In a C^*-algebra B, here taken to be $B(I, M_6)$, let us write $x \sim\sim y$ if there is $c \in \mathrm{inv}(B)$ such that $y = c^* x c$. It is trivial to check that $\sim\sim$ is an equivalence relation. To prove the lemma, we have to show that $u \sim\sim a$ for some a for which each $a(\theta)$ is lower triangular with diagonal entries equal to 1. Taking c to be an appropriate (constant) permutation matrix obtained from σ, we see that

$$(5) \qquad u \sim\sim \mathrm{diag}\big(\exp(i\alpha_{\sigma(1)}), \ldots, \exp(i\alpha_{\sigma(6)})\big).$$

We now make the following claim. Let $v \in B(I, M_6)$ be an element such that each $v(\theta)$ is lower triangular with diagonal entries $\exp\big(i\beta_1(\theta)\big), \ldots, \exp\big(i\beta_6(\theta)\big)$ for real Borel functions $\beta_1, \ldots, \beta_6$. Suppose that for some j and some $\varepsilon > 0$ we have

$$(6) \qquad |\beta_j(\theta) - \beta_{j+1}(\theta)| \leq \pi - \varepsilon$$

on I. Suppose further we are given real Borel functions γ_j and γ_{j+1} such that

$$(7) \quad |\gamma_j(\theta) - \gamma_{j+1}(\theta)| \leq |\beta_j(\theta) - \beta_{j+1}(\theta)| \quad \text{and} \quad \frac{1}{2}\big(\gamma_j(\theta) + \gamma_{j+1}(\theta)\big) = \frac{1}{2}\big(\beta_j(\theta) + \beta_{j+1}(\theta)\big)$$

on I. Then we claim that there exists $w \in B(I, M_6)$ such that $w \sim\sim v$, each $w(\theta)$ is lower triangular, and the diagonal entries of $w(\theta)$ are the same as those of $v(\theta)$ except that $\beta_j(\theta)$ and $\beta_{j+1}(\theta)$ have been replaced by $\gamma_j(\theta)$ and $\gamma_{j+1}(\theta)$.

We will prove this claim below. We show now that it implies the conclusion of the lemma. We temporarily use the shorthand $(\beta_1, \ldots, \beta_6)$ for a lower triangular matrix with diagonal entries $\exp\big(i\beta_1(\theta)\big), \ldots, \exp\big(i\beta_6(\theta)\big)$ and unspecified entries below the diagonal. Then we have the following sequence of equivalences for the relation $\sim\sim$:

$$\begin{aligned}
u = (\alpha_1, \ldots, \alpha_6) &\sim\sim (\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(6)}) \\
&\sim\sim (-2\pi/3, -2\pi/3, 0, 0, 2\pi/3, 2\pi/3) \\
&\sim\sim (-2\pi/3, -\pi/3, -\pi/3, \pi/3, \pi/3, 2\pi/3) \\
&\sim\sim (-2\pi/3, -\pi/3, 0, 0, \pi/3, 2\pi/3) \\
&\sim\sim (-2\pi/3, 0, -\pi/3, \pi/3, 0, 2\pi/3) \\
&\sim\sim (-2\pi/3, 0, 0, 0, 0, 2\pi/3) \\
&\sim\sim (0, -2\pi/3, 0, 0, 2\pi/3, 0) \\
&\sim\sim (0, 0, 0, 0, 0, 0).
\end{aligned}$$

In this sequence, the first line is the definition of $\alpha_1, \ldots, \alpha_6$ and (5). The second line is gotten by applying the claim with $\beta_1 = \alpha_{\sigma(1)}$, $\beta_2 = \alpha_{\sigma(2)}$, $\gamma_1 = \gamma_2 = -2\pi/3$; then again with $\beta_3 = \alpha_{\sigma(3)}$, $\beta_4 = \alpha_{\sigma(4)}$, $\gamma_3 = \gamma_4 = 0$; and then a third time with $\beta_5 = \alpha_{\sigma(5)}$, $\beta_6 = \alpha_{\sigma(6)}$, $\gamma_5 = \gamma_6 = 2\pi/3$. (Note that (6) and (7) in the claim follow from (3) and

(4) in each of these cases.) Each of the remaining lines, except for the last, is then gotten from the preceding one by applying the claim to several disjoint pairs of neighboring entries. The last line is then gotten by repeating the steps leading from the second line to the second last line. Since it represents a matrix of the form desired for c^*uc, we have the conclusion of the lemma.

It remains only to prove the claim. We first observe that it suffices to prove the analogous claim for 2×2 matrices (rather than 6×6 matrices), with $j = 1$. This is easily checked by considering the block decomposition corresponding to the partition

$$\{1, \ldots, 6\} = \{1, \ldots, j-1\} \cup \{j, j+1\} \cup \{j+2, \ldots, 6\}.$$

(If c_0 implements the relation $\sim\sim$ in the 2×2 case, then $c = 1 \oplus c_0 \oplus 1$, with identities of size $j-1$ and $5-j$, will implement the relation in the 6×6 case.)

Let $v \in B(I, M_2)$ be given by

$$v(\theta) = \begin{pmatrix} \exp\bigl(i\beta_1(\theta)\bigr) & 0 \\ 2\rho(\theta) & \exp\bigl(i\beta_2(\theta)\bigr) \end{pmatrix},$$

with β_1 and β_2 real. We have to show that if (6) holds with $j = 1$ and some $\varepsilon > 0$, and if γ_1 and γ_2 satisfy (7), then there exists $x \in B(I, M_2)$ of the form

$$x(\theta) = \begin{pmatrix} \exp\bigl(i\gamma_1(\theta)\bigr) & 0 \\ 2\mu(\theta) & \exp\bigl(i\gamma_2(\theta)\bigr) \end{pmatrix}, \tag{8}$$

with $x \sim\sim v$.

To simplify the notation, we will omit the argument θ everywhere in the following computation. We also use the (nonstandard) convention that $\operatorname{sgn}(0) = 1$; thus, sgn is a bounded Borel function on $\mathbb{C}$ such that $|\operatorname{sgn}(\zeta)| = 1$ and $\operatorname{sgn}(\zeta)|\zeta| = \zeta$ for all $\zeta \in \mathbb{C}$. Set $\alpha = (\beta_1 - \beta_2)/2$, $\delta = (\beta_1 + \beta_2)/2$, and $\omega = \operatorname{sgn}\bigl(|\rho| - i\sin(\alpha)\bigr)$. Define

$$d_1 = [2|\rho| + 2\cos(\alpha)]^{-1/2} \begin{pmatrix} 1 & -\bar{\omega}\exp(-i\alpha) \\ \operatorname{sgn}(\rho)\exp(-i\delta) & \bar{\omega}\operatorname{sgn}(\rho)\exp(-i\delta)\bigl(2|\rho| + \exp(i\alpha)\bigr) \end{pmatrix}.$$

(This formula is a modification of one in Section 4 of [1].) Condition (6) implies that the factor in front is at most $[2\sin(\varepsilon/2)]^{-1/2}$. Therefore $d_1 \in B(I, M_2)$. One checks that

$$|\det(d_1)| = |\bar{\omega}\operatorname{sgn}(\rho)\exp(-i\delta)| = 1,$$

from which it follows that $d_1^{-1} \in B(I, M_2)$ also. Finally, a computation shows that

$$d_1^* v d_1 = \exp(i\delta)\begin{pmatrix} 1 & 0 \\ 2\bigl(|\rho|^2 + \sin^2(\alpha)\bigr)^{1/2} & 1 \end{pmatrix}.$$

Since $|(\gamma_1 - \gamma_2)/2| \le |(\beta_1 - \beta_2)/2| < \pi/2$, we have $\sin^2\bigl((\gamma_1 - \gamma_2)/2\bigr) \le \sin^2(\alpha)$. Therefore there is a Borel function μ such that

$$|\mu|^2 + \sin^2\bigl((\gamma_1 - \gamma_2)/2\bigr) = |\rho|^2 + \sin^2(\alpha).$$

Note that $|\mu|^2 \le |\rho|^2 + 1$, so that μ is bounded. Let x be given by (8) with this choice of μ. Define d_2 in the same way as d_1, except using γ_1 and γ_2 in place of β_1 and β_2. Note that this changes α and ω but not δ. Then the same computation as in the previous paragraph shows that $d_2^* x d_2 = d_1^* v d_1$ and $d_2 \in \operatorname{inv}\bigl(B(I, M_2)\bigr)$. Therefore $v \sim\sim d_1^* v d_1 \sim\sim x$, as desired. This completes the proof of the claim, and therefore of the lemma. ∎

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LEMMA 2. *Let u be as in Lemma 1. Then u is a product of three positive invertible elements of $B(S^1, M_6)$.*

PROOF. We use ideas already known in the case of ordinary matrices; see for example Section 2 of [1] and Section 2 of [9]. Let c be as in the conclusion of the previous lemma, and let $d = \mathrm{diag}(\alpha_1, \ldots, \alpha_6)$ where $\alpha_1, \ldots, \alpha_6$ are distinct positive real numbers. Let $x = cd$. Then, since c^*uc is lower triangular with diagonal entries equal to 1, one sees that x^*ux is lower triangular with the distinct diagonal entries $\alpha_1^2, \ldots, \alpha_6^2$. Therefore there is $g \in \mathrm{inv}\big(B(S^1, M_6)\big)$ such that

$$g(x^*ux)g^{-1} = \mathrm{diag}(\alpha_1^2, \ldots, \alpha_6^2) = d^2.$$

(The proof is the same as that of Lemma 2.5 of [5]. Also compare Lemma 4.2 of [2].) Thus x^*ux is similar to a positive invertible element, and, as in Section 2 of [9], x^*ux is a product a_1a_2 of two positive invertible elements. Following [1], we now obtain

$$u = (x^{-1})^*a_1a_2x^{-1} = [(x^{-1})^*a_1x^{-1}][xx^*][(x^{-1})^*a_2x^{-1}],$$

which expresses u as a product of three positive invertible elements. ∎

Let M be a properly infinite von Neumann algebra. We say that $u \in M$ is a *shift* if there are mutually orthogonal Murray-von Neumann equivalent projections $p_n \in M$ $(n \in \mathbf{Z})$ such that $\sum_{n\in\mathbf{Z}} p_n = 1$, and there are partial isometries $u_n \in M$ such that $u_n^*u_n = p_n$, $u_nu_n^* = p_{n+1}$ and $u = \sum_{n\in\mathbf{Z}} u_n$. (Convergence is in the strong operator topology. Of course, what we have defined is a bilateral shift.) The *multiplicity* of the shift is the Murray-von Neumann equivalence class $[p_0]$.

LEMMA 3. *Any two shifts in M with the same multiplicity are unitarily equivalent.*

PROOF. Let u, p_n, and u_n be a above, and let $v = \sum_{n\in\mathbf{Z}} v_n$ with $v_n^*v_n = q_n$, $v_nv_n^* = q_{n+1}$ be another shift with q_0 Murray-von Neumann equivalent to p_0. Conjugating v by an appropriate unitary, we can assume $q_n = p_n$ for all n. Now v^*u is a block diagonal unitary, and it is easy to find another block diagonal unitary which conjugates v to u. (Use the method of the proof of Lemma 3.1 of [2].) ∎

LEMMA 4. *Let M be a properly infinite von Neumann algebra, and let $u \in M$ be a shift whose multiplicity is divisible by* 6. *Then u is a product of three positive invertible elements of M.*

PROOF. Let u_n, p_n be as in the discussion preceding Lemma 3. By assumption, we can write $p_0 = e_1 + \cdots + e_6$ with $e_1, \ldots, e_6$ mutually orthogonal Murray-von Neumann equivalent projections. Set $f_j = \sum_{n\in\mathbf{Z}} u^ne_ju^{-n}$. Then $f_1, \ldots, f_6$ are mutually orthogonal Murray-von Neumann equivalent projections which sum to 1 and commute with u. From them we obtain an isomorphism $M \cong M_6(f_1Mf_1)$ which sends u to $v = \mathrm{diag}(s_1, \ldots, s_6)$, where the s_j are shifts in f_1Mf_1, all of the same multiplicity $[e_1]$. Set $A = f_1Mf_1$ and $s = s_1$. By the previous lemma, it suffices to show that $\mathrm{diag}(s, \ldots, s) \in M_6(A)$ is a product of three positive invertible elements. Now ζs and ζs^*, for $\zeta \in S^1$, are also shifts of the same multiplicity as s. Therefore, it actually suffices to show that

$$w = \operatorname{diag}(s, \zeta_0^2, \zeta_0^2 s, s^*, \zeta_0 s^*, \zeta_0^2 s^*)$$

is a product of three positive invertible elements for $\zeta_0 = \exp(2\pi i/3)$.

Bounded Borel functional calculus defines a $*$-homomorphism $\varphi: B(S^1, \mathbb{C}) \longrightarrow A$ which sends the identity function $z(\zeta) = \zeta$ to s. This homomorphism extends in the obvious way to a $*$-homomorphism $\psi: B(S^1, M_6) \longrightarrow M_6(A)$, for which one has

$$\psi\big(\operatorname{diag}(z, \zeta_0 z, \zeta_0^2 z, z^*, \zeta_0 z^*, \zeta_0^2 z^*)\big) = w.$$

But $\operatorname{diag}(z, \zeta_0 z, \zeta_0^2 z, z^*, \zeta_0 z^*, \zeta_0^2 z^*)$ is exactly the element proved in Lemma 2 to be a product of three positive invertible elements of $B(S^1, M_6)$. Therefore w is a product of three positive invertible elements in $M_6(A)$. ■

PROPOSITION 5. *Let M be a properly infinite von Neumann algebra. Then every unitary element of M is a product of* 6 *positive invertible elements.*

PROOF. Let $u \in M$ be unitary. Theorem 3 of [3] provides a countably infinite collection of mutually orthogonal Murray-von Neumann equivalent projections which sum to 1 and commute with u. Label them q_{nj} for $n \in \mathbf{Z}$ and $j \in \{1, \ldots, 6\}$. Set $p_n = \sum_{j=1}^{6} q_{nj}$. Then p_n is equivalent to p_{n+1}, so there is a partial isometry v_n such that $v_n^* v_n = p_n$ and $v_n v_n^* = p_{n+1}$. Let $v = \sum_{n \in \mathbf{Z}} v_n$ with convergence in the strong operator topology. Then both v and $v^* u = \sum_{n \in \mathbf{Z}} v_{n-1}^* p_n u p_n$ are shifts of multiplicity $[p_0]$. Since $[p_0]$ is divisible by 6 by construction, the previous lemma implies that v and $v^* u$ are each products of three positive invertible elements. Therefore u is a product of 6 positive invertible elements. ■

THEOREM 6. *Let M be a properly infinite von Neumann algebra. Then every invertible element of M is a product of* 7 *positive invertible elements.*

PROOF. This follows from the previous proposition by polar decomposition. ■

We can also improve the bound in part of Theorem 1.1 of [10]. Following the usual convention, we say that a not necessarily invertible operator $a \in L(H)$ is *positive* if $\langle a\xi, \xi \rangle \geq 0$ for all $\xi \in H$. (This conflicts with [10], where such operators are called nonnegative.)

PROPOSITION 7. *Let H be a separable infinite dimensional Hilbert space, and let $a \in L(H)$ be a product of finitely many positive operators. Then a is a product of* 8 *positive operators.*

The upper bound 18 is given in [10]. It is also shown there that a is such a product if and only if $a \in \overline{\operatorname{inv}\big(L(H)\big)}$. Quite possibly this proposition also holds for all properly infinite von Neumann algebras. We have not, however, checked whether the arguments from [10] carry over to that case.

PROOF OF PROPOSITION 7. Let $a \in L(H)$ be a finite product of positive operators. We claim that there are $s, p \in L(H)$ such that $\dim(\ker(s)) = \dim(\ker(s^*))$, p is a projection, and $a = sp$ or $a = ps$. If $\dim(\ker(a)) = \dim(\ker(a^*))$, we can take $p = 1$ and $s = a$. Otherwise, Proposition 2.4 of [10] shows that the range of a is not closed. If $\dim(\ker(a)) > \dim(\ker(a^*))$, the proof of Proposition 2.6 of [10] provides p and s with $a = sp$. Taking adjoints, we can write $a = ps$ if the reverse inequality holds.

Using the polar decomposition and $\dim(\ker(s)) = \dim(\ker(s^*))$, we can find a unitary u such that $s = u(s^*s)^{1/2}$. Then $a = u(s^*s)^{1/2}p$ or $a = pu(s^*s)^{1/2}$. Since u is a product of 6 positive operators by Proposition 5, this exhibits a as a product of 8 positive operators. ■

Two questions naturally suggest themselves.

QUESTION 8. What is the smallest n such that every invertible $a \in L(H)$ is a product of n positive elements?

Our result shows that $n \leq 7$. (The best previously known result was $n \leq 17$.) The best known lower bound seems to be $n \geq 5$. The following proof of this bound was pointed out to me by Heydar Radjavi, and is also contained in the proof of Theorem 2 of [9]. Suppose -1 is a product $a_1a_2a_3a_4$ of 4 positive invertible operators. Then $a_1a_2 = -a_4^{-1}a_3^{-1}$. But $a_1a_2 = a_1^{\frac{1}{2}}(a_1^{\frac{1}{2}}a_2a_1^{\frac{1}{2}})a_1^{-\frac{1}{2}}$ is similar to a positive invertible operator, and so has spectrum contained in $(0, \infty)$. Similarly $\mathrm{sp}(-a_4^{-1}a_3^{-1}) \subset (-\infty, 0)$. This is a contradiction.

QUESTION 9. What is the set of finite products of positive invertible elements in a type II_1 factor, and how many are needed?

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