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HALL'S CONDITION AND IDEMPOTENT RANK OF IDEALS OF ENDOMORPHISM MONOIDS

R. GRAY

School of Mathematics, University of Leeds, Leeds LS2 9JT, UK
(robertg@maths.leeds.ac.uk)

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Abstract In 1990, Howie and McFadden showed that every proper two-sided ideal of the full transformation monoid T_n, the set of all maps from an n-set to itself under composition, has a generating set, consisting of idempotents, that is no larger than any other generating set. This fact is a direct consequence of the same property holding in an associated finite 0-simple semigroup. We show a correspondence between finite 0-simple semigroups that have this property and bipartite graphs that satisfy a condition that is similar to, but slightly stronger than, Hall's condition. The results are applied in order to recover the above result for the full transformation monoid and to prove the analogous result for the proper two-sided ideals of the monoid of endomorphisms of a finite vector space.

Keywords: matching; generating set; endomorphism monoid; idempotent

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1. Introduction

For any mathematical structure M, the set of endomorphisms, denoted $\mathrm{End}(M)$, is closed under composition and forms a monoid (a semigroup with identity). When M is simply a finite set, say with $|M| = n$, $\mathrm{End}(M) \cong T_n$ the full transformation monoid. It is known [**6**] that the proper two-sided ideals of T_n, namely the semigroups $K(n, r) = \{\alpha \in T_n : |\mathrm{Im}(\alpha)| \leqslant r : 1 \leqslant r < n\}$, are generated by their sets of idempotents (elements that satisfy $\alpha^2 = \alpha$). Furthermore, in [**13**] Howie and McFadden proved that $K(n, r)$ is 'easy' to generate using idempotents, in the sense that it has a generating set, consisting of idempotents, that is no larger than any other generating set. More precisely, they proved that the smallest number of elements required to generate this semigroup (its *rank*) and the smallest number of idempotents required to generate this semigroup (its *idempotent rank*) are both equal to $S(n, r)$ (the Stirling number of the second kind). A number of other structures M are known to have the property that the proper two-sided ideals of $\mathrm{End}(M)$ are idempotent generated. In particular, it was proven that $\mathrm{End}(M)$ has this property when M is a finite-dimensional vector space [**3**], when M is a finite-dimensional independence algebra (a concept which generalizes sets and vector spaces) [**5, 8**] and when M is a finite chain [**7**]. We are interested in determining when idempotent rank will equal rank for such examples. We say that an idempotent-generated semigroup with idempotent rank equal to rank has an *idempotent basis.* As mentioned above, the semigroup $K(n,r)$ has an idempotent basis. On the other hand, the proper two-sided ideals of the semigroup O_n of order-preserving mappings (endomorphisms of an n element chain) are idempotent generated but do not, in general, have idempotent bases (see [**7**]). We will now introduce some basic terminology and results from semigroup theory. For undefined notions concerning semigroups, the reader is referred to [**12**].

For $s \in S$ the principal right ideal generated by s is $sS \cup \{s\}$, which we denote by sS^1. Similarly, the principal left ideal generated by s is S^1s and the principal two-sided ideal generated by s is S^1sS^1. We say that $s,t \in S$ are $\mathcal{R}$-, $\mathcal{L}$- or $\mathcal{J}$-related if they generate the same principal right-, left- or two-sided ideal, respectively. Also, we say s and t are $\mathcal{H}$-related if they are $\mathcal{R}$-related and $\mathcal{L}$-related. It is easily seen that $\mathcal{R}$, $\mathcal{L}$, $\mathcal{J}$ and $\mathcal{H}$ are equivalence relations on S (known as Green's relations) and thus divide S into equivalence classes called the $\mathcal{R}$-, $\mathcal{L}$-, $\mathcal{J}$- and $\mathcal{H}$-classes, respectively. If H is an $\mathcal{H}$-class in a semigroup, then either $H^2 \cap H = \varnothing$ or $H^2 = H$, in which case H is a subgroup of S. In addition to this, H is a subgroup if and only if it contains an idempotent. The $\mathcal{J}$-classes may be ordered in a natural way, where $J_a \leqslant J_b$ if $S^1aS^1 \subseteq S^1bS^1$ (where J_x denotes the $\mathcal{J}$-class of x). The proper two-sided ideals of the endomorphism monoids mentioned above have unique maximal $\mathcal{J}$-classes with respect to this order and, it turns out, they are generated by the elements contained within their respective maximal $\mathcal{J}$-classes. It follows from this that the minimal (with respect to set theoretic inclusion) generating sets of these semigroups are contained completely within this $\mathcal{J}$-class. In order to study a particular $\mathcal{J}$-class of a semigroup in detail, we will need the notion of 0-simple semigroup. A semigroup with 0 is called 0-*simple* if 0 and S are its only two-sided ideals (and $S^2 \neq \{0\}$). A semigroup S is said to be *completely* 0-*simple* if it is 0-simple and has 0-minimal left and right ideals. Every finite 0-simple semigroup is completely 0-simple. These semigroups may be described by the following construction. Let G be a group, let I and Λ be non-empty finite index sets and let $P = (p_{\lambda i})$ be a *regular* $\Lambda \times I$ matrix over $G \cup \{0\}$ (where regular means that every row and every column of P has at least one non-zero entry). Then $S = \mathcal{M}^0[G; I, \Lambda; P]$, the $I \times \Lambda$ *Rees matrix semigroup over the* 0-*group* $G \cup \{0\}$ *with structure matrix* P, is the semigroup $(I \times G \times \Lambda) \cup \{0\}$ with multiplication defined by

$$(i,g,\lambda)(j,h,\mu) = \begin{cases} (i, gp_{\lambda j}h, \mu) & \text{if } p_{\lambda j} \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$
$$(i,g,\lambda)0 = 0(i,g,\lambda) = 00 = 0.$$

It is known, by the structure theorem of Rees [**12**, Theorem 3.2.3], that every completely 0-simple semigroup is isomorphic to some Rees matrix semigroup $\mathcal{M}^0[G; I, \Lambda; P]$ over a group G. The 0-simple semigroups occur naturally 'inside' arbitrary semigroups appearing as principal factors of $\mathcal{J}$-classes. Let J be some $\mathcal{J}$-class of a semigroup S. Then the principal factor of S corresponding to J is the set $J^* = J \cup \{0\}$ with multiplication

$$s * t = \begin{cases} st & \text{if } s, t, st \in J, \\ 0 & \text{otherwise.} \end{cases}$$

The semigroup J^* is either a semigroup with zero multiplication or is a 0-simple semigroup [**12**, Chapter 3]. In a Rees matrix semigroup the set I indexes the non-zero $\mathcal{R}$-classes, Λ indexes the non-zero $\mathcal{L}$-classes and the non-zero group $\mathcal{H}$-classes are all isomorphic to G. We will use $E(S)$ to denote the set of idempotents of a semigroup S. Also, given a subset A of a semigroup S, we use $\langle A \rangle$ to denote the set of all elements that can be written as products of elements of A. If S is a semigroup with a zero element, then $\langle A \rangle$ will denote the set of all elements that can be written as products of elements of A together with $\{0\}$ if it is not already generated. We write $\operatorname{rank}(S)$ to denote the rank of S, and $\operatorname{id rank}(S)$ to denote its idempotent rank. With the above notation we have:

$$\operatorname{rank}(S) = \min\{|A| : A \subseteq S,\ \langle A \rangle = S\}, \qquad \operatorname{id rank}(S) = \min\{|A| : A \subseteq E(S),\ \langle A \rangle = S\}.$$

We will also need a few notions from graph theory, in particular the idea of a perfect matching. We define a *graph* to consist of a set $\mathcal{V}$ of vertices and a set $\mathcal{E}$ of 2-subsets of $\mathcal{V}$ called edges. The *degree* of a vertex v is the number of edges adjacent to it and will be denoted by $d(v)$. We call a graph k*-regular* if every vertex has degree k for some number k. A graph is *bipartite* if $\mathcal{V}$ can be written as the disjoint union of two sets A and B in such a way that every edge in $\mathcal{E}$ has one vertex in A and the other in B. We say that the bipartite graph $\Gamma = A \cup B$ is *balanced* if $|A| = |B|$. A walk in a graph is a sequence $(v_0, e_1, v_1, e_2, v_2, \dots, e_n, v_n)$, where e_i is the edge $\{v_{i-1}, v_i\}$ for $i = 1, \dots, n$. A graph is *Hamiltonian* if it has a walk that starts and ends at the same vertex and passes through every other vertex of the graph precisely once. The *neighbourhood* of a vertex v in a graph G is the set of vertices of G to which v is adjacent and is denoted $N_G(v)$ (or sometimes just $N(v)$ when the meaning is clear). Given a set of vertices $\mathcal{W}$, $N_G(\mathcal{W})$ is defined to be the set of all vertices of $\mathcal{V} \setminus \mathcal{W}$ that are neighbours of at least one vertex from $\mathcal{W}$. A *matching* in a graph is a collection of edges of which no two have a vertex in common. A *perfect matching* is a matching on $|\mathcal{V}|/2$ edges. In particular, perfect matchings in balanced bipartite graphs $\Gamma = X \cup Y$ correspond to bijections $\pi : X \to Y$ satisfying $\{x, \pi(x)\} \in \mathcal{E}$ for all $x \in X$. Finally, we recall the following well-known result of Hall that tells us precisely when a bipartite graph has a perfect matching.

Theorem 1.1 (Hall [11]). *The bipartite graph $G = X \cup Y$ has a perfect matching if and only if $|N(A)| \geqslant |A|$ for all subsets A of X.*

A bipartite graph $G = X \cup Y$ satisfying the condition given in Theorem 1.1 will be said to satisfy Hall's condition.

2. Idempotent-generated completely 0-simple semigroups

Associated with any Rees matrix semigroup S is the so-called *rectangular* 0*-band* T given by replacing all the non-zero entries in the structure matrix P of S by the number 1, and by replacing the underlying group by the trivial group. For a finite 0-simple semigroup S, Green's $\mathcal{H}$ relation is a (two-sided) congruence on S and $S/\mathcal{H}$ is isomorphic to this rectangular 0-band T.

Definition 2.1. Given $S = \mathcal{M}^0[G; I, \Lambda; P]$, by the natural rectangular 0-band homomorphic image of S we mean $S/\mathcal{H}$. This semigroup can be concretely represented as $T = \mathcal{M}^0[\{1\}; I, \Lambda; Q]$, where $q_{\lambda i} = 1$ if and only if $p_{\lambda i} \neq 0$. We will use $\natural$ to denote the corresponding epimorphism from S to T with $0\natural = 0$ and $(i, g, \lambda)\natural = (i, 1, \lambda)$.

Since the middle components of the semigroups $T = \mathcal{M}^0[\{1\}; I, \Lambda; Q]$ always equal 1, we can effectively ignore them and consider T as a semigroup of pairs $(I \times \Lambda) \cup \{0\}$ with multiplication:

$$(i, \lambda)(j, \mu) = \begin{cases} (i, \mu) & \text{if } q_{\lambda j} = 1, \\ 0 & \text{if } q_{\lambda j} = 0, \end{cases}$$
$$(i, \lambda)0 = 0(i, \lambda) = 00 = 0.$$

The relevance to our problem of the natural rectangular 0-band homomorphic image is demonstrated by the following results.

Lemma 2.2. *Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a finite idempotent-generated completely 0-simple semigroup and let $T = S\natural$ be the natural rectangular 0-band homomorphic image of S. Let $B \subseteq T$ be a generating set for T. If A is a subset of S such that $A\natural = B$, then $\langle A \rangle = S$.*

Proof. Since $\langle B \rangle = T$, it follows that A generates at least one element in every $\mathcal{H}$-class of S. In particular, every group $\mathcal{H}$-class of S has non-empty intersection with $\langle A \rangle$. Let H be one of these group $\mathcal{H}$-classes. Then, given $h \in H \cap \langle A \rangle$, since H is a finite group, there is a natural number i such that $h^i = 1_H$, the identity of the group $\mathcal{H}$-class. Since the idempotents of S are precisely the identities of the group $\mathcal{H}$-classes of S, it follows that $E(S) \subseteq \langle A \rangle$. Since S is idempotent generated, it follows that $\langle A \rangle = \langle E(S) \rangle = S$. □

Lemma 2.3. *Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a finite idempotent-generated completely 0-simple semigroup and let $T = S\natural$ be the natural rectangular 0-band homomorphic image of S. Then*

(i) $\operatorname{rank}(S) = \operatorname{rank}(T) = \max(|I|, |\Lambda|)$,

(ii) $\operatorname{id rank}(S) = \operatorname{id rank}(T)$.

In particular, S has an idempotent basis if and only if T has an idempotent basis.

Proof. (i) Let Y be a generating set for T. Then, by Lemma 2.2, $X = \{(i, 1, \lambda) : (i, \lambda) \in Y\}$ generates S, since $X\natural = Y$. Observing that $|X| = |Y|$, we conclude that $\operatorname{rank}(S) \leqslant \operatorname{rank}(T)$. On the other hand, we know that $\operatorname{rank}(S) \geqslant \operatorname{rank}(T)$, since T is a homomorphic image of S. The fact that $\operatorname{rank}(T) = \max(|I|, |\Lambda|)$ is proven in [**10**, § 2].

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(ii) If $X \subseteq E(S)$ generates S, then $X\natural \subseteq E(T)$ with $|X\natural| \leqslant |X|$ and $\langle X\natural \rangle = T$. Therefore, $\operatorname{id rank}(S) \geqslant \operatorname{id rank}(T)$. For the converse, let $Y \subseteq E(T)$ generate T. Then $X = \{(i, p_{\lambda i}^{-1}, \lambda) : (i, \lambda) \in Y\}$ is a subset of the non-zero idempotents of S and, by Lemma 2.2, it generates S. Since $|X| = |Y|$, we conclude that $\operatorname{id rank}(S) \leqslant \operatorname{id rank}(T)$. □

When working with completely 0-simple semigroups we will find it useful to associate with them the following bipartite graph.

Definition 2.4. Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a completely 0-simple semigroup. Let $\Delta(P)$ denote the undirected bipartite graph with set of vertices $I \cup \Lambda$ and an edge connecting i to λ if and only if $p_{\lambda i} \neq 0$.

Given a completely 0-simple semigroup S, not already expressed as a Rees matrix semigroup, we will write $\Delta(S)$ to denote the bipartite graph with vertices $\mathfrak{R} \cup \mathfrak{L}$, where $\mathfrak{R}$ and $\mathfrak{L}$ are the non-zero $\mathcal{R}$- and $\mathcal{L}$-classes of S, respectively, and vertex $R \in \mathfrak{R}$ is connected to the vertex $L \in \mathfrak{L}$ if and only if $H = R \cap L$ contains an idempotent. Clearly, given $S = \mathcal{M}^0[G; I, \Lambda; P]$, the graphs $\Delta(P)$ and $\Delta(S)$ are isomorphic.

There is a straightforward correspondence between non-zero products of idempotents in $T = S\natural$ and paths from I to Λ in $\Delta(P)$. Indeed, the equality $(i_1, \lambda_1) \cdots (i_k, \lambda_k) = (i_1, \lambda_k)$, with $(i_l, \lambda_l) \in E(T)$ for $1 \leqslant l \leqslant k$, holds in T if and only if $p_{\lambda_1 i_2}, p_{\lambda_2 i_3}, \dots, p_{\lambda_{k-1} i_k}$ are all non-zero, which is equivalent to saying that $i_1 \to \lambda_1 \to i_2 \to \lambda_2 \to \cdots \to i_k \to \lambda_k$ is a path in the graph $\Delta(P)$.

Lemma 2.5. *The set $E' \subseteq E(T)$ of idempotents generates $(i_1, \lambda_k) \in T$ if and only if there is a path $i_1 \to \lambda_1 \to i_2 \to \lambda_2 \to \cdots \to i_k \to \lambda_k$ in $\Delta(P)$ such that $(i_m, \lambda_m) \in E'$ for $1 \leqslant m \leqslant k$.*

We start by considering square idempotent-generated Rees matrix semigroups (i.e. where $|I| = |\Lambda|$) and then extend our results to deal with non-square ones. Note that if S is a square idempotent-generated Rees matrix semigroup with structure matrix P, then $\Delta(P)$ is a connected, balanced bipartite graph. If $\Delta(P)$ is Hamiltonian, then S will have an idempotent basis. Indeed, if $i_1 \to \lambda_1 \to \cdots \to i_n \to \lambda_n \to i_1$, with $n = |I| = |\Lambda|$, is a Hamiltonian circuit, then, by Lemma 2.5, the subset $E' = \{(i_j, \lambda_j) : 1 \leqslant j \leqslant n\}$ of $E(S)$ generates T. It is, however, not necessary that $\Delta(P)$ is Hamiltonian in order for S to have an idempotent basis.

Example 2.6. Let S be the rectangular 0-band with structure matrix

$$P = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

It is then an easy exercise to check that $\Delta(P)$ is not Hamiltonian but that $E' = \{(1,1), (2,2), (3,3), (4,4)\}$ generates S.

On the other hand, it is clear that for a square idempotent-generated Rees matrix semigroup S to have an idempotent basis, the graph $\Delta(P)$ must have a perfect matching. Given a generating set consisting of $|I| = |\Lambda|$ idempotents, the edges corresponding to these elements will constitute a perfect matching in $\Delta(P)$. This is equivalent, as a consequence of Hall's marriage theorem, to saying that for every subset X of I we have $|N(X)| \geqslant |X|$. It is, however, not sufficient that $\Delta(P)$ has a perfect matching (and is connected) to conclude that S has an idempotent basis.

Example 2.7. Consider the rectangular 0-band S with structure matrix

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The graph $\Delta(P)$ is connected and has a perfect matching but $\operatorname{id}\operatorname{rank}(S) = 3 > 2 = \operatorname{rank}(S)$.

Definition 2.8. We call a subset A of the Rees matrix semigroup $S = \mathcal{M}^0[S; I, \Lambda; P]$ a *sparse cover* of S if $|A| = \max(|I|, |\Lambda|)$ and A intersects every non-zero $\mathcal{R}$-class and every non-zero $\mathcal{L}$-class of S.

The main result in [**10**, § 2] says that every rectangular 0-band, and as a consequence every idempotent-generated completely 0-simple semigroup, has at least one sparse cover that generates it. At the other extreme it is easy to see that, when $|I| = |\Lambda| = n$, the semigroup S has at most $|G|^n n!$ generating sets of size n and this number is attained precisely when any sparse cover of S generates S.

Lemma 2.9. *Let $\Gamma = I \cup \Lambda$ be a connected, balanced bipartite graph. Then the following are equivalent:*

(i) *for every non-empty proper subset X of I we have $|N(X)| > |X|$;*

(ii) *for every non-empty proper subset Y of Λ we have $|N(Y)| > |Y|$.*

Proof. Let X be a non-empty proper subset of I. Suppose that $|N(X)| \leqslant |X|$. It then follows that $(|\Lambda| - |N(X)|) - |\Lambda| \geqslant (|I| - |X|) - |I|$ and so $|\Lambda \setminus N(X)| \geqslant |I \setminus X| \geqslant |N(\Lambda \setminus N(X))|$. □

If a connected balanced bipartite graph $\Gamma = I \cup \Lambda$ satisfies either, and hence both, of the conditions given in Lemma 2.9, we say that Γ satisfies the *strong Hall condition* (SHC). We say that $S = \mathcal{M}^0[G; I, \Lambda; P]$, with $|I| = |\Lambda|$, satisfies the SHC if the graph $\Delta(P)$ does.

We will now describe the class of square idempotent-generated completely 0-simple semigroups that have idempotent bases.

Theorem 2.10. *Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a finite idempotent-generated completely 0-simple semigroup with $|I| = |\Lambda|$. Then the following are equivalent:*

(i) $\mathrm{rank}(S) = \mathrm{id}\,\mathrm{rank}(S)$;

(ii) *any sparse cover of S generates S;*

(iii) *S satisfies the SHC.*

Proof. First we show that it is sufficient to prove the result just for rectangular 0-bands. Let S be a finite idempotent-generated completely 0-simple semigroup with $|I| = |\Lambda|$, and let $T = S\natural$ be the natural rectangular 0-band homomorphic image of S. It follows from Lemma 2.3 that $\mathrm{rank}(S) = \mathrm{id}\,\mathrm{rank}(S)$ if and only if $\mathrm{rank}(T) = \mathrm{id}\,\mathrm{rank}(T)$. By definition, the graphs $\Delta(S)$ and $\Delta(T)$ are isomorphic and so S satisfies the SHC if and only if T does. Also, if A is a sparse cover of S, then $A\natural$ is a sparse cover of T and, by Lemma 2.2, A generates S if and only if $A\natural$ generates T. Conversely, if B is a sparse cover of T, then $B\natural^{-1}$ is a union of $\mathcal{H}$-classes of S and any transversal B' of these $\mathcal{H}$-classes is a sparse cover of S. Again by Lemma 2.2, B generates T if and only if B' generates S. It follows that any sparse cover of S generates S if and only if any sparse cover of T generates T. It follows from these observations that it is sufficient to prove the result for rectangular 0-bands.

Let $S = (I \times \Lambda) \cup \{0\}$ be an $n \times n$ rectangular 0-band with $I = \Lambda = \{1, \dots, n\}$. Condition (ii) for rectangular 0-bands says that for every $\beta \in S_n$ (the symmetric group on $\{1, \dots, n\}$) the set $X(\beta) = \{(1, 1\beta), \dots, (n, n\beta)\}$ generates S.

(i) $\Rightarrow$ (ii). Suppose without loss of generality that $E' = \{(1, 1), \dots, (n, n)\}$ is an idempotent basis for S. Let $X(\beta) = \{(1, 1\beta), \dots, (n, n\beta)\}$ for some $\beta \in S_n$. Let k be the order of β in S_n. Then

$$(i, i) = (i, i\beta)(i\beta, i\beta^2) \cdots (i\beta^{k-1}, i\beta^k),$$

where each of the terms on the right-hand side belong to $X(\beta)$. Therefore, $\langle X(\beta)\rangle \supseteq \langle E'\rangle = S$ and we conclude that $X(\beta)$ generates S.

(ii) $\Rightarrow$ (iii). Suppose that S does not satisfy the SHC and let J be a non-empty proper subset of I such that $|N(J)| \leqslant |J|$. We will find a sparse cover of S that does not generate S. Let $I = \{i_1, \dots, i_n\}$, $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ with $J = \{i_1, \dots, i_q\}$ and $N(J) = \{\lambda_1, \dots \lambda_l\}$, where $l \leqslant q$. We claim that the sparse cover $Y = \{(i_1, \lambda_1), \dots, (i_n, \lambda_n)\}$ does not generate S; in particular, $(i_n, \lambda_1) \notin \langle Y\rangle$. Consider a path $i_n \to \lambda_n \to i_{q_1} \to \lambda_{q_1} \to \cdots \to i_{q_t} \to \lambda_{q_t}$, corresponding to a product of elements of Y, starting at i_n. We know that $N(\Lambda \setminus N(J)) \subseteq I \setminus J$, which implies that $i_{q_1} \in I \setminus J$. It follows that $\lambda_{q_1} \in \Lambda \setminus N(J)$ and so $i_{q_2} \in I \setminus J$. Continuing in this way, we see that $\lambda_{q_t} \in \Lambda \setminus N(J)$, giving $\lambda_{q_t} \in \Lambda \setminus N(J)$ and, as a consequence, $\lambda_{q_t} \neq \lambda_1$.

(iii) $\Rightarrow$ (i). Suppose that S satisfies the SHC. Then, in particular, S satisfies Hall's condition and, by Hall's marriage theorem, the graph $\Delta(P)$ has a perfect matching, $\pi : I \to \Lambda$. We claim that $M = \{(i, \pi(i)) : i \in I\}$ generates S. Let $(i, \lambda) \in S$ be arbitrary. We will show the existence of a path from i to λ with every second edge in M. Start with the path $i \to \pi(i)$. If $\pi(i) = \lambda$, we are done. Otherwise, by the SHC we know that $N(\{\pi(i)\}) > 1$, which allows us to choose $i_2 \in N(\{\pi(i)\}) \setminus \{i\}$ and extend our path to $i \to \pi(i) \to i_2 \to \pi(i_2)$. If $\pi(i_2) = \lambda$, we are done. Otherwise, by the SHC we know that

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$|N(\{\pi(i), \pi(i_2)\})| > 2$ and so there exists $i_3 \in N(\{\pi(i), \pi(i_2)\}) \setminus \{i, i_2\}$, to which we can extend the path. Continuing in this way, since $|I| = |\Lambda|$ is finite, we will eventually obtain a path from i to λ, as required. □

The above result can be extended to give the general non-square result. First we note that if we can find a 'subsquare' that satisfies the SHC, then we are done.

Lemma 2.11. *Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be an idempotent-generated completely 0-simple semigroup. If $\Delta(P)$ has a connected and balanced, bipartite subgraph on $2\min(|I|, |\Lambda|)$ vertices that satisfies the SHC, then*

$$\operatorname{id}\operatorname{rank}(S) = \operatorname{rank}(S) = \max(|I|, |\Lambda|).$$

Proof. Without loss of generality assume that $|I| \leqslant |\Lambda|$. As in the proof of Theorem 2.10, it is sufficient to prove the result for rectangular 0-bands. Let T be a rectangular 0-band with structure matrix P. Let $I \cup \Lambda'$ be a connected and balanced subgraph of $\Delta(P)$ that satisfies the SHC and let $\Lambda'' = \Lambda \setminus \Lambda'$. Let $\pi : I \to \Lambda'$ be a perfect matching corresponding to an idempotent basis of the sub-rectangular 0-band $I \times \Lambda'$ guaranteed to exist by Theorem 2.10. Note that $(I \times \Lambda') \cup \{0\}$ is indeed an idempotent-generated rectangular 0-band, since its structure matrix Q is connected. Define $\tau : \Lambda'' \to I$ so that $\{\tau(\lambda), \lambda\}$ is an edge in $\Delta(P)$ for each λ in Λ''; this is possible by regularity of P. We claim that $A = B_1 \cup B_2 = \{(i, \pi(i)) : i \in I\} \cup \{(\tau(\lambda), \lambda) : \lambda \in \Lambda''\}$ is an idempotent basis for T. Let $(i, \lambda) \in I \times \Lambda$ be arbitrary. If $\lambda \in \Lambda'$, then $(i, \lambda) \in \langle B_1 \rangle$. Otherwise $\lambda \in \Lambda''$, in which case $(i, \pi(\tau(\lambda))) \in \langle B_1 \rangle$ and $(\tau(\lambda), \lambda) \in B_2$. Multiplying these elements gives

$$(i, \lambda) = (i, \pi(\tau(\lambda)))(\tau(\lambda), \lambda) \in \langle A \rangle,$$

as required. □

It is not true that if a completely 0-simple semigroup has an idempotent basis, then we can necessarily find a subsquare that satisfies the SHC.

Example 2.12. Let S be the rectangular 0-band with underlying matrix

$$P = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

In this example the set $A = \{(1, 1), (2, 2), (3, 3), (4, 3)\}$ consists entirely of idempotents and generates S, while none of the sub-3×3 rectangular 0-bands of S have idempotent bases.

We now describe a process of extending non-square matrices to square matrices by making copies of existing rows (columns). By a *square extension* of an $m \times n$ matrix P, where $m \leqslant n$ say, we mean an $n \times n$ matrix Q in which the first m rows are the same as the m rows of P and each of the remaining rows is the same as one of these first m rows. More precisely, we have the following definition.

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Definition 2.13. Let $I = \{1, \dots, n\}$ and $\Lambda = \{1, \dots, m\}$ with $m \leqslant n$ and let $P = (p_{\lambda i})$ be an $m \times n$ matrix over $G \cup \{0\}$. Let $F = \{B_\lambda : \lambda \in \Lambda\}$ be a family of disjoint subsets of I such that $\lambda \in B_\lambda$ for all $\lambda \in \Lambda$ and $\bigcup_{\lambda \in \Lambda} B_\lambda = I$. Given such a partition F of I, define $\bar{f} : I \to \Lambda$ so that $i \in B_{\bar{f}(i)}$. By the square extension of P by F we mean the $n \times n$ matrix Q with entries $q_{xy} = p_{\bar{f}(x)y}$. We will use $Sq(P)$ to denote the set of all square extensions of the matrix P.

Example 2.14. If

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix},$$

then $Sq(P) = \{Q_1, Q_2\}$, where

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

In particular, if P is a square matrix, then $Sq(P) = \{P\}$.

Notation 2.15. With $I = \{1, \dots, m\}$, $\Lambda = \{1, \dots, n\}$, $A \subseteq I$ and $b \in \Lambda$ we define

$$(A, b) = \{(a, b) : a \in A\} \subseteq I \times \Lambda.$$

Theorem 2.16. *Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a finite idempotent-generated completely 0-simple semigroup. Then S has an idempotent basis if and only if $\Delta(Q)$ satisfies the SHC for some square extension Q of P.*

Proof. As in the proof of Theorem 2.10, it is sufficient to prove the result for rectangular 0-bands. Let $I = \{1, \dots, n\}$ and $\Lambda = \{1, \dots, m\}$ with $m \leqslant n$. Let S be a rectangular 0-band indexed by I and Λ with structure matrix $P = (p_{\lambda i})$ and idempotent basis

$$B = \{(A_1, 1), (A_2, 2), \dots, (A_m, m)\},$$

where $A_i \neq \varnothing$ for $1 \leqslant i \leqslant m$ and $\{1, \dots, n\}$ is the disjoint union of the sets A_i for $1 \leqslant i \leqslant m$. Also suppose, without loss of generality, that $j \in A_j$ for $1 \leqslant j \leqslant m$. Let Q be the square extension of P by F, where $F = \{A_1, \dots, A_m\}$. Let T be the rectangular 0-band defined by Q. We claim that T has an idempotent basis and thus the graph $\Delta(Q)$ satisfies the SHC. For $1 \leqslant i \leqslant m$, choose and fix a bijection $\phi_i : A_i \to A_i$. Since I is equal to the disjoint union $\bigcup_{i=1}^m A_i$ we can combine these functions together to give $\phi : I \to I$, where $\phi(x) = \phi_i(x)$ for $x \in A_i$. We claim that the set $X = \{(1, \phi(1)), \dots, (n, \phi(n))\}$ is a subset of $E(T)$ and that $\langle X \rangle = T$. For the first part note that if $x, y \in A_i$ for some $1 \leqslant i \leqslant m$, then

$$q_{xy} = p_{\bar{f}(x)y} = p_{iy} = p_{\bar{f}(y)y} = 1,$$

since $(y, \bar{f}(y))$ belongs to B and is therefore an idempotent. In particular, $q_{\phi(j)j} = 1$ for $1 \leqslant j \leqslant n$ since j and $\phi(j)$ both belong to the set $A_{\bar{f}(j)}$. It follows from this that $A_j \times A_j$

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is a subset of the idempotents of T for $1 \leqslant j \leqslant m$. Now we observe that $A_i \times A_i$ is contained in $\langle X\rangle$ for $1 \leqslant i \leqslant m$ since, given $(x,y) \in A_i \times A_i$, we have

$$(x,y) = (x,\phi_i(x))(\phi_i^{-1}(y),y) = (x,\phi_i(x))(\phi_i^{-1}(y),\phi_i(\phi_i^{-1}(y))),$$

where $q_{\phi_i^{-1}(y)\phi_i(x)} = 1$ since $\phi_i(x), \phi_i^{-1}(y) \in A_i$. Given any element (x,y) of T, the subset $\{y\}$ of $\{1,\dots,n\}$ can be extended to a transversal J of the sets $A_1,\dots,A_m$. Consider the subset $I \times J$ of T. By construction, $\{I \times J\} \cup \{0\}$ is isomorphic to S and since B generates S it follows that

$$\langle((A_1 \times A_1) \cup \dots \cup (A_m \times A_m)) \cap (I \times J)\rangle = \{I \times J\} \cup \{0\}.$$

We conclude that

$$(x,y) \in I \times J \subseteq \langle((A_1 \times A_1) \cup \dots \cup (A_m \times A_m))\rangle \subseteq \langle X\rangle,$$

and since (x,y) was arbitrary it follows that X generates T as required. For the converse let Q be a square extension of P by $F = \{B_1,\dots,B_m\}$, where $\Delta(Q)$ satisfies the SHC. Let T be the rectangular 0-band defined by Q. By Theorem 2.10 we know that T has an idempotent basis. Let $\tau \in S_n$ so that $Y = \{(1,\tau(1)),\dots,(n,\tau(n))\}$ is an idempotent basis of T. Define a map $\psi : T \to S$ by $\psi((i,\lambda)) = (i,\bar{f}(\lambda))$ and $\psi(0) = 0$. It is an easy exercise to check that ψ is an onto homomorphism and that $|\psi(Y)| = n$. Since $Y \subseteq E(T)$ and ψ is a homomorphism, it follows that $\psi(Y) \subseteq E(S)$. Also, since $\langle Y\rangle = T$ and ψ is a homomorphism, it follows that $\langle\psi(Y)\rangle = S$ with $|\psi(Y)| = n$ and therefore S has an idempotent basis as required. □

3. Useful corollaries

The result of the previous section gives necessary and sufficient conditions for an idempotent-generated completely 0-simple semigroup to have an idempotent basis. Unfortunately, testing the SHC directly is a time-consuming business since it involves looking at all the subsets of a set. In practice it will be useful for us to note a few conditions that are slightly stronger than the SHC but that are easier to check.

Lemma 3.1. *If $\Gamma = X \cup Y$ is a k-regular, connected, balanced bipartite graph, then Γ satisfies the SHC.*

Proof. The number of edges adjacent to the vertices of X and $N(X)$ are $k|X|$ and $k|N(X)|$, respectively. The set of edges adjacent to X is a subset of the edges that are adjacent to $N(X)$ giving $k|N(X)| \geqslant k|X|$, which implies that $|N(X)| \geqslant |X|$. When $|N(X)| = |X|$, the edges that are adjacent to X are precisely those that are adjacent to $N(X)$. It follows that $X \cup N(X)$ is a connected component of Γ, which, since Γ is connected, implies that $|X| = |I|$. □

Definition 3.2. We say that a balanced bipartite graph $\Gamma = X \cup Y$ with a perfect matching $\pi : X \to Y$ (a bijection such that $\{x,\pi(x)\} \in E(\Gamma)$ for every x in X) has a *symmetric* distribution of edges with respect to the matching π if $d(x) = d(\pi(x))$ for every x in X.

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Note that if $\Gamma = X \cup Y$ is k-regular and connected, then Γ satisfies the HC, which implies that Γ has a perfect matching and clearly has a symmetric distribution of idempotents with respect to this matching. Thus, Γ being k-regular and connected is a stronger property than Γ having a symmetric distribution of idempotents with respect to some perfect matching.

Lemma 3.3. *If $\Gamma = X \cup Y$ is a connected and balanced bipartite graph that has a symmetric distribution of edges with respect to some perfect matching $\pi : X \to Y$, then Γ satisfies the SHC.*

Proof. Let A be a non-empty (not necessarily proper) subset of X. Since $|\pi(A)| = |A|$ and $\pi(A) \subseteq N(A)$, it follows that $|N(A)| \geqslant |\pi(A)| = |A|$. In addition to this if $|N(A)| = |A|$, then $|N(A)| = |\pi(A)|$, which means that $N(A) = \pi(A)$. But the total number of edges adjacent to A is $\sum_{a\in A} d(a)$, while the total number of edges adjacent to $N(A) = \pi(A)$ is

$$\sum_{b\in\pi(A)} d(b) = \sum_{a\in A} d(\pi(a)) = \sum_{a\in A} d(a).$$

It follows that $A \cup N(A) = A \cup \pi(A)$ is a connected component of $X \cup Y$, but, since Γ is connected, this means that $A = X$ and we are done. $\square$

Definition 3.4. We say that $S = \mathcal{M}^0[G; I, \Lambda; P]$ has a k-uniform distribution of idempotents if the graph $\Delta(P)$ is k-regular. We say that $S = \mathcal{M}^0[G; I, \Lambda; P]$ has a symmetric distribution of idempotents with respect to some perfect matching if $\Delta(P)$ has a symmetric distribution of edges with respect to some perfect matching.

4. Applications

4.1. Ideals of the full transformation semigroup

Our first application of the results of the previous two sections is to prove that the semigroup $K(n,r)$ has an idempotent basis. The original proof [**13**] used a fairly complicated Pascal's triangle type induction. It is known that $K(n,r)$ is generated by the elements in its top $\mathcal{J}$-class (see, for example, [**5**, Lemma 2.2]).

Lemma 4.1. *Let J_r denote the top $\mathcal{J}$-class of the semigroup $K(n,r)$ and $P_r = K(n,r)/K(n,r-1)$ for $1 < r < n$. Then*

$$\operatorname{rank}(K(n,r)) = \operatorname{rank}(P_r), \qquad \operatorname{id}\operatorname{rank}(K(n,r)) = \operatorname{id}\operatorname{rank}(P_r).$$

Our aim is to prove that the $\bigl(\binom{n}{r}, S(n,r)\bigr)$ bipartite graph $\Delta(P_r)$ (with $1 < r < n$) has a balanced $\bigl(\binom{n}{r}, \binom{n}{r}\bigr)$ bipartite subgraph that is connected and has a perfect matching $\pi : X \to Y$ with respect to which S has a symmetric distribution of idempotents. Then, as a consequence of Theorem 2.10 and Lemma 3.3, it will follow that $K(n,r)$ has an idempotent basis. In order to prove this result we need to consider $\Delta(P_r)$ in more detail.

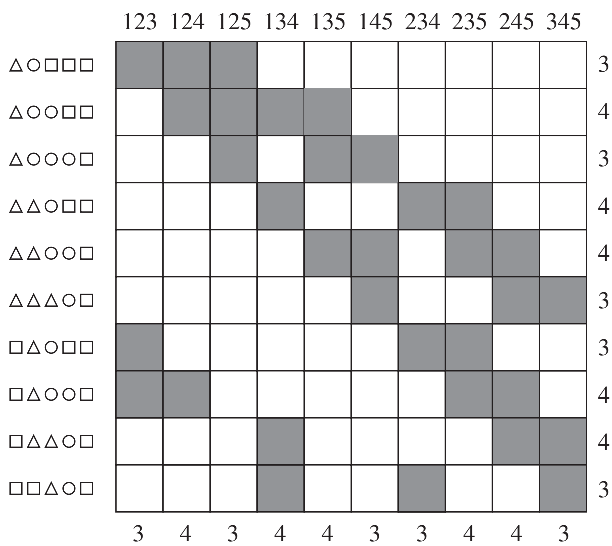

Figure 1. The symmetric distribution of idempotents with respect to the perfect matching ϕ corresponding to the idempotents along the main diagonal of a subsquare of J_3 in T_5. The kernel labelling row i is $\phi(I)$, where I is the image labelling column i. The shaded boxes give the positions of the idempotents.

Notation 4.2. We will use $\mathcal{F}_r$ to denote the family of all subsets of $X_n = \{1, \dots, n\}$ with size r

$$\mathcal{F}_r = \{A \subseteq X_n : |A| = r\}.$$

We will use $\mathcal{K}_r \subseteq X_n \times X_n$ to denote the family of all partitions of X_n into r classes.

The graph $\Delta(P_r)$ has vertex set $\mathcal{F}_r \cup \mathcal{K}_r$ (disjoint union) with $A \in \mathcal{F}_r$ connected to $K \in \mathcal{K}_r$ if and only if A is a transversal of the classes of K. We are mainly concerned with subsets and partitions of $X_n = \{1, \dots, n\}$ and will want to use modular arithmetic on these symbols rather than the usual $\{0, \dots, n-1\}$. In view of this fact we will use the convention that $n \bmod n = n$ rather than $n \bmod n = 0$. Unless otherwise stated, this is the way in which we will use modular arithmetic.

Definition 4.3. Given $a, b \in X_n$ we define

$$[a, b] = \{a, a+1, a+2, \dots, b\},$$

with all entries reduced $\bmod n$. We will call $[a, b]$ the *interval* between a and b.

Example 4.4. Given $2, 4 \in X_5$, we have $[4, 2] = \{4, 5, 1, 2\}$, while $[2, 4] = \{2, 3, 4\}$ and $[1, 1] = \{1\}$.

Definition 4.5. Define $\phi : \mathcal{F}_r \to \mathcal{K}_r$ by

$$\phi(I) = \{[i_1, i_2 - 1], [i_2, i_3 - 1], \dots, [i_r, i_1 - 1]\}$$

with all entries reduced $\bmod n$ and where $I = \{i_1, \dots, i_r\} \subseteq X_n$ and $i_1 < \dots < i_r$.

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It is clear from the definition that the map ϕ is injective. It associates (in a natural way) a kernel (with r classes) to each r-subset of X_n (see, for example, figure 1).

Lemma 4.6. *The balanced bipartite graph $\Gamma' = \mathcal{F}_r \cup \phi(\mathcal{F}_r)$ is connected and has a symmetric distribution of edges with respect to the perfect matching ϕ.*

Proof. Firstly, we will show that the graph is connected. Note that by definition each $A \in \mathcal{F}_r$ is connected to the corresponding $\phi(A) \in \mathcal{K}_r$ and as a consequence the graph Γ' is connected precisely when the homomorphic image given by collapsing the pairs $\{A, \phi(A)\}$ to single vertices is connected. Let G denote this new graph, which is made up of $\binom{n}{r}$ vertices labelled by the r-subsets of X_n. In particular, the vertices corresponding to the subsets A and B are connected if B is obtained by adding the number 1 to one of the entries of A. As a consequence of this we can show that an arbitrary r-set $I = \{i_1, \dots, i_r\}$ with $i_1 < \dots < i_r$ is connected to $\{1, 2, \dots, r\}$, since (with the symbol '$\sim$' to be read as 'is connected to')

$$\begin{aligned}
\{1, \dots r-2, r-1, r\} &\sim \{1, \dots, r-2, r-1, r+1\} \sim \dots \sim \{1, \dots r-2, r-1, i_r\} \\
&\sim \{1, \dots, r-2, r, i_r\} \qquad\qquad \sim \dots \sim \{1, \dots r-2, i_{r-1}, i_r\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad \sim \dots \sim \{1, \dots i_{r-2}, i_{r-1}, i_r\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \ddots \qquad\qquad \vdots \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \sim \{i_1, \dots i_{r-2}, i_{r-1}, i_r\}.
\end{aligned}$$

Therefore, every vertex is connected to the one labelled with $\{1, \dots, r\}$. We conclude that the graph G is connected and thus the graph Γ' is connected.

Secondly, we have to check that Γ' has a symmetric distribution of idempotents with respect to the perfect matching ϕ (i.e. that for every $A \in \mathcal{F}_r$ we have $d(A) = d(\phi(A))$). On the one hand if we fix the partition $K = \{[i_1, i_2 - 1], [i_2, i_3 - 1], \dots, [i_r, i_1 - 1]\}$, then the number of images of size r that form a transversal of this partition is equal to the product $\prod_{j=1}^{r} |[i_j, i_{j+1} - 1]|$. On the other hand, if we fix the image $I = \{i_1, \dots, i_r\}$, with $i_1 < i_2 < \dots < i_r$, and consider the partitions in $\phi(\mathcal{K}_r)$ of which this image is a transversal, we see that I is a transversal of $\{[j_1, j_2 - 1], [j_2, j_3 - 1], \dots, [j_r, j_1 - 1]\}$, with $i_1 \in [j_1, j_2 - 1]$, say, if and only if $j_l \in [i_{l-1} + 1, i_l]$ (subscripts reduced $\bmod r$) for $1 \leqslant l \leqslant r$. We conclude that

$$d(\phi(A)) = \prod_{j=1}^{r} |[i_j, i_{j+1} - 1]| = \prod_{j=1}^{r} |[i_j + 1, i_{j+1}]| = d(A),$$

where all subscripts are reduced $\bmod r$. $\square$

Theorem 4.7 (Howie and McFadden [13, Theorem 5]). *The semigroup*

$$K(n, r) = \{\alpha \in T_n : |\mathrm{Im}(\alpha)| \leqslant r\}$$

satisfies

$$\mathrm{id\,rank}(K(n, r)) = \mathrm{rank}(K(n, r)) = S(n, r)$$

for $1 < r < n$.

Proof. It follows from Lemmas 3.3 and 4.6 that P_r has an idempotent basis and then, as a consequence of Lemma 4.1, so does $K(n,r)$. $\square$

4.2. Ideals of the endomorphism monoid of a finite vector space

Now we apply our results to give an analogous result to Theorem 4.7 for the proper ideals of the monoid of endomorphisms of a finite vector space. In this case the $\mathcal{J}$-classes are square, allowing us to apply Theorem 2.10 directly. Let V be an n-dimensional vector space over a field F with $|F| = q$. Let $\mathrm{End}(V)$ denote the monoid of all linear transformations of V which can be represented concretely as the monoid of all $n \times n$ matrices with entries in the field F. The ideals of $\mathrm{End}(V)$ are given by

$$I(r,n,q) = \{A \in \mathrm{End}(V) : \dim(\mathrm{Im}(A)) \leqslant r\} \quad \text{for } 1 \leqslant r \leqslant n.$$

We denote the top $\mathcal{J}$-class of this subsemigroup, the $\mathcal{J}$-class consisting of all linear maps with $\dim(\mathrm{Im}(A)) = r$, by $J(r,n,q)$ and the completely 0-simple semigroup $I(r,n,q)/I(r-1,n,q)$ by $PF(r,n,q)^0$. A summary of some elementary properties of $\mathrm{End}(V)$ is given in the following lemma.

Lemma 4.8. *Let V be an n-dimensional vector space over the finite field F where $|F| = q$.*

(i) *Green's relations are given by*

$$\begin{aligned} A\mathcal{R}B &\iff \mathrm{Null}(A) = \mathrm{Null}(B), \\ A\mathcal{L}B &\iff \mathrm{Im}(A) = \mathrm{Im}(B), \\ A\mathcal{J}B &\iff \dim(\mathrm{Im}(A)) = \dim(\mathrm{Im}(B)). \end{aligned}$$

(ii) *The number of non-zero $\mathcal{L}$-classes ($\mathcal{R}$-classes) in $J(r,n,q)$ is $\left[{n \atop r}\right]_q$, where*

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-r+1}-1)}{(q^r-1)(q^{r-1}-1)\cdots(q-1)}.$$

(iii) *The number of idempotents in any non-zero $\mathcal{L}$-class ($\mathcal{R}$-class) of $J(r,n,q)$ is $q^{r(n-r)}$.*

(iv) *The semigroup $I(r,n,q)$ is idempotent generated and in particular is generated by the idempotents in its top $\mathcal{J}$-class $J(r,n,q)$.*

Proof. (i) See [**2**, Exercise 2.2.6] for the proof.

(ii) This is the formula for the number of r-dimensional (and also $(n-r)$-dimensional) subspaces of an n-dimensional vector space over a field F with size q.

(iii) Idempotents appear in $\mathcal{H}$-classes H_A, where A satisfies $\mathrm{Im}(A) \oplus \mathrm{Null}(A) = V$. We are therefore counting the number of $(n-r)$-dimensional subspaces that are orthogonal to some fixed r-dimensional subspace. This number equals $q^{r(n-r)}$.

(iv) This is a consequence of [**5**, Lemma 2.2]. $\square$

These facts along with the general results from §§ 2 and 3 allow us, immediately, to deduce the following.

Theorem 4.9. *Let V be an n-dimensional vector space over the finite field F where $|F| = q$. Then the semigroup $I(r, n, q) = \{A \in \mathrm{End}(V) : \dim(\mathrm{Im}(A)) \leqslant r\}$ satisfies*

$$\mathrm{rank}(I(r,n,q)) = \mathrm{id}\,\mathrm{rank}(I(r,n,q)) = \begin{bmatrix} n \\ r \end{bmatrix}_q$$

for $1 \leqslant r < n$.

Proof. By Lemma 4.8 the completely 0-simple semigroup $PF(r,n,q)^0$ has a $q^{r(n-r)}$-uniform distribution of idempotents which, by Lemma 3.1, tells us that $PF(r,n,q)^0$ has an idempotent basis, which in turn implies that $I(r,n,q)$ has an idempotent basis. □

Also, as a by-product of the above discussion we get the following result which characterizes generating sets of minimum cardinality of the semigroup $I(r,n,q)$.

Corollary 4.10. *A subset of $I(r,n,q)$ with size $[{n \atop r}]_q$ is a generating set for $I(r,n,q)$ if and only if it consists of matrices all of whose images have dimension r no two of which have the same null space or the same image space.*

Proof. The completely 0-simple semigroup $PF(r,n,q)^0$ is square and has an idempotent basis. By Theorem 2.10, any sparse cover of $PF(r,n,q)^0$ generates $PF(r,n,q)^0$ which is equivalent to the statement in the corollary. □

We have seen that the graph $\Delta(PF(r,n,q)^0)$ satisfies the SHC for all $1 \leqslant r < n$. In fact, for the particular case when $r = n-1$, we can show that the graph $\Delta(PF(n-1,n,q)^0)$ is Hamiltonian. The first step is to verify that $2q^{n-1} > (q^n - 1)/(q-1)$ (this follows from the fact that $q \geqslant 2$), which tells us that in the graph $\Delta(PF(n-1,n,q)^0)$ every vertex has degree strictly greater than $|I|/2$. Then consider the following result, which gives a sufficient condition for a bipartite graph to have a Hamiltonian circuit.

Theorem 4.11 (Moon and Moser [14]). *If $G = (X, Y)$ is a bipartite graph with $|X| = |Y| = n$ such that, for any non-adjacent pair of vertices, $(x, y) \in V(X) \times V(Y)$ satisfies $d(x) + d(y) \geqslant n + 1$, then G has a Hamiltonian cycle.*

From this we conclude that $\Delta(PF(n-1,n,q)^0)$ is Hamiltonian. Note that this method cannot be used to prove the same thing for graphs associated with principal factors that are lower down in the semigroup.

Example 4.12 (the semigroup $I(2,4,2)$). Let $V = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and consider the semigroup PF_2^0. Then, according to Lemma 4.8, the graph $\Delta(P)$ is a bipartite graph with $|I| = |\Lambda| = 35$ and each vertex has degree 16. In this example we cannot show that the graph is Hamiltonian by applying Moon and Moser's result.

This leaves open the following question: is $\Delta(PF(r,n,q)^0)$ Hamiltonian for all $1 \leqslant r < n$?

We have seen that it is true when $r = n-1$ and in [**1**] the author shows that it is true for some other special cases. Also note that the information given in Lemma 4.8 is not, on its own, sufficient to guarantee a Hamiltonian circuit. It is not true in general that every connected, k-regular bipartite graph necessarily has a Hamiltonian circuit. In fact, even stronger than this, there is an example of a 3-connected (a stronger condition than being connected), 3-regular bipartite graph that is non-Hamiltonian (see, for example, [**4**]).

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