DISCONTINUOUS HOMOMORPHISMS OF $C(X)$ WITH $2^{\aleph_0} > \aleph_2$

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Abstract. Assume that M is a transitive model of $ZFC + CH$ containing a simplified $(\omega_1, 2)$-morass, $P \in M$ is the poset adding $\aleph_3$ generic reals and G is P-generic over M. In M we construct a function between sets of terms in the forcing language, that interpreted in $M[G]$ is an $\mathbb{R}$-linear order-preserving monomorphism from the finite elements of an ultrapower of the reals, over a non-principal ultrafilter on ω, into the Esterle algebra of formal power series. Therefore it is consistent that $2^{\aleph_0} > \aleph_2$ and, for any infinite compact Hausdorff space X, there exists a discontinuous homomorphism of $C(X)$, the algebra of continuous real-valued functions on X.

§1. Introduction. This paper addresses Kaplansky's conjecture in the theory of Banach algebras concerning the existence of discontinuous homomorphisms of $C(X)$, the algebra of continuous real-valued functions with domain X, where X is an infinite compact Hausdorff space. This problem sits squarely in the extensive history of the question of automatic continuity—whether algebraic tameness implies topological tameness. Among the open questions in this area of research is whether the existence of a discontinuous homomorphism of $C(X)$ implies that the cardinality of the continuum is at most $\aleph_2$. We prove in this paper that the existence of a discontinuous homomorphism of $C(X)$ does not imply that $2^{\aleph_0} \leq \aleph_2$.

In [10], B. Johnson proved that there is a discontinuous homomorphism of $C(X)$ provided that there is a nontrivial submultiplicative norm on the finite elements of an ultrapower of $\mathbb{R}$ over ω. In [7], J. Esterle constructs an algebra of formal power series, $\mathcal{E}$, and shows in [6] that the infinitesimal elements of $\mathcal{E}$ admit a nontrivial submultiplicative norm. By results of Esterle in [8], it is known that $\mathcal{E}$ is an η_1-ordering of cardinality $2^{\aleph_0}$. Furthermore, $\mathcal{E}$ is a totally ordered field by a result of Hahn in 1907 [9], and is real-closed by a result of Maclane [11].

It is a theorem of P. Erdös, L. Gillman, and M. Henriksen in [5] that any pair of η_1-ordered real-closed fields of cardinality $\aleph_1$ are isomorphic as ordered fields.[1] In fact, it is shown using a back-and-forth argument that any order-preserving field isomorphism between countable subsets of η_1-ordered real-closed fields may be extended to an order-isomorphism. It is a standard result of model

[1]An η_1- ordered field, F, is one for which any countable gap, (A, B) has a witness (that is, if $A, B \subseteq F, |A| \leq \aleph_0 |B| \leq \aleph_0$ and $(\forall x \in A, y \in B)\ x < y$, then there is $z \in F$ such that $(\forall x \in A)\ x < z$ and $(\forall y \in B)\ z < y$. If F is a real-closed field it is an η_1-ordered field if and only if it is $\aleph_1$-saturated.

theory that for any non-principal ultrafilter U on ω, $\mathbb{R}^\omega/U$ is an $\aleph_1$-saturated real-closed field (and hence an η_1-ordering). By a result of Johnson [10], between any pair of η_1-ordered real-closed fields of cardinality $\aleph_1$ there is an $\mathbb{R}$-linear order-preserving field isomorphism (hereafter $\mathbb{R}$-isomorphism). This implies, in a model of the continuum hypothesis (CH), that there is an $\mathbb{R}$-linear order-preserving monomorphism (hereafter $\mathbb{R}$-monomorphism) from the finite elements of $\mathbb{R}^\omega/U$ to $\mathcal{E}$, and hence in models of ZFC+CH there exists a discontinuous homomorphism of $C(X)$. The proof that in a model of ZFC+CH there exists a discontinuous homomorphism of $C(X)$ is due independently to Dales [1, 2] and Esterle [6].

Shortly thereafter R. Solovay found a model of ZFC+¬CH in which all homomorphisms of $C(X)$ are continuous. Later, in his Ph.D. thesis, W.H. Woodin constructed a model of ZFC+Martin's Axiom in which all homomorphisms of $C(X)$ are continuous. This naturally gave rise to the question of whether there is a model of set theory in which CH fails and there is a discontinuous homomorphism of $C(X)$. Woodin subsequently showed that in the Cohen extension of a model of ZFC+CH by generic reals indexed by ω_2, there is a discontinuous homomorphism of $C(X)$ [15]. Woodin shows that in this model the gaps in $\mathcal{E}$ that must be witnessed in a classical back-and-forth construction are always countable. He observes that this construction may not be extended to a Cohen extension by more than $\aleph_2$ generic reals. He suggests the plausibility of using morasses to construct an $\mathbb{R}$-monomorphism from the finite elements of an ultrapower of the reals to the Esterle algebra in generic extensions with more than $\aleph_2$ generic reals. Woodin's argument does not extend to higher powers of the continuum and leaves open the question of whether there exists a discontinuous homomorphism of $C(X)$ in models of set theory in which $2^{\aleph_0} > \aleph_2$. In this paper we show that the existence of a simplified $(\omega_1, 2)$-morass in a model of $ZFC + CH$ is sufficient for the existence of a discontinuous homomorphism of $C(X)$ in a model in which $2^{\aleph_0} = \aleph_3$.

We show that in the Cohen extension adding $\aleph_2$ generic reals to a transitive model of $ZFC + CH$ containing a simplified $(\omega_1, 1)$-morass, there is a level, morass-commutative term function that, interpreted in the Cohen extension, is an $\mathbb{R}$-monomorphism of the finite elements of an ultrapower of $\mathbb{R}$ over ω into the Esterle algebra. This is achieved with a transfinite construction of length ω_1, utilizing the morass functions from the gap-one morass to complete the construction of size $\aleph_2$ by commutativity with morass maps. Using the techniques of this argument, we construct a term function with a transfinite argument of length ω_1 and utilize morass-commutativity with the embeddings of a gap-2 morass to complete the construction of an $\mathbb{R}$-monomorphism from the finite elements of a standard ultrapower of $\mathbb{R}$ over ω to the Esterle algebra in the Cohen extension adding $\aleph_3$ generic reals.

The technical obstacles to such a construction may be reduced to conditions we call morass-extendibility. This paper is dependent on the results of [3] and [4], in which term functions are constructed that are forced to be order-preserving functions. In this paper we construct a term function that is forced to be order-preserving and is simultaneously an $\mathbb{R}$-linear ring-monomorphism.

§2. Preliminaries. In our initial construction we use a simplified $(\omega_1, 1)$-morass. We construct a function on terms in the forcing language for adding $\aleph_2$ generic reals that is forced, in all generic extensions, to be an $\mathbb{R}$-monomorphism from

the finite elements of $\mathbb{R}^\omega/U$, where U is a a non-principal ultrafilter meeting certain technical conditions, into the Esterle algebra, $\mathcal{E}$. In some sense we follow the classical route to such constructions—extension by transcendental elements in an inductive construction of length ω_1. We will require commutativity with morass maps to construct a function on a domain of cardinality $\aleph_2$ making only $\aleph_1$ many explicit commitments. However with each commitment of the construction, there are uncountably many future commitments implied by commutativity with morass maps.

In [13] D. Velleman defines a simplified $(\omega_1, 1)$-morass.

DEFINITION 2.1 (Velleman)(Simplified$(\omega_1, 1)$-morass). A simplified $(\omega, 1)$-morass is a structure

$$\mathcal{M} = \langle (\theta_\alpha \mid \alpha \leq \omega_1), (\mathcal{F}_{\alpha\beta} \mid \alpha < \beta \leq \omega_1) \rangle$$

that satisfies the following conditions:
(P0) (a) $\theta_0 = 1$, $\theta_{\omega_1} = \omega_2$, $(\forall \alpha < \omega_1)\, 0 < \theta_\alpha < \omega_1$.
(b) $\mathcal{F}_{\alpha\beta}$ is a set of order-preserving functions $f : \theta_\alpha \to \theta_\beta$.
(P1) $\left| \mathcal{F}_{\alpha\beta} \right| \leq \omega$ for all $\alpha < \beta < \omega_1$.
(P2) If $\alpha < \beta < \gamma$, then $\mathcal{F}_{\alpha\gamma} = \{ f \circ g \mid f \in \mathcal{F}_{\beta\gamma}, g \in \mathcal{F}_{\alpha\beta} \}$.
(P3) If $\alpha < \omega_1$, then $\mathcal{F}_{\alpha(\alpha+1)} = \{ \text{id} \restriction \theta_\alpha, f_\alpha \}$ where f_α satisfies:

$$(\exists \delta_\alpha < \theta_\alpha)\, f_\alpha \restriction \delta_\alpha = \text{id} \restriction \delta_\alpha \ \text{ and } \ f_\alpha(\delta_\alpha) \geq \theta_\alpha.$$

(P4) If $\alpha \leq \omega_1$ is a limit ordinal, $\beta_1, \beta_2 < \alpha$, $f_1 \in \mathcal{F}_{\beta_1\alpha}$ and $f_2 \in \mathcal{F}_{\beta_2\alpha}$, then there is $\gamma < \alpha$, $\gamma > \beta_1, \beta_2$, and there is $f_1' \in \mathcal{F}_{\beta_1\gamma}$, $f_2' \in \mathcal{F}_{\beta_2\gamma}$, $g \in \mathcal{F}_{\gamma\alpha}$ such that $f_1 = g \circ f_1'$ and $f_2 = g \circ f_2'$.
(P5) For all $\alpha > 0$, $\theta_\alpha = \bigcup \{ f[\theta_\beta] \mid \beta < \alpha, f \in \mathcal{F}_{\beta\alpha} \}$.

Simplified gap-1 morasses, as well as higher gap simplified morasses, are known to exist in L.

We will construct, by an inductive argument of length ω_1, a function between sets of terms in the forcing language adding $\aleph_2$ generic reals. We interpret the morass functions on ordinals as functions between terms in the forcing language and require that the set of terms under construction satisfy certain commutativity constraints with the morass functions. It is implicit that any commitment to an ordered pair of terms in the construction is *de facto* a commitment to uncountably many commitments to ordered pairs in mutually generic extensions. In [3] we worked explicitly with terms in the forcing language. We wish to simplify the details of the construction by working with objects in a forcing extension, where that is possible.

NOTATION 2.2 $(P(A))$. *If A is a set of ordinals, we let $P(A)$ be the poset adding generic reals indexed by the ordinals of A. That is,*

$$P(A) := Fn(A \times \omega, 2),$$

the finite partial functions from $A \times \omega$ to 2.

NOTATION 2.3 (P_v). *If $\mathcal{M}$ is a simplified $(\omega_1, 1) - $morass, and $v \leq \omega_1$, we let P_v be the poset that adds generic reals indexed by θ_v(the ordinal associated with the vertex, v, in $\mathcal{M}$).*

Let M be a transitive model of ZFC, β be an ordinal and P be the poset adding generic reals indexed by β, then $P(\beta) = Fn(\beta \times \omega, 2)$.

We use the notion of strict level of a term in the forcing language defined in [3], and apply it to objects in a forcing extension.

DEFINITION 2.4 (Strict level). Let $\alpha \leq \beta \leq \omega_2$ be ordinals and $\tau \in M^{P(\beta)}$ be a term in the forcing language adding generic reals indexed by β. Then τ has strict level α provided that:

1. There is a term $\bar{\tau} \in M^{P(\alpha)}$ such that $\Vdash \bar{\tau} = \tau$.
2. For any $\gamma < \alpha$, and term, $\bar{\tau} \in M^{P(\gamma)}$, $\Vdash \tau \neq \bar{\tau}$.

Alternatively, for $\gamma < \alpha < \beta$, we consider $P(\beta)$ as the product forcing $P(\gamma) \times P(\alpha \setminus \gamma) \times P(\beta \setminus \alpha)$. Suppose $G(\gamma)$ is $P(\gamma)$-generic over M and G is $P(\alpha \setminus \gamma)$-generic over $M[G(\gamma)]$, and H is $P(\beta \setminus \alpha)$-generic over $M[G(\gamma), G]$. We say that an object in a forcing extension, $a \in M[G(\gamma), G, H]$, has strict level α if and only if there is a $\tau \in M^{P(\alpha)}$ with strict level α, such that $\tau_{G(\gamma)G} = a$. If $a \in M[G(\gamma), G, H]$ has strict level α, then for any $\gamma < \alpha$, $a \notin M[G(\gamma)]$.

Not every term in the forcing language has strict level. However, every object in a forcing extension is the interpretation of a term of strict level. Consequently in our construction we pass freely between objects of strict level α in a generic extension and terms of $M^{P(\alpha)}$.

Many of the constraints required for commutativity with morass maps are expressed in terms of the strict level of objects in a forcing extension (or correspondingly, terms in a forcing language). For instance, in Section 5 we define a term function to be level if the strict level of any term in the domain equals the strict level of its image under the function. Such maps will commute with morass maps in the manner required by our construction.

§3. Constructing an $\mathbb{R}$-monomorphism on a real-closed field.

We wish to construct a function between sets of terms in the forcing language for adding $\aleph_2$ generic reals, that is forced in all generic extensions to be an order-preserving $\mathbb{R}$-linear monomorphism from the finite elements of an non-principal ultrapower of $\mathbb{R}$, $\mathbb{R}^\omega / U$ to $\mathcal{E}$.

DEFINITION 3.1 (Archimedean valuation). If x and y are non-zero elements of a real-closed field, they have the same Archimedean valuation, $x \sim y$, provided that there are $m, n \in \mathbb{N}$ such that

$$|x| < n \cdot |y|$$

and

$$|y| < m \cdot |x|.$$

If $|x| < |y|$ and $x \nsim y$, then x has Archimedean valuation greater than y.

Archimedean valuation induces an equivalence relation on the non-zero elements of a real-closed field (RCF). The non-zero real numbers have the same valuation. Elements with the same valuation as a real number are said to have real valuation. In a non-standard real-closed field, elements with valuation greater than a real

valuation are infinitesimal. The finite elements of a real closed field are the infinitesimal elements and those with real valuation.

At any stage of the construction, we will have defined a partial function on $\mathbb{R}^\omega / U$, so we extend the definition of an $\mathbb{R}$-monomorphism to include a partial function that may not have all real numbers in its domain.

DEFINITION 3.2 ($\mathbb{R}$-monomorphism). Let X and Y be subrings of real-closed fields that contain a real-closed subfield of $\mathbb{R}$, and let $\mathbb{R}_0 = \mathbb{R} \cap X = \mathbb{R} \cap Y$. A function, $\phi : X \to Y$, is an $\mathbb{R}$-monomorphism if it is an order-preserving ring-monomorphism such that $\phi \restriction_{\mathbb{R}_0} = id \restriction_{\mathbb{R}_0}$.

It is a result of B. Johnson [10] that η_1-ordered real-closed fields with cardinality $\aleph_1$ are $\mathbb{R}$-isomorphic in models of ZFC+CH. This result strengthens the classical result that $\aleph_1$-saturated real closed fields of cardinality $\aleph_1$ are isomorphic. We review some of the basic results from this work, which we require for extension of these results to higher powers of the continuum.

DEFINITION 3.3 (Full subfield). Let D^* be a field extending $\mathbb{R}$. A subfield $D \subseteq D^*$ is full iff for every finite element, $r + \delta \in D$, where $r \in \mathbb{R}$ and δ is infinitesimal, $r \in D$.

We will need to extend two results due to B. Johnson [10] to meet the requirement of $\mathbb{R}$-linearity in the context of constructing term functions using a morass.

LEMMA 3.4 (B. Johnson). *Assume that D and I are full real-closed subfields of η_1-ordered real-closed fields $D^*(\supseteq \mathbb{R})$ and $I^*(\supseteq \mathbb{R})$ (resp.), $\phi : D \to I$ is an $\mathbb{R}$-monomorphism, and $r \in \mathbb{R}$. Then there is an extension of ϕ, ϕ^*, that is an $\mathbb{R}$-monomorphism of the real closure of the field generated by D and r, $F(D, r)$, onto the real closure of the field generated by I and r, $F(I, r)$. Furthermore $F(D, r)$ (and consequently, $F(I, r)$) is full.*

LEMMA 3.5 (B. Johnson). *Let D, D^*, I, I^* and ϕ be as in Lemma 3.4, $x \in D^*$ and assume that the real closure of the field generated by D and x, $F(D, x)$, is full. Let $y \in I^*$ be such that*

$$(\forall d \in D)(d < x \iff \phi(d) < y).$$

Then there is an $\mathbb{R}$-monomorphism extending ϕ, $\phi^* : F(D, x) \to I^*$, such that $\phi^*(x) = y$.

§4. The Esterle algebra.

We define the Esterle algebra [6] and review some basic properties.

DEFINITION 4.1 (S_{ω_1}). S_{ω_1} is the lexicographic linear-ordering with domain $\{s : \omega_1 \to 2 \mid s$ has countable support and the support of s has a largest element $\}$.

DEFINITION 4.2 (G_{ω_1}). G_{ω_1} is the ordered group with domain $\{g : S_{\omega_1} \to \mathbb{R} \mid g$ has countable well-ordered support$\}$, lexicographic ordering, and group operation pointwise addition.

We define an ordered algebra of formal power series, $\mathcal{E}$. The universe of $\mathcal{E}$ is the set of formal power series, $\sum_{\lambda < \gamma} \alpha_\lambda x^{a_\lambda}$, where:

1. $\gamma < \omega_1$.
2. $(\forall \lambda < \gamma) \, \alpha_\lambda \in \mathbb{R}$.
3. $\{a_\lambda \mid \lambda < \gamma\}$ is a countable well-ordered subset of G_{ω_1} and $\lambda_1 < \lambda_2 < \gamma \Rightarrow$ $a_{\lambda_1} < a_{\lambda_2}$.

The ordered algebra, $\mathcal{E}$, is isomorphic to the set of functions, with countable well-ordered support, from G_{ω_1} to $\mathbb{R}$. The lexicographic order linearly-orders $\mathcal{E}$. Addition is pointwise and multiplication is defined as follows: Suppose $a = \sum_{\lambda < \gamma_1} \alpha_\lambda x^{a_\lambda}$ and $b = \sum_{\kappa < \lambda_2} \beta_\kappa x^{b_\kappa}$ are members of $\mathcal{E}$. Let

$$C = \{c \mid (\exists \lambda < \gamma_1)(\exists \kappa < \gamma_2) \ c = a_\lambda + b_\kappa\}.$$

Then

$$a \cdot b = \sum_{c \in C} \left(\left(\sum_{a_\lambda + b_\kappa = c} \alpha_\lambda \cdot \beta_\kappa \right) x^c \right).$$

DEFINITION 4.3 (Esterle algebra, $\mathcal{E}$).   The Esterle algebra, $\mathcal{E}$, is $\{f : G_{\omega_1} \to \mathbb{R} \mid f$ has countable well-ordered support$\}$. $\mathcal{E}$ is lexicographically ordered, with pointwise addition, and multiplication defined above.

$\mathbb{R}$ may be embedded in $\mathcal{E}$ by $\alpha \longmapsto \alpha x^e$, where e is the group identity in G_{ω_1}. Exponents in G_{ω_1} larger than e (called positive exponents) correspond to infinitesimal Archimedean valuations, and those smaller than e (called negative exponents) correspond to infinite valuations. The finite elements of $\mathcal{E}$ are those with leading exponent $\geq e$. The Archimedean valuations of the Esterle algebra are represented by the group of exponents of $\mathcal{E}$.

THEOREM 4.4 (J. Esterle [6]). $\mathcal{E}$ is an η_1-ordered real-closed field.

A norm, $\| \ \|$, on an algebra A is submultiplicative if for any $a, b \in A$,

$$\|a \cdot b\| \leq \|a\| \cdot \|b\|.$$

THEOREM 4.5 (G. Dales [2], J. Esterle [8]).   The set of finite elements of $\mathcal{E}$ bears a non-trivial submultiplicative norm.

It is a standard result of model theory that if U is a non-principal ultrafilter on ω, the ultrapower $\mathbb{R}^\omega / U$ is an $\aleph_1$-saturated real-closed field. Any two $\aleph_1$-saturated, or η_1-ordered, real-closed fields with cardinality of the continuum are isomorphic in models of ZFC+CH.

THEOREM 4.6 (B. Johnson [10]).   (CH) If U is a non-principal ultrafilter, there is an $\mathbb{R}$-monomorphism from the finite elements of $\mathbb{R}^\omega / U$ into $\mathcal{E}$.

Hence CH implies that there is a non-trivial submultiplicative norm on the infinitesimal elements of $\mathbb{R}^\omega / U$.

We turn our attention to terms in a forcing language M^P that are forced to be members of the Esterle algebra. In [3] and [4], we found sufficient conditions for morass constructions. The aggregate of these conditions were characterized as morass-definability and gap-2 morass-definability. The central theorem of the papers were that morass-definable η_1-orderings are order-isomorphic in the Cohen extension adding $\aleph_2$ generic reals of a model of $ZFC + CH$ containing a simplified

$(\omega_1, 1)$-morass; and gap-2 morass definable η_1-orderings are order-isomorphic in the Cohen extension adding $\aleph_3$ generic reals of a model of $ZFC + CH$ containing a simplified $(\omega_1, 2)$-morass.

In the definitions that follow, we assume P is the poset adding generic reals indexed by an ordinal, α.

DEFINITION 4.7 (Countable support). Let M be a transitive model of ZFC and G be P-generic over M and $S \subseteq \alpha$. Then $x \in M[G]$ has support S if there is $\tau \in M^{P(S)}$ such that $\tau_G = x$. If there is a countable $S \subseteq \alpha$, and $\tau \in M^{P(S)}$ such that $\tau_G = x$, then we say that x has countable support.

For $\alpha < \omega_1$, every element of $M[G(\alpha)]$ has countable support. The definition may be generalized to posets adding arbitrary sets of ordinals. We note the possibility of confusion with the support of power series in $\mathcal{E}$ treated as functions from G_{ω_1} to $\mathbb{R}$, exponents in G_{ω_1} treated as functions from S_{ω_1} to $\mathbb{R}$ and transfinite binary sequences of S_{ω_1}. Presumably the context will clarify the use of terminology.

DEFINITION 4.8 (Level-dense). Let κ be an ordinal, $P = P(\kappa)$, $T_X \in M^P$ be forced to be a linear-ordering and $X \subseteq M^P$ be a set of terms of strict level for the domain of T_X. X is level-dense if for $x, y \in X$, where the support of x and the support of y are disjoint, and G is P-generic over M, there is $z \in X \cap M$ such that $M[G] \models x < z < y$.

DEFINITION 4.9 (Upward level-dense). Let P be the poset that adds generic reals indexed by an ordinal κ and $\beta \leq \kappa$. Let $T_X \in M^P$ be forced to be a linear-ordering, and $X \subseteq M^P$ be a set of terms of strict level for the domain of T_X. X is upward level-dense provided that for every $x, y, z \in X$ and $p \in P$, in which z has strict level $\alpha \leq \beta$ with $p \Vdash x < z < y$, there is a term $w \in X$ of strict level β such that $p \Vdash x < w < y$.

If $\zeta < \xi \leq \omega_2$ and $f \in \mathcal{F}_{\zeta\xi}$, then f is an order-preserving injection from θ_ζ to θ_ξ. We define a function $g : P(\theta_\zeta) \to P(\theta_\xi)$ as follows. If $x \in P(\theta_\zeta)$, then x is a finite partial function from $\theta_\zeta \times \omega \to 2$. That is, x is composed of finitely many ordered triples, $\langle \alpha, n, s \rangle$ where $\alpha \in \theta_\zeta$, $n \in \omega$ and s is either 0 or 1. For $x \in P(\theta_\zeta)$, let $g(x) = \{\langle f(\alpha), n, s \rangle \mid \langle \alpha, n, s \rangle \in x\}$. Then $g : P(\theta_\zeta) \to P(\theta_\xi)$ replaces forcing conditions of $P(\theta_\zeta)$ with forcing conditions of $P(\theta_\xi)$. Finally, we define, by recursion, a function $h : M^{P(\theta_\zeta)} \to M^{P(\theta_\xi)}$ such that for any term, $\tau \in M^{P(\theta_\zeta)}$, $h(\tau) = \{\langle h(\sigma), g(q) \rangle \mid \langle \sigma, q \rangle \in \tau\}$. We will refer to h as the index replacement function on $M^{P(\theta_\zeta)}$ induced by f. For notational convenience we use f for both the order monomorphism on θ_ζ, and the index replacement function it induces.

DEFINITION 4.10 (Morass-closed). Suppose $\langle (\theta_\alpha \mid \alpha \leq \omega_1), (\mathcal{F}_{\alpha\beta} \mid \alpha < \beta \leq \omega_1) \rangle$ is a simplified $(\omega_1, 1)$-morass, $v < \lambda \leq \omega_1$ and $X \subseteq M^P$. We say that X is morass-closed at stage v beneath λ if for any $f \in \mathcal{F}_{v\lambda}$ and $x \in X \cap M^{P_v}$, $f(x) \in X \cap M^{P_\lambda}$. If for any $v < \lambda$, X is morass-closed at stage v beneath λ, then X is morass-closed beneath λ. If X is morass-closed at stage v beneath ω_1, then X is morass-closed at stage v.

Morass-closure extends naturally to structures with operations and relations, with the following useful consequence. Let X be a morass-closed structure at stage v

beneath λ. Suppose that S is a set of terms for the universe of X that is morass-closed at stage v beneath λ. If R is a binary relation of X we may consider R as terms in the forcing language that are comprised of ordered pairs of members of S with forcing conditions. That is, for $s, t \in S \cap M^{P_v}$ and $p \in P_v$, the term $\langle\langle s, t \rangle, p \rangle \in R$ just in case $p \Vdash sRt$. For the structure to be morass-closed at stage v beneath λ, the relation R must be morass-closed at stage v beneath λ. If $p \in P_v$, $f \in \mathcal{F}_{v\lambda}$, $s, t \in S \cap M^{P_v}$ and $p \Vdash sRt$, then $f(p) \Vdash f(s)Rf(t)$. Similarly for operations. For instance, if $+$ is a morass-closed operation of X; $s, t, u \in S \cap M^{P_v}$ and $p \Vdash s + t = u$, then $f(p) \Vdash f(s) + f(t) = f(u)$.

DEFINITION 4.11 (Morass-definable). Let G be P generic over M and $\langle X, +, \cdot, < \rangle \in M[G]$ be a linearly-ordered ring. X is morass-definable if there is a set of terms $T \subseteq M^P$ satisfying:

1. T is a morass-closed set of terms with strict level.
2. The linear order is morass-closed.
3. The ring operations are morass-closed.
4. T is level dense and upward level dense.
5. Every term of T has countable support.

The satisfaction of the first three conditions of the definition implies that the structure, $\langle X, +, \cdot, < \rangle$, is morass-closed.

We say that an object in a forcing extension satisfies the definitions above provided that there are terms in the forcing language that satisfy the definitions and that are interpreted as the object of interest. For instance, in any forcing extension of M, $M[G]$, the interpretation of $\langle \mathbb{R}, +, \cdot, < \rangle$ is morass-definable. In earlier papers see [3, 4] we showed that certain ultrapowers of $\mathbb{R}$ over ω were also morass-definable.

Let $E \subseteq M^P$ be the set of terms of strict level for elements in the Esterle algebra in the forcing language of the poset P. It is routine to check that $\mathcal{E}$ is morass-closed and that every element of the Esterle algebra is the interpretation of a term with countable support.

LEMMA 4.12. *The Esterle algebra is level dense.*

PROOF. Let $a, b \in M^P$ be terms of strict level for elements of the Esterle algebra bearing disjoint supports. Let G be P-generic over M and $M[G] \models a < b$. We wish to show that a and b are separated in $M[G]$ by an element of $\mathcal{E} \cap M$. We work in $M[G]$. Let γ_1 and γ_2 be countable ordinals and

$$a = \sum_{\lambda < \gamma_1} \alpha_\lambda x^{a_\lambda}$$

and

$$b = \sum_{\lambda < \gamma_2} \beta_\lambda x^{b_\lambda}.$$

If a and b are equal on a partial sum[2] then that partial sum is in M so subtracting the largest common partial sum of a and b, we may assume that a and b differ on

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[2]We intend for "partial sum" to mean initial partial sum.

the first term of the sums, and

$$\alpha_0 x^{a_0} \neq \beta_0 x^{b_0}.$$

If $a_0 = b_0$, then $a_0 \in M$ and there is $q \in \mathbb{Q}$ such that

$$\alpha_0 < q < \beta_0.$$

Then $qx^{a_0} \in M$ and

$$a < qx^{a_0} < b.$$

Hence we assume that a and b have distinct Archimedean valuations.

We consider the case $a_0 < b_0$. Then the sign of $b - a$ is determined by the sign of α_0. Hence $\alpha_0 < 0$. It is sufficient to prove that there is an element of $\mathcal{E} \cap M$ that has valuation between a_0 and b_0. Let

$$a_0 = f : S_{\omega_1} \to \mathbb{R}$$

and

$$b_0 = g : S_{\omega_1} \to \mathbb{R}.$$

If a_0 and b_0 are equal on an initial segment of their supports, then this initial segment is in the ground model. We can therefore assume that a_0 and b_0 either have distinct least members, or have the same least member of their supports, $s \in M$, and

$$f(s) < g(s).$$

In the latter case there is $q \in \mathbb{Q}$ such that

$$f(s) < q < g(s).$$

Then $\{(s, q)\} \in M$ and

$$a_0 < \{(s, q)\} < b_0.$$

Hence we have left to consider the case in which s_{a_0} is the least member of the support of a_0, s_{b_0} is the least member of the support of b_0 and $s_{a_0} \neq s_{b_0}$. If either s_{a_0} or s_{b_0} are in M we can find a member of $G_{\omega_1} \cap M$ that is a valuation between a_0 and b_0. So we may assume that neither s_{a_0} nor s_{b_0} are in M. As we shall see, it is sufficient to show that between any distinct elements of S_{ω_1} from mutually generic extensions, there is a member of S_{ω_1} in the ground model. We assume without loss of generality that $s_{b_0} < s_{a_0}$ (the case $s_{a_0} < s_{b_0}$ is altogether similar). Treating s_{b_0} and s_{a_0} as countable subsets of ω_1, let μ be the least element of s_{a_0} that is not a member of s_{b_0}. Let $\Delta = s_{a_0} \cap \mu$. Then $\Delta = s_{b_0} \cap \mu \in M$. We note that $b_0(\mu) > 0$, otherwise $b_0 < a_0$, contrary to assumption. Let

$$s_{c_0} = \Delta \cup \{\mu\}.$$

Then $s_{c_0} \in M$ and

$$s_{b_0} < s_{c_0} < s_{a_0}.$$

Let $c_0 \in M$ be defined so that, for $\rho < \mu$,

$$b_0(\rho) = c_0(\rho)$$

and

$$0 < c_0(\mu) < b_0(\mu).$$

Then $c_0 \in M$ and

$$a_0 < c_0 < b_0.$$

Since $a < b$, the coefficient of $a_0 > 0$. Let $q \in \mathbb{Q}$ and $0 < q < a_0$. Then $qx^{c_0} \in M$ and

$$a < qx^{c_0} < b.$$

Therefore $\mathcal{E}$ is level dense. ⊣

LEMMA 4.13. *The Esterle algebra is upward level-dense.*

Let $\alpha \leq \beta \leq \omega_2$. Suppose $x, y, z \in M^P$, with $x < z < y$, and z has strict level α. It is sufficient to show that there is an element $w \in \mathcal{E}$ of strict level β such that

$$x < w < y.$$

Assume that $G \times H$ is $P(\alpha) \times P(\beta \setminus \alpha)$-generic, and work in $M[G]$. Then $z \in M[G]$. If x and y have an identical partial sums, then z must share that partial sum, and the partial sum is in $M[G]$. We may subtract the largest partial sum shared by x and y and pass to x and y that disagree on the first term of the formal power series. If x and y have the same Archimedean valuation, a, then the first term of z has valuation a. Let $r \in \mathbb{R}$ be of strict level β and lie between the initial coefficients of x and y. Let $w = r \cdot x^a$. Then w has strict level β and

$$x < w < y.$$

So we assume that x and y have distinct valuation. If x and z have the same valuation, a, and the initial coefficient of x is negative, let $r \in \mathbb{R}$ be negative and greater than the initial coefficient of x and have strict level β. If the initial coefficient of x is positive, let r be positive and greater than the initial coefficient of x and have strict level β. In either case, let $w = r \cdot x^a$. Then w has strict level β and

$$x < w < y.$$

The cases for x and y having the same valuation are similar.

If x and y have distinct valuation, let $r \in \mathbb{R}$ be positive and have strict level β. Let $w = r \cdot z$. Then w has strict level β and

$$x < w < y.$$

Therefore $\mathcal{E}$ is upward level dense.

THEOREM 4.14. *The Esterle algebra is morass-definable.*

PROOF. Let T be the set of terms of strict level in $M^{P(\omega_3)}$ for elements of the Esterle algebra. We note that every element of the Esterle algebra in a $P(\omega_2)$-generic extension of a transitive model of ZFC has countable support. Hence the interpretation of T in any generic extension will be the Esterle algebra. T is clearly morass-closed. By Lemma 4.12, $\mathcal{E}$ is level dense. By Lemma 4.13, $\mathcal{E}$ is upward level dense. Therefore the Esterle algebra is morass-definable. ⊣

§5. Extendible functions. We use a simplified gap-1 morass to construct a function between sets of terms of a forcing language adding Cohen generic reals so that the interpretation of that function in a generic extension will satisfy certain conditions. We require the term function under construction to satisfy properties so that commutativity with morass maps will automatically extend the function on a countable domain to a function on an uncountable domain. Throughout this section we assume M is a transitive model of $ZFC + CH$ that contains a simplified $(\omega_1, 1)$-morass.

Definition 5.1 (Level term function). Let θ be an ordinal, $X \subseteq M^{P(\theta)}$ a set of terms having strict level and $\phi : X \to M^{P(\theta)}$. Then ϕ is a level term function provided that the range of ϕ is a set of terms of strict level, and for any $x \in X$, the strict level of x and the strict level of $\phi(x)$ are equal.

Definition 5.2 (Morass-commutative term function). Assume:

1. $\langle(\theta_\alpha \mid \alpha \le \omega_1), (\mathcal{F}_{\alpha\beta} \mid \alpha < \beta \le \omega_1)\rangle$ is a simplified $(\omega_1, 1)$-morass.
2. $v < \bar{v} \le \omega_1$.
3. $X \subseteq M^{P_{\bar{v}}}$ is morass closed at stage v beneath $\bar{v}$.
4. $\phi : X \to M^{P_{\bar{v}}}$.

If $\sigma \in \mathcal{F}_{v\bar{v}}$, then ϕ commutes with σ provided that for every $x \in X \cap M^{P_v}, \sigma[\phi(x)] = \phi(\sigma[x])$. A function $\phi : X \to M^{P_{\bar{v}}}$ is morass-commutative at stage v beneath $\bar{v}$ if for every $\sigma \in \mathcal{F}_{v\bar{v}}$, ϕ commutes with σ. If ϕ is morass-commutative at stage v beneath ω_1, then ϕ is morass-commutative at stage v.

In order to extend an R-monomorphism by commutativity with a splitting map we must satisfy both algebraic and order constraints.

In the next section we show that the morass-commutative extension of an $\mathbb{R}$-monomorphism, satisfying certain technical constraints (extendibility), may be extended to an $\mathbb{R}$-monomorphism satisfying those same constraints. The technical constraints are those required for an inductive construction along the vertices of a simplified morass.

We state the technical conditions that permit the inductive construction of the following sections.

Definition 5.3 (Standard term for a subset of ω). A standard term for a subset of ω, $x \in M^P$, is a term of strict level such that for each $(\tau, p) \in x$, τ is a canonical term for a natural number.

Definition 5.4 (Standard term for an ultrafilter). Let $U \subset M^P$ be a morass-closed set of standard terms for subsets of ω such that, for all $\alpha \le \omega_1$, $U \cap M^{P_\alpha}$ is forced to be an ultrafilter in all $P(\alpha)$-generic extensions of M. Then U is a standard term for an ultrafilter.

We will refer to an ultrapower of $\mathbb{R}$ over a standard ultrafilter U as a standard ultrapower. We restrict our attention to functions from terms for a standard ultrapower of $\mathbb{R}$ to terms for elements of the Esterle algebra.

Definition 5.5 (Extendible function). Let $v < \omega_1$, $X, Y \subseteq M^{P_v}$ have elements of strict level and $\phi : X \to Y \in M^{P_v}$ be a level term function. Then ϕ is extendible provided that the following are satisfied:

1. It is forced that X is a subring of a standard ultrapower of $\mathbb{R}$ that has countable transcendence degree over $\mathbb{R}$.
2. Y is forced to be a subset of the Esterle algebra.
3. It is forced that Y is closed under partial sums. That is, every initial series of a power series of Y is a member of Y.
4. Y contains canonical forcing terms for every real coefficient appearing in a power series of Y.
5. ϕ is forced to be an $\mathbb{R}$-isomorphism.

If v is a vertex of a morass, θ_v is the ordinal associated with v and $\sigma \in \mathcal{F}_{vv+1}$ is the splitting function on θ_v then $X \cup \sigma[X]$ is morass-closed at stage v beneath $v + 1$. We will show that if ϕ is an extendible term function on X, then $\phi \cup \sigma[\phi]$ may be extended to extendible term function.

§6. Commutative extensions of extendible term functions.

In the inductive construction of the following sections we will need technical lemmas of two types: those insuring that commutativity with morass maps may be used to extend extendible functions to extendible functions, and those allowing the extension of the domain by a specified element to an extendible function. Throughout this section we assume:

1. P is the poset adding generic reals indexed by ω_2.
2. σ is a splitting map on $\theta < \omega_1$ with splitting point $\delta = 0$. That is, $\sigma : \theta \to \omega_1$ and $\sigma[\theta] \cap \theta = \emptyset$.
3. $G(\theta)$ is $P(\theta)$-generic over M.
4. H is $P[\sigma(\theta)]$-generic over $M[G]$.
5. $\phi : X \to Y \in M^{P(\theta)}$ is an extendible function.

We wish to show that the extension of $\phi \cup \sigma[\phi]$ to the ring generated by $X \cup \sigma[X]$ is extendible.

6.1. Splitting maps and algebraic independence. The central result of this subsection states, roughly, that a subset of a field in a generic extension that is algebraically independent (AI) over the restriction of the field to the ground model, will be AI over the restriction of an extension field to a mutually generic extension. It will follow that the union of an AI subset of a field in a generic extension with its morass "split" in a mutually generic extension will be AI over the restriction of the field to the ground model. We consider the special case in which a morass-map, "σ," "splits" a poset $P(\theta)$, for some ordinal θ. Then there is $\delta < \theta$ so that the poset $P(\theta) = P(\delta) \times P(\theta \setminus \delta)$. Then $\sigma(\theta) = \delta \cup (\sigma[\theta] \setminus \delta)$, $\theta \cap \sigma[\theta] = \delta$, and $\sigma(P(\theta)) = P(\delta) \times P(\sigma[\theta] \setminus \delta)$. Hence we may consider the case in which $G(\delta)$ is $P(\delta)$-generic over M, and we are forcing over $M[G(\delta)]$. Then

$$\sigma[\theta \setminus \delta] \cap (\theta \setminus \delta) = \emptyset.$$

Hence, without loss of generality, we may assume that the splitting point is $\delta = 0$.

LEMMA 6.1. *Let P be the poset adding generic reals indexed by ω_2, G be P-generic over M and F be a morass-definable field in $M[G]$. Let $\theta < \omega_1$, P_0 be the poset adding generic reals indexed by θ, G_0 be the P_0-generic factor of G, and σ be a splitting map on θ (with splitting point δ). If $\chi = \{x_1, \ldots, x_n\} \subseteq (F \cap M[G_0])])$ is linearly independent (LI) over $M \cap F$, then χ is LI over $M[G(\sigma[\theta])] \cap F$.*

Proof. Let M, P, F, χ, θ, σ, and G satisfy the hypotheses of the lemma. As discussed at the beginning of this section, without loss of generality, we assume that $\delta = 0$. Let $H = G(\sigma[\theta])$. So H is $P(\sigma[\theta])$-generic over $M[G_0]$. We observe that, since F is morass-definable, it is the interpretation in $M[G]$ of a set of terms with strict level, T. For any $\alpha < \omega_2$, the interpretation of $T \cap M^{P(\alpha)}$ is just $F \cap M[G(\alpha)]$, and is a subfield of F. Assume $\chi \subseteq (F \cap M[G_0])$ is linearly dependent over $F \cap M[H]$ and that $\{y_1, \dots y_n\} \in M[H]$ (all non-zero) are such that $\sum_{i=1}^n x_i \cdot y_i = 0$ in $M[G_0][H]$. For $1 \leq i \leq n$, let $\mathbf{x}_i \in M^{P(\theta)}$ and $\mathbf{y}_i \in M^{P(\sigma[\theta])}$ be terms for x_i and y_i, respectively. Then there is $(p, q) \in G_0 \times H$ such that, in M, $(p, q) \Vdash \sum_{i=1}^n \mathbf{x}_i \cdot \mathbf{y}_i = 0$. We force beneath (p, q).

Let β be countable, Q_0 be a poset for adding countably many generic reals indexed by $\beta \setminus \theta$ and H_0 be Q_0-generic over $M[G_0]$ such that $(p, q_0) \in G_0 \times H_0$, and the orthogonal complement of $\mathbf{x} = (x_1, \dots, x_n)$ in $F^n \cap M[H_0]$, $\mathbf{x}_0^\perp$, has maximum possible dimension, m, in $F^n \cap M[H_0]$, where $0 < m < n$. Let $q_0 \in H_0$ force that the $\mathbf{x}^\perp$ has dimension m, where $q_0 < q$. Let σ be a splitting map on Q_0 with splitting point $\delta = \theta$. Let $Q_1 = \sigma(Q_0)$, $q_1 = \sigma(q_0) < \sigma(q)$ and $\mathbf{x}_1^\perp = \sigma(\mathbf{x}_0^\perp)$. So Q_1 and Q_0 are disjoint. If H_1 is Q_1-generic over $M[G_0][H_0]$ and $q_1 \in H_1$, then in $M[H_1]$, $\mathbf{x}_1^\perp$ has dimension m. Furthermore, in $M[H_0][H_2]$, $\mathbf{x}^\perp$ has dimension m. Then $\mathbf{x}_0^\perp = \mathbf{x}_1^\perp$ are elements of mutually generic extensions of M, and are thereby members of M. Therefore $\mathbf{x}^\perp$, computed in M has dimension $m > 0$, and the components of x are linearly dependent over $F \cap M$. ⊣

Lemma 6.2. *Let M, P, F, G, θ, σ, and χ satisfy the hypotheses of Lemma* 6.1, *and $H = G(\sigma[\theta])$. Assume $\bar{x} = \langle x_1, \dots, x_n \rangle \in M[G(\theta)]$ and $\bar{y} = \langle y_1, \dots, y_n \rangle \in M[H]$ are such that*

$$M[G][H] \models \sum_{i=1}^n x_i \cdot y_i = 0.$$

Then in $M[H]$, $\bar{y}$ is in the span of vectors over $F^n \cap M$, all of which are orthogonal to $\bar{x}$.

Proof. Let $M[G][H] \models \sum_{i=1}^n x_i \cdot y_i = 0$. Let A_0 be the row-reduced echelon form of the orthogonal complement of $\bar{x}$, $\bar{x}^\perp$, computed in $M[G]$. We force over ground model, M. Let $(p, q) \in G \times H$ be such that $(p, q) \Vdash \sum_{i=1}^n x_i \cdot y_i = 0$. We force below (p, q). Let $\theta_1 \geq \sigma(\theta)$, $Q_1 = P(\sigma(\theta_1) \setminus \theta)$ and H_1 be Q_1-generic over $M[G]$ be such that the dimension of $\bar{x}^\perp$ computed in $M[H_1]$, $\bar{x}_1^\perp$, is maximal for all possible choices of θ_1 and H_1 (where $q \in H_1$). Let $A_1 \in M[H_1]$ have row space equal to $\bar{x}_1^\perp$ and be in row-reduced echelon form. Since we are forcing below (p, q), A_1 has a non-zero row.

Let the rank of A_1 be m. In any generic extension of $M[G]$ by a poset for adding countably many ordinals, the rank of $\bar{x}^\perp$ computed in that extension is no greater than m. Let $q_1 \in Q_1$ and $q_1 \Vdash (A_1$ is in row-reduced echelon form with row space equal to $\bar{x}^\perp$ and $Rank(A_1) = m)$.

Let σ be a splitting function on θ_1 with splitting point θ. Let $Q_2 = P(\sigma(\theta) \setminus \theta)$ and H_2 be Q_2 generic over $M[G][H_1]$ with $q_2 = \sigma(q_1) \in H_2$. Let $A_2 = \sigma(A_1)$. Then $q_2 \Vdash A_2$ is in row-reduced echelon form with row space equal to $\bar{x}^\perp$, and has rank m.

Since m is the maximum possible rank for $\bar{x}^{\perp}$, and row-reduced echelon form is canonical,

$$M[G][H_1][H_2] \models A_1 = A_2.$$

A_1 and A_2 are in mutually generic extensions of $M[G]$, so

$$A_1 = A_2 \in M[G].$$

But $A_1 \in M[H_1]$ and $A_2 \in M[H_2]$, so $A_1 \in M$. Then $\bar{y}$ is in the row space of A_1. Hence $\bar{y}$ is in the span of vectors in $F^n \cap M$.   ⊣

COROLLARY 6.3. *Let F, G, and H satisfy the hypotheses of Lemma* 6.1. *If $\chi \subseteq M[G] \cap F$ is algebraically independent* (*AI*) *over $M \cap F$, then χ is AI over $M[H] \cap F$.*

PROOF. Let χ^* be the multiplicative semi-group generated by the elements of χ. Then χ^* is LI over $M \cap F$. By Lemma 6.1, χ^* is LI over $M[H] \cap F$. Therefore χ is AI over $M[H] \cap F$.   ⊣

COROLLARY 6.4. *Let F, G, σ, θ, and H satisfy the hypotheses of Lemma* 6.1. *Assume that it is forced in all $P(\theta)$-generic extensions of M that $\chi = \{\mathbf{x_1}, \ldots, \mathbf{x_n}\} \subseteq M^{P(\theta)}$ is AI over $M \cap F$. Then $\chi \cup \sigma[\chi]$ is forced in all P-generic extensions of M to be AI over $M \cap F$. In particular, $val_G(\chi \cup \sigma[\chi])$ is AI over $M \cap F$.*

PROOF. F is morass-closed, so in all $P(\sigma[\theta])$-generic extensions of M, $\sigma[\chi]$ is AI over $M \cap F$. By Corollary 6.3, $\sigma[\chi]$ is AI over $M \cap F$ in $M[G]$. Suppose there is a nontrivial linear combination (over $M \cap F$) of distinct elements of the semigroup generated by $\chi \cup \sigma[\chi]$ that equals 0. By Corollary 6.3, χ is AI over $M[H] \cap F$. This implies that there is a nontrivial linear combination (over $M \cap F$) of elements of the semigroup generated by $\sigma[\chi]$ that equals 0. However, $\sigma[\chi]$ is AI over $M \cap F$, so $val_G(\chi \cup \sigma[\chi])$ is AI over $M \cap F$.   ⊣

LEMMA 6.5. *Let X be a subring of finite elements of a standard ultrapower of $\mathbb{R}$, $\mathbb{R}^{\omega}/U$ and $Y \subseteq \mathcal{E}$. Let X^* be the ring generated by $X \cup \sigma[X]$, where σ is a splitting function. If $\phi : X \to Y$ is an extendible $\mathbb{R}$-monomorphism, then there is an extendible $\mathbb{R}$-monomorphism, $\phi^* : X^* \to Y^*$, extending ϕ and $\sigma[\phi]$.*

PROOF. Without loss of generality we assume that the splitting point of σ, $\delta = 0$. An element of X^* may be expressed as $\sum_{i=1}^{n} x_i \cdot \sigma(y_i)$, for some $n \in \mathbb{N}$, and $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$. Let S be the set of expressions of this form. We define a function $\psi : S \to \mathcal{E}$ where

$$\psi\left(\sum_{i=1}^{n} x_i \cdot \sigma(y_i)\right) = \sum_{i=1}^{n} \phi(x_i) \cdot \sigma(\phi(y_i)).$$

Let $\iota : S \to X^*$ be the natural quotient map from the expressions of S to X^*. The kernel of ι is the set of expressions of S that sum to 0 in X^*. ψ defines a ring homomorphism on X^* if and only if, for any s in the kernel of ι,

$$\psi(s) = 0.$$

For $i \leq n$ let $z_i = \sigma(y_i)$ and $s = \sum_{i=1}^{n} x_i \cdot z_i$ be in the kernel of ι. Then in X^*,

$$\sum_{i=1}^{n} x_i \cdot z_i = 0.$$

Let

$$\bar{x} = (x_1, \ldots, x_n) \in X^n \cap M[G],$$

$$\bar{y} = (y_1, \ldots, y_n) \in X^n \cap M[G],$$

and

$$\bar{z} = (z_1, \ldots, z_n) \in \sigma([X])^n \cap M[H].$$

By Lemma 6.2, $\bar{z}$ is in the span of elements of $X^n \cap M$ that are orthogonal to $\bar{x}$. Let $\{b_1, \ldots, b_m\}$ be an LI set of vectors of $X^n \cap M$ orthogonal to $\bar{x}$ that contains $\bar{z}$ in its span. Let $\langle \cdot, \cdot \rangle$ be the dot product and $(\alpha_1, \ldots, \alpha_m) \in \sigma([X])^n \cap M[H]$ be such that

$$\sum_{i=1}^{m} \alpha_i \cdot b_i = \bar{z}.$$

Let $\bar{\phi} : X^n \to \mathcal{E}^n$ be defined by

$$\bar{\phi}(s_1, \ldots, s_n) = (\phi(s_1), \ldots, \phi(s_n)).$$

Recall that for a splitting map σ, $\sigma(\phi) : \sigma[X] \to \mathcal{E}$ is defined so that σ and ϕ commute. Then

$$\psi(\langle \bar{x}, \bar{z} \rangle) = \left\langle \bar{\phi}(\bar{x}), \sigma(\bar{\phi})\left(\sum_{i=1}^{m} \alpha_i \cdot b_i\right)\right\rangle = \sum_{i=1}^{m} \sigma(\alpha_i) \cdot \langle \bar{\phi}(\bar{x}), \sigma(\bar{\phi})(b_i) \rangle.$$

However, $b_i \in X^n \cap M$ for all $i \leq n$, so

$$\sigma(\bar{\phi})(b_i) = \bar{\phi}(b_i).$$

Hence

$$\sum_{i=1}^{m} \sigma(\alpha_i) \cdot \langle \bar{\phi}(\bar{x}), \sigma(\bar{\phi})(b_i) \rangle = \sum_{i=1}^{m} \sigma(\alpha_i) \cdot \langle \bar{\phi}(\bar{x}), \bar{\phi}(b_i) \rangle = \sum_{i=1}^{m} \sigma(\alpha_i) \cdot \phi(\langle \bar{x}, b_i \rangle).$$

For all $i \leq m$, $b_i \perp \bar{x}$. So for all $i \leq m$,

$$\phi(\langle \bar{x}, b_i \rangle) = 0$$

and $\psi(\langle \bar{x}, \bar{z} \rangle) = 0$. Therefore ψ defines a ring homomorphism on X^*, ϕ^*, that extends $\phi \cup \sigma(\phi)$ to X^*.

Assume that $\bar{x}, \bar{y} \in M[G] \cap X^n$, $\bar{z} = \sigma(\bar{y})$ and

$$\langle \bar{x}, \bar{z} \rangle \neq 0.$$

We consider X as a vector space over $M \cap X$. Let B be a basis for X over $M \cap X$. There is a finite subset of basis vectors, $X' = \{x'_1, \ldots, x'_m\} \subseteq B$ so that every component of $\bar{x}$ is in the span of X'. Hence $\langle \bar{x}, \bar{y} \rangle$ may be expressed as a linear

combination of X' over $\sigma[X]$. For $1 \leq i \leq m$, let z_i' be the sum of coefficients of x_i' in the linear combination over X'. Then $z_i' \in \sigma[X]$ and

$$\sum_{i=1}^{m} x_i' \cdot z_i' = \sum_{i=1}^{n} x_i \cdot z_i \neq 0.$$

Then $\{\phi(x_1'), \ldots, \phi(x_m')\}$ is LI over $\phi[X] \cap M$. By Lemma 6.1, $\{\phi(x_1'), \ldots, \phi(x_m')\}$ is LI over $\mathcal{E} \cap M[H]$. Therefore

$$\sum_{i=1}^{n} \phi^*(x_i \cdot z_i) = \sum_{i=1}^{m} \phi^*(x_i' \cdot z_i') \neq 0.$$

Thus ϕ^* is a monomorphism.

We show that ϕ^* is $\mathbb{R}$-linear. Let $r \in \mathbb{R} \cap X^*$. Since X is a subring of finite elements of $\mathbb{R}^\omega / U$, every element of X may be expressed as the sum of a real number and an infinitesimal.

Then there are $x_1 \ldots, x_n, y_1, \ldots y_n \in X$ such that

$$r = \sum_{i=1}^{n} x_i \cdot \sigma(y_i).$$

For $1 \leq i \leq n$, each x_i and y_i may be written as $r_i + \delta_i$ and $s_i + \varepsilon_i$, resp., where for $r_i, s_i \in \mathbb{R}$ and δ_i, ε_i are infinitesimal. So

$$\sum_{i=1}^{n} r_i \cdot \sigma(s_i) = r$$

and

$$\sum_{i=1}^{n} r_i \cdot \sigma(\varepsilon_i) + \sigma(s_i) \cdot \delta_i + \delta_i \cdot \sigma(\varepsilon_i) = 0.$$

Since ϕ is extendible, for $1 \leq i \leq n$, $\phi(r_i) = r_i$ and $\phi(\delta_i)$ is infinitesimal in $\mathcal{E}$. Therefore $r_i \in X$. Similarly, $s_i \in X$. Hence r is in the ring generated by $(\mathbb{R} \cap X) \cup (\sigma[\mathbb{R} \cap X])$. So $\phi^*(r) = r$, and ϕ^* is $\mathbb{R}$-linear.

We show that ϕ^* is order-preserving. For $n \in \mathbb{N}$ and $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$, we show that $\sum_{i=1}^{n} x_i \cdot \sigma(y_i) > 0$ iff $\sum_{i=1}^{n} \phi^*(x_i) \cdot \phi^*(\sigma(y_i)) > 0$.

If $n = 1$, then the sign of $x_1 \cdot \sigma(y_1)$ is the sign of the product of the leading coefficients of x_1 and $\sigma(y_1)$, which are preserved by ϕ^*. So

$$x_1 \cdot \sigma(y_1) > 0 \iff \phi(x_1) \cdot \sigma(\phi(y_1)) > 0.$$

Assume that $n \geq 2$, and

$$\sum_{i=1}^{n} x_i \cdot \sigma(y_i) > 0.$$

Let

$$y = \phi^* \left(\sum_{i=1}^{n} x_i \cdot \sigma(y_i) \right) = \sum_{\lambda < \gamma} \alpha_\lambda x^{a_\lambda}.$$

By assumption, $\phi[X]$ is a ring of finite elements of $\mathcal{E}$ that is closed under partial sums and contains all coefficients that appear in members of the range of ϕ. Since ϕ is extendible, there are $b_0 \in M[G] \cap G_{\omega_1}$ and $c_0 \in M[H] \cap G_{\omega_1}$ such that $x^{b_0} \in \phi[X]$, $x^{c_0} \in \sigma[\phi[X]]$, $b_0, c_0 \geq e$ (in G_{ω_1}) and

$$a_0 = b_0 + c_0.$$

Let $u, v \in X$ be such that

$$\phi(u) = x^{b_0}$$

and

$$\sigma(\phi(v)) = x^{c_0}.$$

Let $\hat{X}$ be the ring generated by X, $1/u$ and $1/v$ in $\mathbb{R}^\omega/U$. Let $\psi : \hat{X} \to \mathcal{E}$ be the unique ring homomorphism extending ϕ satisfying $\psi(1/u) = 1/\phi(u) = x^{-b_0} \in \mathcal{E}$ and $\psi(\sigma(1/v)) = x^{-c_0} \in \mathcal{E}$. Then ψ is extendible. Hence the extension of ψ to the ring homomorphism, ψ^*, on the ring generated by $\hat{X} \cup \sigma[\hat{X}]$ is an $\mathbb{R}$-linear monomorphism. Furthermore, since ϕ is an $\mathbb{R}$-linear order-monomorphism, ϕ extends uniquely to an $\mathbb{R}$-linear order-preserving field monomorphism on the field generated by $\hat{X}$. Hence ψ and $\sigma[\psi]$ are $\mathbb{R}$-monomorphisms on $\hat{X}$ and $\sigma[\hat{X}]$, resp. Then

$$\psi^* \left((1/u) \cdot (\sigma(1/v)) \cdot \left(\sum_{i=1}^n x_i \cdot \sigma(y_i) \right) \right) =$$

$$\alpha_0 + x^{-a_0} \cdot \phi^* \left(\sum_{i=1}^n x_i \cdot \sigma(y_i) \right) =$$

$$\alpha_0 + x^{-a_0} \cdot \left(\sum_{\lambda=1}^\gamma \alpha_\lambda x^{a_\lambda} \right) =$$

$$\alpha_0 + \sum_{\lambda=1}^\gamma \alpha_\lambda x^{a_\lambda - a_0}.$$

For $1 \leq i \leq n$, $(1/u) \cdot (\sigma(1/v)) \cdot x_i \cdot \sigma(y_i)$ is finite. Therefore, for $1 \leq i \leq n$,

$$(1/u) \cdot (\sigma(1/v)) \cdot x_i \cdot \sigma(y_i) = r_i \cdot \sigma(s_i) + \delta_i \cdot \sigma(s_i) + r_i \cdot \sigma(\varepsilon_i) + \delta_i \cdot \sigma(\varepsilon_i),$$

where $r_i, s_i \in \mathbb{R} \cap X$ and $(1/u) \cdot \delta_i, \sigma(1/v) \cdot \sigma(\varepsilon_i) \in \hat{X}$ are infinitesimal. If the ith term is infinitesimal, then $r_i = 0 = s_i$. Hence,

$$\alpha_0 = \sum_{i=1}^n r_i \cdot s_i.$$

Since ψ^* is $\mathbb{R}$-linear, the real part of $(1/u) \cdot (\sigma(1/v)) \cdot \sum_{i=1}^n x_i \cdot \sigma(y_i)$ is α_0, and therefore $\alpha_0 > 0$. Furthermore, $\psi^*((1/u) \cdot \sigma(1/v)) > 0$. Hence

$$\psi^* \left(\sum_{i=1}^n x_i \cdot \sigma(y_i) \right) = \phi^* \left(\sum_{i=1}^n x_i \cdot \sigma(y_i) \right) > 0.$$

Therefore ϕ^* is order-preserving and ϕ^* is extendible. $\dashv$

LEMMA 6.6. *Let*:

1. $\bar{v} < v \le \omega_1$.
2. $D \in M[G_{\bar{v}}]$ *be a subring of a standard ultrapower of* $\mathbb{R}$ *over* ω.
3. $\phi : D \to \mathcal{E} \in M[G_{\bar{v}}]$ *be an extendible* $\mathbb{R}$-*monomorphism on* D.
4. $D^* = \bigcup_{\sigma \in \mathcal{F}_{\bar{v}v}} \sigma[D]$.

Then there is a unique extendible $\mathbb{R}$-*monomorphism,* ϕ^*, *on the ring generated by* D^* *which, for any* $\sigma \in \mathcal{F}_{\bar{v}v}$, *extends* $\sigma[\phi]$.

PROOF. If $v = \bar{v} + 1$, then the result follows from Lemma 6.5.

If there is no limit ordinal λ, $\bar{v} < \lambda \le v$. Then there is $n \in \omega$ such that

$$v = \bar{v} + n.$$

By Lemma 6.5, for any extendible $\phi : D \to \mathcal{E}$ and splitting function σ, the ring monomorphism on the ring generated by $D \cup \sigma[D]$ extending $\phi \cup \sigma[\phi]$ is extendible. By n iterated applications of Lemma 6.5, there is a unique extendible $\mathbb{R}$-monomorphic extension of ϕ, $\phi^* \supset \bigcup_{\sigma \in \mathcal{F}_{\bar{v}v}} \sigma[\phi]$, to the ring generated by $\bigcup_{\sigma \in \mathcal{F}_{\bar{v}v}} \sigma[D]$.

So assume there is a limit ordinal λ, $v_{\bar{\alpha}} < \lambda \le v_{\alpha}$. Let λ be the least limit ordinal greater than $v_{\bar{\alpha}}$. Let

$$D_\lambda = \bigcup_{\sigma \in \mathcal{F}_{\bar{v},\lambda}} \sigma[D].$$

Let D_λ^* be the ring generated by D_λ, We show that there is an extendible $\mathbb{R}$-monomorphism of D_λ^* which, for any $\sigma \in \mathcal{F}_{\bar{v}\lambda}$, extends $\sigma[\phi]$.

Let F be a finitely generated subring of D_λ^*. Let $\{d_1, \dots, d_n\}$ generate F, $\sigma_1, \dots, \sigma_n \in \mathcal{F}_{\bar{v},\lambda}$ and for all $i \le n$, $c_i \in D$ be such that

$$d_i = \sigma_i(c_i).$$

By condition P4 in the definition of the simplified morass, there is $N \in \omega$, $g \in \mathcal{F}_{v_{\bar{\alpha}}+N,\lambda}$ and $f_1, \dots, f_n \in \mathcal{F}_{v_{\bar{\alpha}},v_{\bar{\alpha}}+N}$ such that, for $i \le n$,

$$\sigma_i = g \circ f_i.$$

For each $m < n$, let h_m be the splitting function of $\mathcal{F}_{\bar{v}+m,\bar{v}+m+1}$. By Lemma 6.5, $\phi \cup h_1[\phi]$ may be extended to an extendible $\mathbb{R}$-monomorphism. Furthermore this ring monomorphism may be extended by the splitting functions h_2 through h_m. Let ψ be the function on $\bigcup_{\sigma \in \mathcal{F}_{\bar{v}\lambda}} \sigma[D]$ resulting after the n splits. Then ψ is extendible and $f_i(c_i)$ is in the domain of ψ for all $i \le n$. Therefore $g \circ \psi$ is an $\mathbb{R}$-linear order monomorphism and is the restriction of ϕ^* to a ring containing F. Thus

$$\phi_\lambda^* = \bigcup_{\sigma \in \mathcal{F}_{\bar{v}\lambda}} \sigma[\phi]$$

is a well-defined extendible $\mathbb{R}$-monomorphism of D_λ^*.

By an inductive argument on v, invoking condition P2, and the results above at limits and Lemma 6.5 at successor ordinals, it is straightforward to show that $\bigcup_{\sigma \in \mathcal{F}_{\lambda,v}} \sigma[\phi_\lambda^*]$ has a unique extension to an extendible $\mathbb{R}$-monomorphism of the ring generated by $\bigcup_{\sigma \in \mathcal{F}_{\lambda,v}} \sigma[D_\lambda^*]$. ⊣

6.2. Extensions by a specified element. Because ordered subrings of real closed fields have unique extensions to real closed subfields, we will be able to restrict our attention to extending domains of extendible functions by algebraically independent elements. We wish to prove analogues of Johnson's theorems that extendible functions may be extended by a specified element. Throughout the arguments of this section, we will commonly use x to represent an element of F, and also to represent the variable in the power series representations of member of $\mathcal{E}$. Presumably the context will make clear which use is intended.

If X is a subring of a ring R and $Z \subset R$, we let $X[Z]$ be the subring of R generated by $X \cup Z$.

LEMMA 6.7. *Suppose* $\phi : X \to Y$ *is an extendible function,* $r \in \mathbb{R}$ *and* r *is transcendental over* X. *Then there is an extension of* ϕ *to an extendible function,* $\psi : X[r] \to Y[r]$.

PROOF. Since ϕ is extendible, X is full and for all $r_0 \in \mathbb{R} \cap X$, $\phi(r_0) = r_0 \cdot x^e$ (where e is the group identity of $\mathcal{E}$). Then the real closure of X is full and the set of real numbers of the real-closure of X is the real closure of $\mathbb{R} \cap X$. Let X^* be the real closure of X and $\phi^* : X^* \to \mathcal{E}$ be the unique $\mathbb{R}$-monomorphism extending ϕ to X^*. Then for all $r_0 \in \mathbb{R} \cap X^*$, $\phi(r_0) = r_0 \cdot x^e$. If r is transcendental over X, then r is transcendental over X^*. The real closure of Y and X^* contain precisely the same real numbers and are full. By Lemma 3.4, there is an $\mathbb{R}$-monomorphic extension of ϕ^*, ψ^*, to the real closure of the field generated by $X[r]$. Let $\psi = \psi^* \restriction_{X[r]}$. Then ψ is extendible. $\dashv$

LEMMA 6.8. *Suppose* $\phi : X \to Y$ *is an extendible function and* $x^* \in \mathbb{R}^\omega / U$ *is transcendental over* X. *Then there is* $\bar{X} \supseteq X$, *with* $x^* \in \mathbb{R}^\omega / U$ *in the real closure of* $\bar{X}$, *and extendible* $\psi : \bar{X} \to \mathcal{E}$ *extending* ϕ.

PROOF. If $x^* \in \mathbb{R}$, then apply Lemma 6.7. Assume $x^* \notin \mathbb{R}$. If x^* is not infinitesimal, then $x^* = r + \delta$, where $r \in \mathbb{R}$ and δ is infinitesimal. If $r \notin X$, then we may extend ϕ to an extendible function, $\phi^* : X[r] \to Y[r]$. We consider the case in which $r \in X$. Since X is extendible, $x^* - r = \delta \in X$. Therefore we may assume that x^* is infinitesimal. Let (l, u) be the gap formed by x^* in X, $L = \phi[l]$ and $U = \phi[u]$. The Esterle algebra is an η_1-ordering, so there is $y \in \mathcal{E}$ that witnesses the gap (L, U). By application of Johnson's Lemma 3.5, there is $y \in \mathcal{E}$ that witnesses the gap (L, U) and such that the real closure of the field extending $Y[y]$ is full. Although the existence of such an element can be used to advantage, the element y may fail to have some of the properties we require for a morass construction.

Let $\mu < \omega_1$ be the strict level of x^*. We seek an element of $\mathcal{E}$ that witnesses the gap (L, U) and is a candidate for the image of x^*. The candidate must have strict level μ and be transcendental over Y, among other requirements. The Esterle algebra is level dense and upward level dense, so by Lemma 4.5 of [3] there is $y \in \mathcal{E}$, with the strict level μ, such that for all $z \in X$,

$$x^* < z \iff y < \phi(z).$$

Let Z be the countable set of real coefficients appearing in power series of $Y[y]$. Then by iterated applications of Lemma 6.7 there is an extendible extension of ϕ to $\phi^* : X[Z] \to Y[Z]$. Let $X^* = X[Z]$ and $Y^* = Y[Z]$. If x^* is in the real-closure of

X^*, let $\bar{X} = X^*$ and $\psi : \bar{X} \to \mathcal{E}$ be the unique ring monomorphism extending ϕ^*. Then ψ is extendible. So we assume that x^* is transcendental over X^*.

It is possible that $Y^*[y]$ is not closed under partial sums. We show that there is $y^* \in \mathcal{E}$ that witnesses (L, U) and such that $Y^*[y^*]$ is full and closed under partial sums.

Case 1: There is a largest partial sum of y that is a member of Y^*.

We include in this case that the first term of y is a monomial not in Y^*. Let t be the largest partial sum of y that is also a member of Y^* and $s = (\phi^*)^{-1}(t) \in X^*$. Then $y - t \notin Y^*$ and $x^* - s \notin X^*$. We shift our attention to the gap formed by $x^* - s$ in X^*. Then for all $z \in X^*$

$$x^* - s < z \iff y - t < \phi(z).$$

If $x^* - s$ were algebraic over X^*, then it would be in the real-closure of X^*. Since X^* is a ring, and $s \in X^*$, this would imply that x^* is in the real-closure of X^*, contrary to assumption that x^* is transcendental over X^*. Therefore $x^* - s$ is transcendental over X^*. We claim that $x^* - s$ must have an Archimedean valuation distinct from the Archimedean valuations of members of X^*. Let $y - t \in \mathcal{E}$ have leading term αx^a. Then $\alpha \in Z \subseteq Y^*$. If x^a were in the range of ϕ^*, then $t + \alpha x^a \in Y^*$ would be a partial sum of y, contrary to assumption. So a must be a valuation distinct from the valuations of Y^*. Therefore $x^* - s$ must have a valuation distinct from the valuations of members of X^*.

Let (L^*, U^*) be the gap formed by $y - t$ in Y^*. If $\alpha > 0$, then it is sufficient to show that there is $c \in G_{\omega_1}$ such that x^c has strict level μ and witnesses the gap (L^*, U^*). The case $\alpha < 0$ is altogether similar. Let $y_0 = x^a$. The strict level of y_0 equals the strict level of $a \in G_{\omega_1}$. It is straightforward to see that there is $b \in G_{\omega_1}$ such that:

1. b is positive.
2. Any element of the support of b is greater than any element of the support of any exponent occurring in any power series of Y^*.
3. b has strict level μ.

The exponent $b \in G_{\omega_1}$ is greater than 0 but less than any positive exponent in Y^*. Furthermore, b is greater than the constant 0 function, the additive identity of G_{ω_1} (and the valuation of standard reals in $\mathcal{E}$). However b is greater than any positive valuation occurring in Y^*. Consequently x^b is less than any infinitesimal of Y^*. Let $c = a + b$. Then,

1. x^c witnesses the gap (L, U).
2. x^c is transcendental over X.
3. x^c has strict level μ.
4. The subring of $\mathcal{E}$ generated by $Y^* \cup \{x^c\}$ is full and is closed under partial sums.

Let $\psi : X^*[x^*] \to Y^*[x^c]$ be the unique $\mathbb{R}$-monomorphism extending ϕ such that

$$\psi(x^*) = t + x^c.$$

Then ψ is extendible.

Case 2: There is no largest partial sum of y that is a member of Y^*.

Let D be the well-ordering by ascending valuation of the exponents of y. Then D has a countable order-type. Let D' be the smallest initial segment of D such that

$y \upharpoonright_{D'}$ (the partial sum of y with exponents from D') is not a member of Y. Let $y' = y \upharpoonright_{D'}$. Then y' witnesses the gap (L, U). The strict level of y' is no greater than μ. If the strict level of y' equals μ, let $\psi : X^*[x^*] \to Y^*[y']$ be the unique $\mathbb{R}$-monomorphism extending ϕ such that

$$\psi(x^*) = y'.$$

Otherwise, let $a \in G_{\omega_1}$ be an exponent of $\mathcal{E}$ that has strict level μ and is greater than all exponents occurring in Y^*. Let

$$y^* = y' + x^a.$$

Then y^* witnesses the gap (L, U) and has strict level μ. Let $\psi^* : X^*[x^*] \to Y^*[y^*]$ be the unique $\mathbb{R}$-monomorphism extending ϕ such that

$$\psi(x^*) = y^*.$$

Then ψ^* satisfies the conditions for an extendible function, except closure of the range of ψ^* under partial sums. In particular y' and x^a are not in the range of ψ^*. Let (L^*, U^*) be the gap formed by x^a in $Y^*[y^*]$. We observe that x^a is infinitesimal with respect to every member of $Y^*[y^*]$. Since F is level dense and upward level dense, there is a positive element of F, ε, with strict level μ that is infinitesimal with respect to all elements of $X^*[x^*]$. Therefore there is an $\mathbb{R}$-monomorphism, $\psi : X^*[x^*, \varepsilon] \to Y^*[y^*, x^a]$, extending ψ^* and such that

$$\psi(\varepsilon) = x^a.$$

We note that $y' \in Y^*[y^*, x^a]$ and

$$Y^*[y^*, x^a] = Y^*[y', x^a].$$

Then $Y^*[y', x^a]$ is closed under coefficients and partial sums, so ψ is extendible.   ⊣

These results permit a simplification of the construction. Given an extendible $\mathbb{R}$-monomorphism, $\phi : D \to \mathcal{E}$, we may extend ϕ to an $\mathbb{R}$-monomorphism of $D[\mathbb{R}]$, and then extend by algebraically independent infinitesimals.

§7. An $\mathbb{R}$-monomorphism from the finite elements of $\mathbb{R}^\omega/U$ into the Esterle algebra. In the Cohen extension adding $\aleph_2$ generic reals, we construct a level $\mathbb{R}$-monomorphism, ϕ, from the finite elements of a standard ultrapower of $\mathbb{R}$ into $\mathcal{E}$. This construction differs significantly from the construction of Woodin [15]. The construction of Woodin relies on the fact that in the Cohen extension by $\aleph_2$-generic reals of a model of $ZFC + CH$, any cut of $\mathcal{E}$ has a countable, cofinal subcut. As Woodin observes, this argument is not generalizable to models of ZFC with higher powers of the continuum. Our construction yields a monomorphism that is level, and therefore is sensitive to the "complexity" (with respect to the index of Cohen reals) of the elements of the ultrapower and the Esterle algebra. Consequently, for any $\mu \leq \omega_2$, $\phi \cap M[G(\mu)]$ is an $\mathbb{R}$-monomorphism of $\mathbb{R}^\omega/U \cap M[G(\mu)]$ to $\mathcal{E} \cap M[G(\mu)]$.

THEOREM 7.1. *Suppose M is a transitive model of ZFC+CH containing a simplified $(\omega_1, 1)$-morass and P is the poset adding generic reals indexed by ordinals less than*

ω_2. *Let G be P-generic over M, $F \in M[G]$ be the ring of finite elements of a standard ultrapower of $\mathbb{R}$ over ω, and $\mathcal{E} \in M[G]$ be the Esterle algebra computed in $M[G]$. Then there is a level $\mathbb{R}$-monomorphism, $\phi : F \to \mathcal{E}$.*

PROOF. For each $v, \alpha \le \omega_1$, let G_v be the factor of G adding generic reals indexed by θ_v (the ordinal associated with the vertex v in the morass), and $G(\alpha)$ be the factor of G adding generic reals indexed by α. Let

$$F_v = F \cap M[G_v],$$

$$F(\alpha) = F \cap M[G(\alpha)].$$

In Lemma 3.3 of [4], it is proved that, given a simplified $(\omega_2, 1)$-morass, any countable subset of ω_3 is in the image of a single morass map from a vertex below ω_2. The proof does not depend on ω_2 and generalizes to simplified $(\omega_n, 1)$-morasses. Since every element of F has countable support, we observe that $F(\omega_1) = \bigcup_{v < \omega_1} F_v$ and, since any countable subset of ω_2 is a subset of the range of a morass map from $\mathcal{F}_{v\omega_1}$ $(v < \omega_1)$, $F_{\omega_1} = \bigcup_{v < \omega_1, \sigma \in \mathcal{F}_{v\omega_1}} \sigma[F_v] = F$. Any commitment to the construction entails uncountably many subsequent commitments by way of commutativity with morass maps. Terms for members of F are in the image of morass maps from lower vertices of the morass. ⊣

DEFINITION 7.2 (Morass generator). If $\beta < \alpha \le \omega_1$, $\sigma \in \mathcal{F}_{\beta\alpha}$, $x \in M^{P_\beta}$, and $y = \sigma(x) \in M^{P_\alpha}$, then x is a morass ancestor of y and y is a morass descendant of x. If x has no morass ancestors, then we say x is a morass generator.

Let X be a a set of strict terms for members of $F(\omega_1)$ that is forced to be a maximal algebraically independent subset of morass-generators. We may assume that every member of X has Archimedean valuation 0 (is in $\mathbb{R}$) or is infinitesimal. We note that the morass-closure of X beneath ω_1 contains a transcendental basis for F. Although the morass-closure of X has cardinality $\aleph_2$ in $M[G]$, X has cardinality $\aleph_1$.

Let $\langle x_\alpha \mid \alpha < \omega_1 \rangle$ be a well-ordering of X. Let $\langle v_\alpha \mid \alpha < \omega_1 \rangle$ be a weakly ascending transfinite sequence of countable ordinals such that $x_\alpha \in M[G_{v_\alpha}]$. We will construct by transfinite recursion (on ω_1) an ascending sequence of functions (ordered by inclusion), $\langle \phi_\alpha : D_\alpha \to \mathcal{E}_\alpha \mid \alpha < \omega_1 \rangle$, such that for all $\alpha < \omega_1$,

1. $x_\alpha \in D_\alpha$.
2. D_α is morass-closed beneath v_α.
3. $\phi_\alpha : D_\alpha \to E_\alpha$ is extendible.

At each stage of the construction, $\alpha < \omega_1$, the domain of ϕ_α extends $\bigcup_{\beta < \alpha, f \in \mathcal{F}_{\beta\alpha}} f(D_\beta)$, so that it contains x_α and is morass-closed beneath v_α, and ϕ_α is extendible.

Case: $\alpha = 0$.

Let $x^* = x_0$ have strict level $\mu \le v_0$, v be the least ordinal such that $\mu \in \theta_v$, and $X_0 = \langle x_{0,n} \rangle$ be a maximal algebraically independent sequence of morass descendents of $x^* \in M[G_{v_0}]$. Then X_0 is countable, and possibly finite. If $v \le \eta < v_0$ and $x' \in M[G_\eta]$ is a morass-descendent of x^*, then since $id \upharpoonright_{\theta_\eta} \in \mathcal{F}_{\eta v_0}$, x' is a morass-descendent of x^* in $M[G_{v_0}]$, and $x' \in X_0$. Since x^* has strict level $\mu \in \theta_v$, for $\sigma \in \mathcal{F}_{v v_0}$, the strict level of $\sigma(x^*) = \sigma(\mu)$. If $y, z \in X_0$ both have the same strict level, then morass functions in $\mathcal{F}_{v v_0}$ that witness that y and z are morass descendents

of x^*, must agree up to $\mu \leq \theta_v$, and $y = z$. Therefore all members of X_0 have pairwise distinct strict levels.

We construct a sequence of $\mathbb{R}$-monomorphisms, $\langle \phi_{0,n} \rangle$, with the order-type of X_0, such that, for all n less than the order-type of X_0,

1. $D_{0,0} = \mathbb{Q}[x_{0,0}]$.
2. $D_{0,n}[x_{0,n+1}] \subseteq D_{0,n+1}$.
3. $\phi_{0,n} : D_{0,n} \to \mathcal{E} \in M[G_{v_0}]$.
4. For all $m < n$, $\phi_{0,m} \subseteq \phi_{0,n}$.
5. $\phi_{0,n}$ is extendible.

Let $z = x_{0,0}$, and μ be the strict level of z. We may assume that z is positive. If $z \in \mathbb{R}$, let $D_{0,0} = \mathbb{Q}[z]$ and $\phi_{0,0} : D_{0,0} \to \mathcal{E}$ be the identity restricted to $D_{0,0}$.

If z is infinitesimal, let $a \in G_{\omega_1}$ be positive and have strict level μ, and

$$y = x^a \in \mathcal{E}.$$

Let $D_{0,0} = \mathbb{Q}[z]$ and $\phi_{0,0} : D_{0,0} \to \mathcal{E}$ be the $\mathbb{R}$-linear ring monomorphism such that

$$\phi_{0,0}(z) = y.$$

Let $N \in \omega$ and assume that $\langle \phi_{0,n} \mid n \leq N \rangle$ satisfies conditions 1–5 above (below $N + 1$). If X_0 has length $N + 1$, Then $D_0 = D_{0,N}$ and $\phi_0 = \phi_{0,N}$. Otherwise $\phi_{0,N}$ is extendible and Lemmas 6.7 and 6.8 apply. Let $z = x_{0,N+1}$. If $z \in \mathbb{R}$, let

$$D_{0,N+1} = D_{0,N}[z]$$

and

$$\phi^* = \phi_{0,N} \cup \{(z,z)\}.$$

By Lemma 6.7, there is an $\mathbb{R}$-monomorphic extension of ϕ^*, $\phi_{0,N+1} : D_{0,N+1} \to \mathcal{E}$. The sub-ring of $\mathcal{E}$ generated by a set closed under partial sums and coefficients is closed under partial sums and coefficients so $\phi_{0,N+1}$ is extendible.

If z is non-standard, let R_0 be the set of reals contained in the smallest full, real closure of $D_{0,N} \cup \{z\}$. By Lemma 6.7 there is an extendible level $\mathbb{R}$-monomorphism extending $\phi_{0,N}$, $\phi^* : D_{0,N}[R_0] \to \mathcal{E}$. Let

$$D_{0,N+1} = D_{0,N}[R_0, z].$$

Then $D_{0,N+1} \in M[G_{v_0}]$. By Lemma 6.8 there is an extendible extension of ϕ^*,

$$\phi_{0,N+1} : D_{0,N+1} \to \mathcal{E}.$$

Let

$$D_0 = \bigcup_{n \in \omega} D_{0,n}$$

and

$$\phi_0 = \bigcup_{n \in \omega} \phi_{0,n}.$$

Then the morass descendants of x_0 (in $M[G_{v_0}]$) are elements of D_0, and ϕ_0 is extendible.

Assume α a successor.

Let $\alpha = \bar{\alpha} + 1$. Assume that $\langle D_\beta \mid \beta < \alpha \rangle$ and $\langle \phi_\beta \mid \beta < \alpha \rangle$ have been defined so that for all $\gamma < \beta \le \bar{\alpha}$,

1. $x_\gamma \in D_\gamma$.
2. $D_\gamma \subseteq D_\beta$ and D_β is morass-closed beneath v_β.
3. $\phi_\gamma \subseteq \phi_\beta$.
4. $\phi_\beta : D_\beta \to \mathcal{E}$ is an extendible $\mathbb{R}$-monomorphism.

If $v_\alpha = v_{\bar{\alpha}}$, then we may argue as in the previous case. Let $\langle x_{\alpha,n} \rangle$ be an enumeration of a maximal AI set of morass descendants of x_α in $M[G_{v_\alpha}]$. We may extend $D_{\bar{\alpha}}$ to D_α containing the morass descendants of x_α, and $\phi_{\bar{\alpha}}$ to $\phi_\alpha : D_\alpha \to \mathcal{E}$ so that for all $\gamma < \beta \le \alpha$,

1. $D_\gamma \subseteq D_\beta$ and D_β is morass-closed beneath v_β.
2. $\phi_\gamma \subseteq \phi_\beta$.
3. ϕ_β is extendible.

If $v_{\bar{\alpha}} < v_\alpha$, then by Lemma 6.6 there is an extendible $\mathbb{R}$-monomorphism ϕ^* extending $\bigcup_{\sigma \in \mathcal{F}_{v_{\bar{\alpha}} v_\alpha}} \sigma[\phi_{\bar{\alpha}}]$ to the ring generated by $\bigcup_{\sigma \in \mathcal{F}_{v_{\bar{\alpha}} v_\alpha}} \sigma[D_{\bar{\alpha}}]$, D^*. Let D' be ring generated by D^* and the morass descendants of x_α in $M[G_{v_\alpha}]$. Let $\mathbb{R}_0$ be the real numbers of the smallest full extension of the real closure of D'. Then the real closure of $D'[\mathbb{R}_0]$ is full. Let

$$D_\alpha = D'[\mathbb{R}_0].$$

By the preceding case, there is an extendible $\mathbb{R}$-monomorphism $\phi_\alpha : D_\alpha \to \mathcal{E}$ with $\phi_\alpha \supseteq \phi^*$.

Finally, assume $\alpha < \omega_1$ is a limit ordinal.

If there is $\beta < \alpha$ such that $v_\alpha = v_\beta$ then we may proceed as in the case $\alpha = 0$ to define D_α and ϕ_α.

So we assume that $v_\beta < v_\alpha$ for all $\beta < \alpha$. Let

$$\lambda = \bigcup_{\beta < \alpha} v_\beta.$$

If $\lambda = v_\alpha$, then let

$$D^* = \bigcup_{\beta < \alpha} \left(\bigcup_{\sigma \in \mathcal{F}_{v_\beta v_\alpha}} \sigma[D_\beta] \right).$$

Let D be a finitely generated subring of D^* with generators $\{d_1, \dots, d_n\}$. Then there is $\beta < \alpha$,

$$C = \{c_1, \dots, c_n\}$$

and for, $i \le n$, morass functions $\sigma_i \in \mathcal{F}_{v_\beta v_\alpha}$ such that

$$\sigma_i(c_i) = d_i.$$

By condition P4 of Definition 2.1, there is $v_\beta \le \gamma < \lambda$, $f_1, \dots, f_n \in \mathcal{F}_{v_\beta \gamma}$ and $g \in \mathcal{F}_{\gamma \lambda}$ such that for all $i \le n$,

$$\sigma_i(c_i) = g \circ f_i(c_i) = d_i.$$

By Lemma 6.6 there is an extendible $\mathbb{R}$-monomorphism ϕ^* on the ring generated by $\bigcup_{\sigma \in \mathcal{F}_{v_\beta \lambda}} \sigma[\phi_\beta]$. It follows that there is an extendible $\mathbb{R}$-monomorphism on the ring generated by $\bigcup_{\beta < \alpha}(\bigcup_{\sigma \in \mathcal{F}_{v_\beta v_\alpha}} \sigma[D_\beta])$ extending $\bigcup_{\beta < \alpha}(\bigcup_{\sigma \in \mathcal{F}_{v_\beta v_\alpha}} \sigma[\phi_\beta])$. We may then proceed as in earlier cases to define an extendible $\mathbb{R}$-monomorphism $\phi_\alpha : D_\alpha \to \mathcal{E}$, with D_α containing the morass descendants of x_α in $M[G_{v_\alpha}]$.

Finally, assume that

$$\lim_{\beta < \alpha} v_\beta = \lambda < v_\alpha.$$

By the previous argument, there is an extendible $\phi^* : \bigcup_{\beta < \alpha}(\bigcup_{\sigma \in \mathcal{F}_{v_\beta \lambda}} \sigma[D_\beta]) \to \mathcal{E}$ extending the morass images in $M[G_\lambda]$ of the ϕ_β. By Lemma 6.6 there is an extendible $\mathbb{R}$-monomorphism $\phi'_\alpha : D'_\alpha \to \mathcal{E}$ such that

$$D'_\alpha \supseteq \bigcup_{\beta < \alpha}(\bigcup_{\sigma \in \mathcal{F}_{v_\beta v_\alpha}} \sigma[D_\beta])$$

and

$$\phi'_\alpha \supseteq \bigcup_{\beta < \alpha}(\bigcup_{\sigma \in \mathcal{F}_{v_\beta v_\alpha}} \sigma[\phi_\beta]).$$

We proceed as in earlier cases to extend ϕ'_α to an extendible $\mathbb{R}$-monomorphism, $\phi_\alpha : D_\alpha \to \mathcal{E}$, where D_α extends D' and contains a maximal AI set of morass-descendants of x_α in $M[G_{v_\alpha}]$.

Let $D_{\omega_1} = \bigcup_{\alpha < \omega_1}(\bigcup_{\sigma \in \mathcal{F}_{v_\alpha \omega_1}} \sigma[D_\alpha])$ and $\phi_{\omega_1} = \bigcup_{\alpha < \omega_1}(\bigcup_{\sigma \in \mathcal{F}_{v_\alpha \omega_1}} \sigma[\phi_\alpha])$. Then D_{ω_1} is a subring of F and contains the morass-closure of X (a maximal algebraically independent subset of F). Additionally, $\phi_{\omega_1} : D_{\omega_1} \to \mathcal{E}$ is an $\mathbb{R}$-monomorphism on a domain that contains a transcendental basis for F. Therefore ϕ_{ω_1} extends uniquely to an $\mathbb{R}$-monomorphism, $\phi : F \to \mathcal{E}$.

THEOREM 7.3. *Suppose M is a transitive model of $ZFC + CH$ containing a simplified $(\omega_1, 1)$-morass and $M[G]$ is the Cohen extensions adding $\aleph_2$ generic reals. Then if X is an infinite compact Hausdorff space in $M[G]$, there is a discontinuous homomorphism of $C(X)$ in $M[G]$.*

PROOF. By Corollary 6.9 of [3], any non-principal ultrafilter on ω in M may be extended to a standard ultrafilter in the Cohen extension adding $\aleph_2$-generic reals. If U is a standard ultrafilter, then there is a level $\mathbb{R}$-monomorphism from the finite elements of $\mathbb{R}^\omega/U$ into $\mathcal{E}$. Hence the finite elements of $\mathbb{R}^\omega/U$ bear a non-trivial submultiplicative norm. The theorem follows from results of B. Johnson [10].   ⊣

§8. Discontinuous homomorphisms of $C(X)$ in a Cohen extension adding $\aleph_3$-generic reals. We turn to generic extensions adding more than $\aleph_2$ generic reals. We require a simplified $(\omega_1, 2)$-morass for the next construction (Velleman [14]). The simplified $(\omega_1, 2)$-morass will allow us to use morass maps to construct an $\mathbb{R}$-monomorphism from the finite elements of a standard ultrapower to $\mathcal{E}$ in a model with $\aleph_3$ generic reals in a manner similar to Theorem 7.1. The construction of a term function on a domain with cardinality $\aleph_3$, requiring only $\aleph_1$ many explicit

commitments, allows us to exploit that standard ultrapowers and $\mathcal{E}$ are η_1-orderings. We will rely heavily on the definitions and results of D. Velleman [14] in this section.

DEFINITION 8.1 (D. Velleman) (Simplified $(\kappa, 2)$-morass). The structure $\langle \overrightarrow{\varphi}, \overrightarrow{\mathcal{G}}, \overrightarrow{\theta}, \overrightarrow{\mathcal{F}} \rangle$ is a simplified $(\kappa, 2)$-morass provided it has the following properties:

1. $\langle \overrightarrow{\varphi}, \overrightarrow{\mathcal{G}} \rangle$ is a neat simplified $(\kappa^+, 1)$-morass.
2. $\forall \alpha < \beta \leq \kappa$, $\mathcal{F}_{\alpha\beta}$ is a family of embeddings (see page 172 [14]) from $\langle \langle \varphi_\zeta \mid \zeta < \theta_\alpha \rangle, \langle \mathcal{G}_{\zeta\xi} \mid \zeta < \xi \leq \theta_\alpha \rangle \rangle$ to $\langle \langle \varphi_\zeta \mid \zeta < \theta_\beta \rangle, \langle \mathcal{G}_{\zeta\xi} \mid \zeta < \xi \leq \theta_\beta \rangle \rangle$.
3. $\forall \alpha < \beta < \kappa$ $(|\mathcal{F}_{\alpha\beta}| < \kappa)$.
4. $\forall \alpha < \beta < \gamma \leq \kappa$ $(\mathcal{F}_{\alpha\gamma} = \{ f \circ g \mid f \in \mathcal{F}_{\beta\gamma}, g \in \mathcal{F}_{\alpha\beta} \})$. Here $f \circ g$ is defined by:

$$(f \circ g)_\zeta = f_{g(\zeta)} \circ g_\zeta \quad for \ \zeta \leq \theta_\alpha,$$

$$(f \circ g)_{\zeta\xi} = f_{g(\zeta)g(\xi)} \circ g_{\zeta\xi} \quad for \ \zeta < \xi \leq \theta_\alpha.$$

5. $\forall \alpha < \kappa$, $\mathcal{F}_{\alpha\alpha+1}$ is an amalgamation (see page 173 [14]).
6. If $\beta_1, \beta_2 < \alpha \leq \kappa$, α a limit ordinal, $f_1 \in \mathcal{F}_{\beta_1\alpha}$ and $f_2 \in \mathcal{F}_{\beta_2,\alpha}$, then $\exists \beta (\beta_1, \beta_2 < \beta < \alpha$ and $\exists f_1' \in \mathcal{F}_{\beta_1\beta} \exists f_2' \in \mathcal{F}_{\beta_2\beta} \exists g \in \mathcal{F}_{\gamma\alpha} (f_1 = g \circ f_1'$ and $f_2 = g \circ f_2'))$.
7. If $\alpha \leq \kappa$ and α is a limit ordinal, then:
   (a) $\theta_\alpha = \bigcup \{ f[\theta_\beta] \mid \beta < \alpha, f \in \mathcal{F}_{\beta\alpha} \}$.
   (b) $\forall \zeta \leq \theta_\alpha, \varphi_\zeta = \bigcup \{ f_{\bar\zeta}[\varphi_{\bar\zeta}] \mid \exists \beta < \alpha (f \in \mathcal{F}_{\beta\alpha}, f(\bar\zeta) = \zeta) \}$.
   (c) $\forall \zeta < \xi \leq \theta_\alpha, \mathcal{G}_{\zeta\xi} = \bigcup \{ f_{\bar\zeta\bar\xi}[\mathcal{G}_{\bar\zeta\bar\xi}] \mid \exists \beta < \alpha (f \in \mathcal{F}_{\beta\alpha}, f(\bar\zeta) = \zeta, f(\bar\xi) = \xi) \}$.

THEOREM 8.2. *Let M be a transitive model of $ZFC + CH$ containing a simplified $(\omega_1, 2)$-morass, and $M[G]$ be a generic extension of M adding $\aleph_3$ generic reals. Let X be an infinite compact Haussdorf space in $M[G]$, and $C(X)$ be the algebra of continuous real-valued functions of X in $M[G]$. Then there is a discontinuous homomorphism of $C(X)$ in $M[G]$.*

PROOF. Let M be a transitive model of $ZFC + CH$ containing a simplified $(\omega_1, 2)$-morass, $\langle \overrightarrow{\varphi}, \overrightarrow{\mathcal{G}}, \overrightarrow{\theta}, \overrightarrow{\mathcal{F}} \rangle$. Let P be the poset adding generic reals indexed by ω_3, and G be P-generic over M. Then $\langle \overrightarrow{\varphi}, \overrightarrow{\mathcal{G}} \rangle$ is a simplified $(\omega_2, 1)$-morass, and below ω_1, $\langle \overrightarrow{\varphi}, \overrightarrow{\mathcal{G}} \rangle$ satisfies the axioms of a simplified $(\omega_1, 1)$-morass. Hence, by Lemma 3.3 of [4], any countable subset of ω_3 is in the image of a single morass map of $\overrightarrow{\mathcal{G}}$. The construction of Theorem 7.1 below ω_1 can be completed in M. In particular, if $U_0 \in M$ is a non-principal ultrafilter in M, then by Corollary 6.9 of [4], there is $\overline{U} \subseteq M^{P(\omega_1)}$, a standard term for an ultrafilter below ω_1, that is forced to extend U_0. Furthermore, the morass-closure of $\overline{U}$, U, is a standard ultrafilter. By Theorem 6.4 of [4], $\mathbb{R}^\omega / U$ is a gap-2 morass-definable η_1-ordering.

We construct a level term function from the finite elements of a standard ultrapower to the Esterle algebra, that is closed under morass-embeddings and is forced to be an $\mathbb{R}$-monomorphism. For $\alpha < \omega_1$, let $X_\alpha = (\mathbb{R}^\omega / U) \cap M^{P_\alpha}$ and $Y_\alpha = \mathcal{E} \cap M^{P_\alpha}$. We consider X_α and Y_α as the restrictions of $\mathbb{R}^\omega / U$ and $\mathcal{E}$, resp., to the forcing language adding generic reals indexed by φ_{θ_α}. In any P-generic extension of M, $M[G]$, the interpretation of X_α in $M[G]$ is the interpretation of X_α in $M[G_\alpha]$ where G_α is the factor of G that is P_α-generic over M.

It is sufficient to construct a level, morass-commutative term injection from X_{ω_1} to Y_{ω_1} that is forced to be an $\mathbb{R}$-monomorphism. The closure under embeddings, f_{θ_β}, where $\beta < \omega_1$ and $f \in \mathcal{F}_{\beta\omega_1}$, of this term function will be the term function we seek.

Let $\{x_\beta \mid \beta < \omega_1\} \subseteq X_{\omega_1}$ be a transfinite sequence of terms of strict level for a maximal algebraically independent set of morass-generators for the infinitesimal elements of X_{ω_1}, such that $x_\alpha \in X_\alpha$ for all $\alpha < \omega_1$.

We will inductively construct a transfinite sequence of morass-commutative term functions $\langle F_\beta : D_\beta \to E_\beta \mid \beta < \omega_1 \rangle$ that satisfies the following for all $\alpha \le \beta < \omega_1$:

1. $D_\beta \subseteq X_{\theta_\beta}$ is a subring of finite elements of a standard ultrapower of $\mathbb{R}$, that is morass-closed beneath θ_β.
2. $E_\beta \subseteq Y_{\theta_\beta}$ is full and closed under partial sums.
3. $D_\alpha \subseteq D_\beta$ and $E_\alpha \subseteq E_\beta$.
4. $x_\beta \in D_\beta$.
5. F_β is a level term function that is forced to be an $\mathbb{R}$-monomorphism.
6. $f_{\theta_\alpha}[F_\alpha] \subseteq F_\beta$ for all $f \in \mathcal{F}_{\alpha\beta}$.

We call a sequence of term functions satisfying these conditions an extendible sequence beneath β. We argue be induction on $\gamma < \omega_1$.

Base Case: $\gamma = 0$.

Let y_0 be a positive infinitesimal monomial of Y_0 having the same strict level as x_0, and $\mathbb{R}_0$ be the reals of the ground model. Let D_0 be the ring generated by $\mathbb{R}_0 \cup \{x_0\}$, $\mathbb{R}_0[x_0]$, and $E_0 = \mathbb{R}_0[y_0]$. We observe that E_0 is closed under partial sums. Therefore, there is an R-monomorphism, $F_0 : D_0 \to E_0$, with $F_0(x_0) = y_0$.

Successor Case: $\gamma = \beta + 1$. Let $\langle F_\alpha : D_\alpha \to E_\alpha \mid \alpha \le \beta \rangle$ be an extendible sequence satisfying conditions 1–6 above. Let D^* be the ring generated by $\{g_{\theta_\beta}[D_\beta] \mid g \in \mathcal{G}_{\theta_\beta\theta_\gamma}\}$. Then D^* is generated by the union of the images of D_β under the second components of left-branching embeddings of $\mathcal{F}_{\beta\gamma}$. Let h be the right-branching embedding of $\mathcal{F}_{\beta\gamma}$ and D' be the ring generated by D^* and $h_{\theta_\beta}[D_\beta]$.

By Lemma 5.2 of [4], $\bigcup\{f_{\theta_\beta}[F_\beta] \mid f \in \mathcal{F}_{\beta\gamma}\}$ is a level term function that is forced to be an order-preserving injection. Since F_β is $\mathbb{R}$-linear, for any $f \in \mathcal{F}_{\beta\gamma}$, $f_{\theta_\beta}[F_\beta]$ is $\mathbb{R}$-linear. Therefore $\bigcup\{f_{\theta_\beta}[F_\beta] \mid f \in \mathcal{F}_{\beta\gamma}\}$ is $\mathbb{R}$-linear. If $f \in \mathcal{F}_{\beta\gamma}$, then $f_{\theta_\beta} : \theta_\beta \to \theta_\gamma$ is order-preserving. If f is left-branching, then $f_{\theta_\beta} \in \mathcal{G}_{\theta_\beta\theta_\gamma}$. If f is right branching, then f_θ is order-preserving. By Lemma 6.6, $\bigcup\{f_{\theta_\beta}[F_\beta] \mid f \in \mathcal{F}_{\beta\gamma}\}$ extends to a homomorphism, $F' : D' \to \mathcal{E}$. Therefore there is a unique extendible $\mathbb{R}$-monomorphism, $F^* : D^* \to \mathcal{E}$.

Let B be a transcendental basis of D_β over $\mathbb{R}^\omega/U \cap M$. Let D be the ring generated by B and $D_\beta \cap M$. Let V be the semigroup generated by B. Because $\mathcal{F}_{\beta\gamma}$ is a set of compatible embeddings, Lemma 6.1 applies and, treating D as a vector space (over the field generated by $D_\beta \cap M$), V is a basis of elements of strict level for D. Therefore, if $F : D \to \mathcal{E}$ is the naturally induced $\mathbb{R}$-linear extension of $\bigcup_{f \in \mathcal{F}_{\beta\gamma}} f_{\theta_\beta}[F_\beta]$, the image of V under F is a linear independent subset of $\mathcal{E}$ over $E_\beta \cap M$. Therefore F is a linear transformation and an $\mathbb{R}$-monomorphism of D.

We argue along the lines of Lemma 6.5 to show that F' is forced to be order-preserving. By Lemma 6.5, $F' \restriction_{D^*}$ is forced to be order-preserving. If $z \in D'$, there

is $n \in \mathbb{N}$, $x_1, \ldots, x_n \in D^*$ and $y_1, \ldots, y_n \in f_{\theta_\beta}[D_\beta]$ such that $z = \sum_{i=1}^n x_i \cdot y_i$. We show that $\sum_{i=1}^n x_i \cdot y_i > 0$ iff $\sum_{i=1}^n F'(x_i) \cdot F'(y_i) > 0$.

If $n = 1$, then the sign of $x_1 \cdot y_1$ is the sign of the product of the leading coefficients of $F'(x_1)$ and $F'(y_1)$. However, F' is $\mathbb{R}$-linear. So

$$x_1 \cdot y_1 > 0 \iff F'(x_1) \cdot F'(y_1) > 0.$$

Assume that $n \geq 2$, and

$$\sum_{i=1}^n x_i \cdot y_i > 0.$$

Let

$$z = F'\left(\sum_{i=1}^n x_i \cdot y_i\right) = \sum_{\lambda < \gamma} \alpha_\lambda x^{a_\lambda}.$$

By assumption, the range of F' is closed under partial sums, so there are elements $u_1, \ldots, u_j, u_{j+1}, \ldots u_k, v_1, \ldots, v_k \in Y$ such that:

1. $z = \sum_{i=1}^k u_i \cdot v_i$.
2. For $i \leq j$, every term in the power series expansion of $u_i \cdot v_i$ has power less than a_0.
3. For $j < i \leq k$, every term of the power series expansion of $u_i \cdot v_i$ has power at least a_0.

Every term of z, expressed as a power series of $\mathcal{E}$, has valuation no less than a_0, therefore

$$\sum_{i=1}^j u_i \cdot v_i = 0$$

and

$$z = \sum_{i=j+1}^k u_i \cdot v_i.$$

If $j < i \leq k$, then $s_i = u_i \cdot v_i$ has valuation no less than a_0 in G_{ω_1}. If the leading terms of s_i has exponent greater than a_0, then s_i has Archimedean valuation less than z. Let S be the set of all terms of all s_i, for all $1 \leq j \leq i$, having exponent a_0 and T be the set of all terms of all s_i, $1 \leq i \leq j$, with exponent greater than a_0. S is nonempty and every element of T has Archimedean valuation less than every element of S.

There are $b_0, c_0 \in G_{\omega_1}$ such that x^{b_0} is in the range of F^* and x^{c_0} is in the range of $f_{\theta_\beta}[F_\beta]$ and

$$a_0 = b_0 + c_0.$$

Let $u \in D^*$ and $v \in f_{\theta_\beta}[D_\beta]$ be such that

$$F'(u) = x^{b_0}$$

and

$$F'(v) = x^{c_0}.$$

We observe that the ordered ring D' has a unique extension to the field closure of D', $\bar{D}$. Treated as a field map, the unique monomorphic extension of F', $\bar{F} : \bar{D} \to \mathcal{E}$, is an $\mathbb{R}$-linear field monomorphism of D'. We have previously observed that F' is an $\mathbb{R}$-linear monomorphism, and $F^* \cup f_{\theta_\beta}[F_\beta]$ is an order-preserving injection. So

$$F'\left(\sum_{i=1}^{n} x_i \cdot y_i\right) = F'\left(\sum_{i=j+1}^{k} u_i \cdot v_i\right)$$

and

$$\bar{F}\left(\sum_{i=j+1}^{k} u_i/u \cdot v_i/v\right) = \alpha_0 + \sum_{0<\lambda<\gamma} \alpha_\lambda x^{(a_\lambda - a_0)}.$$

The real number, α_0, is in the domain $\bar{F}$, so $\bar{F}(\alpha_0) = F'(\alpha_0) = \alpha_0$. The range of F' is closed under partial sums, so both $\sum_{0<\lambda<\gamma} \alpha_\lambda x_\lambda^a$ and x^{a_0} are in the range of F'. Thus $\sum_{0<\lambda<\gamma} \alpha_\lambda x^{a_\lambda - a_0}$ is in the range of $\bar{F}$. Let

$$\delta = \sum_{0<\lambda<\gamma} \alpha_\lambda x^{a_\lambda - a_0}.$$

Every term of δ is infinitesimal (has Archimedean valuation greater than elements of $\mathbb{R}$). $\bar{F}$ is order preserving on D', so $\bar{F}^{-1}(\delta)$ is infinitesimal. Hence

$$\sum_{i=1}^{n} (x_i \cdot y_i)/(u \cdot v) = \alpha_0 + \bar{F}^{-1}(\delta)$$

and the $\sum_{i=1}^{n} x_i \cdot y_i > 0$ iff $\alpha_0 > 0$ iff $F'(\sum_{i=1}^{n} x_i \cdot y_i) > 0$.

Therefore $F' : D' \to \mathcal{E}$ is an $\mathbb{R}$-monomorphism, in which the range of F' is full and contains all partial sums of the range of F'. Let $\hat{D} = D'[\mathbb{R}][x_\gamma]$, computed in $M[G(\theta_\gamma)]$. By application of Lemma 6.7, we may extend F' to $\hat{F} : \hat{D} \to \mathcal{E}$, to a level $\mathbb{R}$-monomorphism. Assume that $x_\gamma \notin D'[\mathbb{R}]$. By Lemma 6.8, there is $D_\gamma \supset D'[\mathbb{R}]$, with $x_\gamma \in D_\gamma$, and an extension of $\hat{F}$, $F_\gamma : D_\gamma \to \mathcal{E}$, such that F_γ is an extendible function.

Limit Case: Suppose γ is a limit ordinal. Let $D' = \bigcup_{\beta<\gamma} \cup (\{f_{\theta_\beta}[D_\beta] \mid f \in \mathcal{F}_{\beta\gamma}\})$. Then D' is a subring of D_{ω_1} and has countable transcendence degree over $\mathbb{R}$. By condition 6 of Definition 8.1 it is straightforward to verify that $F' : D' \to \mathcal{E}$ defined by $F' = \bigcup_{\beta<\gamma} \cup (\{f_{\theta_\beta}[F_\beta] \mid f \in \mathcal{F}_{\beta\gamma}\})$ is an extendible function. Since D' is full, we may apply Lemma 6.7 to extend F' to an $\mathbb{R}$-monomorphism of $D'[\mathbb{R}]$ (computed in $M[G_{\theta_\gamma}]$). Let D_γ be the ring $D'[\mathbb{R}][x_\gamma]$. Then by Lemma 6.8 there is an extension of F', $F_\gamma : D_\gamma \to \mathcal{E}$ that is an extendible function. Let $F = \{f_{\omega_1}[F_{\omega_1}] \mid f \in \mathcal{F}_{\omega_1\omega_2}\}$. By Lemma 3.3 of [4], the domain of F is forced to be the finite elements of $\mathbb{R}^\omega/U$. ⊣

COROLLARY 8.3. *Let M be a transitive model of $ZFC + V = L$ containing a simplified $(\omega_2, 2)$-morass, and $M[G]$ be a generic extension of M adding $\aleph_4$ generic reals. Let X be an infinite compact Haussdorf space in $M[G]$. Then there is a discontinuous homomorphism of $C(X)$, the algebra of continuous real-valued functions on X in $M[G]$.*

Proof. Let M be a transitive model of $ZFC + V = L$ containing a simplified $(\omega_2, 2)$-morass, $\langle \overrightarrow{\varphi}, \overrightarrow{\mathcal{G}}, \overrightarrow{\theta}, \overrightarrow{\mathcal{F}} \rangle$. Let P be the poset adding generic reals indexed by ω_4, and G be P-generic over M. Then $\langle \overrightarrow{\varphi}, \overrightarrow{\mathcal{G}} \rangle$ is a simplified $(\omega_3, 1)$-morass, and below ω_2, $\langle \overrightarrow{\varphi}, \overrightarrow{\mathcal{G}} \rangle$ satisfies the axioms of a simplified $(\omega_2, 1)$-morass. Let G_{ω_2} be the factor of G that adds generic reals indexed by ω_2. By a routine extension of Theorem 6.4 of [4] there is $\overline{U} \subseteq M^{P(\omega_2)}$, a standard term for an ultrafilter that is morass-closed below ω_2. Furthermore, the morass-continuation of $\overline{U}$, U, is a standard ultrafilter that is gap-2 morass-closed. By Theorem 6.4 of [4], $\mathbb{R}^\omega/U \in M^P$ is a gap-2 morass-definable η_1-ordering.

We adapt the argument of Theorem 8.2 to the $(\omega_2, 2)$-morass. The argument is altogether similar except for one detail. The transfinite construction we require will necessarily be of order-type ω_2. Hence it is not enough that $\mathcal{E}$ is an η_1-ordering to extend the term function by $\aleph_2$ many specified terms. Here we use Woodin's argument [15], that in the generic extension adding $\aleph_2$ generic reals to L, all cuts of $\mathcal{E}$ bear countable cofinal subcuts. We will construct a morass-closed, level term function from the finite elements of a standard ultrapower, $\mathbb{R}^\omega/U$, to the Esterle algebra that is forced to be an $\mathbb{R}$-monomorphism.

For $\alpha < \omega_2$, let $X_\alpha = (\mathbb{R}^\omega/U) \cap M^{P_\alpha}$ and $Y_\alpha = \mathcal{E} \cap M^{P_\alpha}$. We consider X_α and Y_α as the restrictions of $\mathbb{R}^\omega/U$ and $\mathcal{E}$, resp., to the forcing language adding generic reals indexed by φ_{θ_α}. In any P-generic extension of M, $M[G]$, the interpretation of X_α in $M[G]$ is the interpretation of X_α in $M[G_\alpha]$ where G_α is the factor of G that is P_α-generic over M.

We construct a morass-commutative level term injection from X_{ω_2} to Y_{ω_2} that is forced to be an $\mathbb{R}$-monomorphism. The closure under embeddings, f_{θ_β} where $f \in \mathcal{F}_{\beta\omega_2}$, of this function will be the term function we seek.

Let $B = \{x_\beta \mid \beta < \omega_2\} \subseteq X_{\omega_2}$ be a transfinite sequence of terms of strict level for a maximal algebraically independent set of infinitesimal elements of X_{ω_2}, such that $x_\alpha \in X_\alpha$ for all $\alpha < \omega_2$.

We may inductively construct a transfinite sequence of morass-commutative term functions $\langle F_\beta : D_\beta \to E_\beta \mid \beta < \omega_2 \rangle$ that satisfies the following for all $\alpha \le \beta < \omega_2$:

1. $D_\beta \subseteq X_{\theta_\beta}$ is a subring of finite elements and is morass-closed beneath θ_β.
2. $E_\beta \subseteq Y_{\theta_\beta}$ is full and closed under partial sums.
3. $D_\alpha \subseteq D_\beta$ and $E_\alpha \subseteq E_\beta$.
4. $x_\beta \in D_\beta$.
5. F_β is a level, morass-commutative term function that is forced to be an $\mathbb{R}$-monomorphism.
6. $f_{\theta_\alpha}[F_\alpha] \subseteq F_\beta$ for all $f \in \mathcal{F}_{\alpha\beta}$.

The proof is similar to the proof of Theorem 8.2 with only minor modifications. Given an extendible sequence $\langle F_\beta : D_\beta \to E_\beta \mid \beta < \gamma \rangle$, where γ is a successor ordinal, we may construct an extendible $\mathbb{R}$-monomorphism, F', that extends $f_{\theta_\beta}[F_\beta]$ for all $\beta < \gamma$ and $f \in \mathcal{F}_{\beta\gamma}$. By Lemma 6.7 we may extend F' to $\hat{F}$, a level $\mathbb{R}$-monomorphism of the ring generated by the domain of F' and the real numbers of $M[G_{\theta_\gamma}]$. If x_γ is not in the domain of $\hat{F}$, then by Woodin's argument that all gaps of the Esterle algebra in $M[G_{\omega_2}]$ are (ω, ω^*)-gaps, we may apply Johnson's Lemma 3.5 and Lemma 6.8. Then there is D_γ extending the domain of $\hat{F}$, with $x_\gamma \in D_\gamma$, and

an extension of $\hat{F}$, $F_\gamma : D_\gamma \to \mathcal{E}$ such that $\langle F_\beta : D_\beta \to E_\beta \mid \beta \leq \gamma \rangle$ is an extendible sequence. The argument for γ a limit is similar.   ⊣

§9. Pressing the continuum. The techniques of this paper, [3] and [4] depend on the construction of functions between sets of terms in the forcing language of Cohen extensions, utilizing commutativity with order-preserving injections of indexing ordinals of the Cohen poset. Having presented the details of the construction of level term functions using simplified gap-1 and gap-2 morasses, it is relatively straightforward to see how these constructions extend to simplified higher (finite) gap morasses.

A simplified $(\kappa, n + 1)$-morass, for κ a regular cardinal and integer n, is a family of embeddings between fake simplified (κ, n)-morass segments that satisfies properties analogous to those relating a simplified $(\omega_1, 2)$-morass to embeddings between fake simplified $(\omega_1, 1)$-morass segments. Central to the utility of these constructions is that the embeddings are may be considered as order-preserving injections between ordinals. For a thorough treatment of simplified finite gap morasses, including an inductive definition, see Szalkai [12].

Higher gap morasses will allow the extension of results of this paper and [4] to Cohen extensions adding more than $\aleph_4$ generic reals. The definition of gap-2 morass-definable η_1-orderings and η_1-ordered real-closed fields (resp.) are easily generalized to gap-n morass-definable η_1-orderings and η_1-ordered real-closed fields (resp.).

REFERENCES

[1] H. G. Dales, *Automatic continuity: A survey.* **Bulletin of the London Mathematical Society**, vol. 10 (1978), pp. 129–183.

[2] ———, *A discontinuous homomorphism of $C(X)$.* **American Journal of Mathematics**, vol. 101 (1979), no. 3, pp. 647–734.

[3] B. A. Dumas, *Order-isomorphic η_1-orderings in Cohen extensions.* **Annals of Pure and Applied Logic**, vol. 158 (2009), pp. 1–22.

[4] ———, *Gap-2 Morass-definable η_1-orderings.* **Mathematical Logic Quarterly**, vol. 68 (2022), no. 2, pp. 227–242.

[5] P. Erdös, L. Gillman, and M. Henriksen, *An isomorphism theorem for real closed fields.* **Annals of Mathematics**, vol. 61 (1955), pp. 552–554.

[6] J. Esterle, *Sur l'existence d'un homomorphisme discontinu do $C(K)$.* **Acta Mathematica Academiae Scientiarum Hungarica**, vol. 30 (1977), pp. 113–127.

[7] ———, *Semi-normes on $C(K)$.* **Proceedings of the London Mathematical Society**, vol. 36 (1978), no. 3, pp. 27–45.

[8] ———, *Injections de semi-groupes divisible dans des algébras de convolution et construction d'homomorphismes discontinus des $C(K)$,* **Proceedings of the London Mathematical Society**, vol. 36 (1978), no. 3, pp. 59–85.

[9] H. Hahn, *Ober die nichtarchimedischen Grössensysteme.* **Sitzungsberichte Kaiserlichen der Akademie Wissenschaften, Wien, Mathematisch–Naturwissenschaftliche** Klasse 116 (Abteilung IIa, 1907), pp. 601–655.

[10] B. E. Johnson, *Norming $C(\Omega)$ and related algebras.* **Transactions of the American Mathematical Society**, vol. 220 (1976), pp. 37–58.

[11] S. Maclane, *The universality of power series fields.* **Bulletin of the American Mathematical Society**, vol. 45 (1939), pp. 888–890.

[12] I. Szalkai, *An inductive definition of higher gap simplified morasses*. **Publicationes Mathematicae Debrecen**, vol. 58 (2001), no. 4, pp. 605–634.

[13] D. Velleman, *Simplified morasses*, **Journal of Symbolic Logic**, vol. 49 (1984), pp. 257–271.

[14] ———, *Simplified gap-2 morasses*. **Annals of Pure and Applied Logic**, vol. 34 (1987), pp. 171–208.

[15] H. Woodin, *A discontinuous homomorphism from C(X) without CH*. **Journal of the London Mathematical Society**, vol. s2-48 (1993), no. 2, pp. 299–315.

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