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Group Cohomology and L^p-Cohomology of Finitely Generated Groups

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Abstract. Let G be a finitely generated, infinite group, let $p > 1$, and let $L^p(G)$ denote the Banach space $\{\sum_{x\in G} a_x x \mid \sum_{x\in G} |a_x|^p < \infty\}$. In this paper we will study the first cohomology group of G with coefficients in $L^p(G)$, and the first reduced L^p-cohomology space of G. Most of our results will be for a class of groups that contains all finitely generated, infinite nilpotent groups.

1 Introduction

In this paper G will always be a finitely generated, infinite group and S will always be a symmetric generating set for G. Let M be a right G-module. A 1-cocycle with values in M is a map $\delta\colon G \to M$ such that $\delta(gh) = \big(\delta(h)\big) g + \delta(g)$ for any $g, h \in G$; a 1-coboundary is a 1-cocycle of the form $\delta(g) = xg - x$ for some $x \in M$ and for all $g \in G$. We denote by $Z^1(G, M)$ the vector space of all 1-cocycles and the vector space of all 1-coboundaries will be denoted by $B^1(G, M)$. The factor group $H^1(G, M) = Z^1(G, M)/B^1(G, M)$ is called the *first cohomology group of* G with coefficients in M. Suppose now that M is a topological vector space and that the action of G on M is continuous. Then we give $Z^1(G, M)$ the compact open topology. Assuming that M is Hausdorff, this means that $\delta_n \to \delta$ in $Z^1(G, M)$ if and only if $\delta_n(g) \to \delta(g)$ in M for all $g \in G$. In general $B^1(G, M)$ is not closed in $Z^1(G, M)$. The quotient space $\bar{H}^1(G) = Z^1(G, M)/\overline{B^1(G, M)}$, where $\overline{B^1(G, M)}$ is the closure of $B^1(G, M)$ in $Z^1(G, M)$, is called the first reduced cohomology space.

Let $\mathcal{F}(G)$ be the set of complex-valued functions on G. We may represent each f in $\mathcal{F}(G)$ as a formal sum $\sum_{x\in G} a_x x$ where $a_x \in \mathbb{C}$ and $f(x) = a_x$. For a real number $p \geq 1$, $L^p(G)$ will consist of those formal sums for which $\sum_{x\in G} |a_x|^p < \infty$. Let $\alpha \in \mathcal{F}(G)$ and $g \in G$, the right translation of α by g is the function defined by $\alpha_g(x) = \alpha(xg^{-1})$. Observe that if α is represented formally by $\sum_{x\in G} a_x x$, then α_g is represented by $\sum_{x\in G} a_{xg^{-1}} x$.

In this paper we will study the cohomology theories defined above for the case $M = L^p(G)$, and with G acting on $L^p(G)$ by right translations.

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2 Preliminaries

Let $\mathbb{C}G$ be the group ring of G over $\mathbb{C}$. For $\alpha = \sum_{x \in G} a_x x \in \mathbb{C}G$ and $\beta = \sum_{x\in G} b_x x \in \mathcal{F}(G)$ we define a multiplication $\mathcal{F}(G) \times \mathbb{C}G \to \mathcal{F}(G)$ by

$$\beta * \alpha = \sum_{x,y} b_x a_y xy = \sum_{x \in G}\Big(\sum_{y\in G} b_{xy^{-1}} a_y\Big) x.$$

For $1 \le p \in \mathbb{R}$, let $D^p(G) = \{\beta \in \mathcal{F}(G) \mid \beta * (g-1) \in L^p(G) \text{ for all } g \in S\}$. Recall that S is a symmetric set of generators for G. We define a norm on $L^p(G)$ by $\|\alpha\|_p = (\sum_{x\in G} |a_x|^p)^{\frac{1}{p}}$, where $\alpha = \sum_{x\in G} a_x x \in L^p(G)$. Let $\beta = \sum_{x\in G} b_x x \in D^p(G)$ and let e be the identity element of G. We define a norm on $D^p(G)$ by $\|\beta\|_{D^p(G)} = (\sum_{g\in S} \|\beta * (g-1)\|_p^p + |\beta(e)|^p)^{\frac{1}{p}}$. Under this norm $D^p(G)$ is a Banach space. Let α_1 and α_2 be elements of $D^p(G)$. We will write $\alpha_1 \simeq \alpha_2$ if $\alpha_1 - \alpha_2$ is a constant function. Clearly $\simeq$ is an equivalence relation on $D^p(G)$. Identify the constant functions on G with $\mathbb{C}$. Now $D^p(G)/\mathbb{C}$ is a Banach space under the norm induced from $D^p(G)$. That is, if $[\alpha]$ is an equivalence class from $D^p(G)/\mathbb{C}$ then $\|[\alpha]\|_{D^p(G)/\mathbb{C}} = \big(\sum_{g\in S} \|\alpha * (g-1)\|_p^p\big)^{\frac{1}{p}}$. We shall write $\|\alpha\|_{D(p)}$ for $\|[\alpha]\|_{D^p(G)/\mathbb{C}}$.

Define a linear map T from $D^p(G)$ to $Z^1\big(G, L^p(G)\big)$ by $(T\alpha)(g) = \alpha * (g-1)$. It was shown in [1, Lemma 4.2] that $H^1\big(G, \mathcal{F}(G)\big) = 0$, so for each 1-cocycle $\delta \in Z^1\big(G, \mathcal{F}(G)\big)$ there exists an $\alpha \in \mathcal{F}(G)$ such that $\delta(g) = \alpha * (g-1)$. This implies that T is onto. The kernel of T is $\mathbb{C}$, the constant functions on G. Thus $D^p(G)/\mathbb{C}$ is isometric with $Z^1\big(G, L^p(G)\big)$. Now $B^1\big(G, L^p(G)\big) = T\big(L^p(G)\big)$, so we obtain the following:

(a) The first cohomology group of G with coefficients in $L^p(G)$, $H^1\big(G, L^p(G)\big)$, is isomorphic with $D^p(G)/\big(L^p(G) \bigoplus \mathbb{C}\big)$.

(b) The first reduced L^p-cohomology space of G, denoted by $\bar{H}^1_{(p)}(G)$, is isometric with $D^p(G)/\overline{L^p(G) \bigoplus \mathbb{C}}$, where the closure is taken in $D^p(G)$.

3 A Sufficient Condition For the Vanishing of $\bar{H}^1_{(p)}(G)$

In this section we give sufficient conditions on $L^p(G)$ so that $\bar{H}^1_{(p)}(G) = 0$. We begin with

Lemma 3.1 *Let $1 \le p < \infty$ and let $\alpha \in D^p(G)/\mathbb{C}$ be a non-negative, real-valued function. If $\{\beta_n\}$ is a sequence in $D^p(G)/\mathbb{C}$ such that $\beta_n \ge 0$ on G, $\{\beta_n\}$ converges pointwise to ∞ and $\|\beta_n\|_{D(p)} \to 0$ as $n \to \infty$, then $\|\alpha - \min(\alpha, \beta_n)\|_{D(p)} \to 0$ as $n \to \infty$.*

Proof Let $\alpha_n = \min(\alpha, \beta_n)$, $U_n = \{x \in G \mid \alpha(x) > \beta_n(x)\}$ and $V_n = \{x \mid x \in U_n$ or $xg^{-1} \in U_n$ for some $g \in S\}$. Represent α by $\sum_{x\in G} a_x x$, α_n by $\sum_{x\in G} (\tilde{a}_x)_n x$ and β_n

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by $\sum_{x\in G}(b_x)_n x$. Now

$$\begin{aligned}
\|\alpha-\alpha_n\|_{D(p)}^p &= \sum_{g\in S}\|(\alpha-\alpha_n)*(g-1)\|_p^p\\
&= \sum_{g\in S}\sum_{x\in G}\Big|\,(a_{xg^{-1}}-a_x)-\big((\tilde a_{xg^{-1}})_n-(\tilde a_x)_n\big)\,\Big|^{\,p}\\
&\le \sum_{g\in S}\sum_{x\in V_n}\big(|a_{xg^{-1}}-a_x|+|(b_{xg^{-1}})_n-(b_x)_n|\big)^{\,p}\\
&\le 2^p\sum_{g\in S}\sum_{x\in V_n}\big(|a_{xg^{-1}}-a_x|^p+|(b_{xg^{-1}})_n-(b_x)_n|^p\big)\\
&= 2^p\sum_{g\in G}\Big(\sum_{x\in V_n}|a_{xg^{-1}}-a_x|^p+\sum_{x\in V_n}|(b_{xg^{-1}})_n-(b_x)_n|^p\Big).
\end{aligned}$$

Let $\epsilon > 0$ be given. Since $\alpha \in D^p(G)/\mathbb{C}$ there exists a finite subset F of G such that $\sum_{g\in S}\sum_{x\in G\setminus F}|a_{xg^{-1}}-a_x|^p < \epsilon$. Since $\beta_n(x) \to \infty$ as $n \to \infty$, there exists N such that $V_n \subseteq G\setminus F$ for all $n \ge N$. Thus $\sum_{g\in S}\sum_{x\in V_n}|a_{xg^{-1}}-a_x|^p \to 0$ as $n \to \infty$. By hypothesis $\sum_{g\in S}\sum_{x\in V_n}|(b_{xg^{-1}})_n-(b_x)_n|^p \to 0$ as $n\to\infty$. Therefore $\|\alpha-\min(\alpha,\beta_n)\|_{D(p)} \to 0$ as $n \to \infty$. ■

Let $\alpha \in D^p(G)$ and $x \in G$. Then $|\alpha(x)|$ will denote the modulus of $\alpha(x)$ and $|\alpha|$ will denote the function $|\alpha|(x) = |\alpha(x)|$. We are now ready to give a sufficient condition for the vanishing of $\bar H^1_{(p)}(G) = 0$.

Theorem 3.2 *Let $1 \le p < \infty$. Suppose there exists a sequence $\{\alpha_n\}$ in $L^p(G)$ such that $\|\alpha_n\|_{D(p)} \to 0$ as $n\to\infty$ and $\{\alpha_n(x)\}$ does not converge pointwise to zero for each x in G. Then $\bar H^1_{(p)}(G) = 0$.*

Proof By taking a subsequence if necessary we may assume that $\|\alpha_n\|_{D(p)} < \frac{1}{n^2}$ for all n. Since $\||\alpha|\|_{D(p)} \le \|\alpha\|_{D(p)}$ we may assume that $\alpha_n(x) \ge 0$ for all $x \in G$. Set $\beta_n = n\alpha_n$. Now $\beta_n(x) \ge 0$ for all $x \in G$, $\beta_n(x) \to \infty$ as $n\to\infty$ for every $x \in G$ and $\|\beta_n\|_{D(p)}^p = \|n\alpha_n\|_{D(p)}^p = n^p\|\alpha_n\|_{D(p)}^p \le n^p(\frac{1}{n^{2p}}) = \frac{1}{n^p}$. We now have that $\|\beta_n\|_{D(p)} \to 0$ as $n\to\infty$. Let α be a real-valued, non-negative function in $D^p(G)/\mathbb{C}$. By Lemma 3.1, $\|\alpha-\min(\alpha,\beta_n)\|_{D(p)} \to 0$ as $n\to\infty$. Thus $\alpha \in \overline{L^p(G)}$ since $\min(\alpha,\beta_n) \in L^p(G)$. It now follows by approximation that $\overline{L^p(G)} = D^p(G)/\mathbb{C}$. Hence $\bar H^1_{(p)}(G) = D^p(G)/\overline{\big(L^p(G)\bigoplus\mathbb{C}\big)} = 0$. ■

4 Harmonic Functions

In this section we will give some results about harmonic functions on G. Let $\alpha \in \mathcal{F}(G)$ and represent α by $\sum_{x\in G} a_x x$. Now define

$$(\triangle\alpha)(x) := \sum_{g\in S}\Big(\big(\alpha*(g-1)\big)(x)\Big) = \sum_{g\in S}(a_{xg^{-1}}-a_x).$$

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We shall say that α is harmonic on G if $(\triangle\alpha)(x) = 0$ for each $x \in G$; alternatively α is harmonic if $|S|\big(\alpha(x)\big) = \sum_{g\in S}\alpha(xg^{-1})$ for each x in G. Let $\mathrm{LHD}^p(G) = \{\alpha \mid \alpha$ is harmonic and $\alpha \in D^p(G)\}$. Observe that the constant functions are contained in $\mathrm{LHD}^p(G)$.

Lemma 4.1 *Let $x \in G$. There exists a positive constant M_x such that $|\alpha(x)| \le M_x\|\alpha\|_{D^p(G)}$ for all $\alpha \in D^p(G)$.*

Proof Write $x = g_1g_2\cdots g_n$ where $g_k \in S$ and no subword of $g_1g_2\cdots g_n$ is the identity. Set $w_k = g_1g_2\cdots g_k$. Let $\alpha \in D^p(G)$. Now

$$\begin{aligned}|\alpha(x)| &= \big(|\alpha(w_n) - \alpha(w_{n-1}) + \alpha(w_{n-1}) - \alpha(w_{n-2})\\ &\qquad + \cdots + \alpha(w_2) - \alpha(w_1) + \alpha(w_1) - \alpha(e) + \alpha(e)|^p\big)^{\frac{1}{p}}\\ &\le \Big(\big(|\alpha(w_n) - \alpha(w_{n-1})| + |\alpha(w_{n-1}) - \alpha(w_{n-2})|\\ &\qquad + \cdots + |\alpha(w_2) - \alpha(w_1)| + |\alpha(w_1) - \alpha(e)| + |\alpha(e)|\big)^p\Big)^{\frac{1}{p}}.\end{aligned}$$

If $0 \le a_1, \dots, a_n \in \mathbb{R}$, then by Jensen's inequality [3, p. 189] applied to the function x^p for $x > 0$,

$$(a_1 + \cdots + a_n)^p \le n^{p-1}(a_1^p + \cdots + a_n^p),$$

consequently

$$\begin{aligned}|\alpha(x)| &\le \Big(n^{p-1}\big(|\alpha(w_n) - \alpha(w_{n-1})|^p + |\alpha(w_{n-1}) - \alpha(w_{n-2})|^p\\ &\qquad + \cdots + |\alpha(w_1) - \alpha(e)|^p + |\alpha(e)|^p\big)\Big)^{\frac{1}{p}}\\ &= n^{\frac{p-1}{p}}\Big(\big|\big(\alpha * (g_n^{-1} - 1)\big)(w_{n-1})|^p + \big|\big(\alpha * (g_{n-1}^{-1} - 1)\big)(w_{n-2})\big|^p\\ &\qquad + \cdots + \big|\big(\alpha * (g_1^{-1} - 1)\big)(e)\big|^p + |\alpha(e)|^p\Big)^{\frac{1}{p}}\\ &\le n^{\frac{p-1}{p}}\|\alpha\|_{D^p(G)}.\end{aligned}$$

■

We are now ready to prove:

Lemma 4.2 *The set $\mathrm{LHD}^p(G)$ is closed in $D^p(G)$.*

Proof Let $\{\alpha_n\}$ be a sequence in $\mathrm{LHD}^p(G)$ and suppose that $\{\alpha_n\} \to \alpha$ in $D^p(G)$. Let $x \in G$. By Lemma 4.1 there exists a positive constant M_x such that $|(\alpha - \alpha_n)(x)| \le M_x\|\alpha - \alpha_n\|_{D^p(G)}$. Thus $\{\alpha_n(x)\}$ converges pointwise to $\alpha(x)$ for all $x \in G$. Represent α by $\sum_{x\in G} a_x x$ and α_n by $\sum_{x\in G}(\tilde{a}_x)_n x$. Now $\sum_{g\in S}\big((\tilde{a}_{xg^{-1}})_n - (\tilde{a}_x)_n\big) = 0$ for all natural numbers n and for all $x \in G$. Thus $\sum_{g\in S}(a_{xg^{-1}} - a_x) = 0$ for all $x \in G$. ■

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The following proposition will be used in Sections 5 and 6. The idea behind the proposition and the proof were inspired by [4, Theorem 3.1].

Proposition 4.3 *Let G be a finitely generated group that has a central element of infinite order. If* $1 \leq p < \infty$, *then* $\mathrm{LHD}^p(G) = \mathbb{C}$.

Proof Let y be an element of infinite order that is an element of the center of G. Let $\alpha \in \mathrm{LHD}^p(G)$ and $x \in G$. Now $|S|\big(\alpha_y(x)\big) = |S|\big(\alpha(xy^{-1})\big) = \sum_{g\in S} \alpha(xy^{-1}g^{-1}) = \sum_{g\in S} \alpha(xg^{-1}y^{-1}) = \sum_{g\in S} \alpha_y(xg^{-1})$. Hence α_y is harmonic if α is harmonic. Define a new function $\beta(x) = \alpha_y(x) - \alpha(x)$ on G. Now β is harmonic since it is the sum of harmonic functions. The formal series representation of β is $\sum_{x\in G}(a_{xy^{-1}} - a_x)x$. Since $\alpha \in D^p(G)$ we have that $\beta \in L^p(G)$. Thus for each $\epsilon > 0$, the set $\{x \mid |a_{xy^{-1}} - a_x| > \epsilon\}$ is finite. By the maximum (minimum) principle for harmonic functions it must be the case $|a_{xy^{-1}} - a_x| < \epsilon$ for all $x \in G$. Hence $\alpha_y(x) = a_{xy^{-1}} = a_x = \alpha(x)$ for all $x \in G$.

Let $x \in G$ and $g \in S$. We now have that $\alpha(x) - \alpha(xg) = \alpha_y(x) - \alpha_y(xg) = \alpha_{y^2}(x) - \alpha_{y^2}(xg) = \cdots = \alpha_{y^n}(x) - \alpha_{y^n}(xg)$ for all natural numbers n. In other words, $a_x - a_{xg} = a_{xy^{-1}} - a_{xy^{-1}g} = \cdots = a_{xy^{-n}} - a_{xy^{-n}g}$. Since $\alpha \in D^p(G)$ and $y^n \neq y$ for all natural numbers n, we have that $|a_{xy^{-n}} - a_{xy^{-n}g}| < \epsilon$ for all $\epsilon > 0$. Thus $\alpha(x) = \alpha(xg)$. The proposition now follows since S generates G. ∎

Remark 4.4 The center of a finitely generated, infinite nilpotent group contains an element of infinite order.

5 Groups With a Central Element of Infinite Order

Let $1 < p \in \mathbb{R}$ and let d be a natural number. It was proven in [2] that $\mathbb{Z}^d$ satisfies the hypothesis of Theorem 3.2 if and only if $d \leq p$. Thus, for example, Theorem 3.2 cannot be used to determine if $\bar{H}^1_{(p)}(\mathbb{Z}^d) = 0$ whenever $d > p$. In this section we will prove that $\bar{H}^1_{(p)}(G) = 0$ whenever G is a group that has a central element of infinite order.

Given $1 < p \in \mathbb{R}$, we shall always let q denote the conjugate index of p. Thus if $p > 1$, then $\frac{1}{p} + \frac{1}{q} = 1$. Fix $\beta = \sum_{x\in G} b_x x \in D^q(G)/\mathbb{C}$. We can define a linear functional on $D^p(G)/\mathbb{C}$ by $\langle \alpha, \beta\rangle = \sum_{x\in G}\sum_{g\in S}\Big(\big(\alpha*(g-1)\big)(x)\Big)\Big(\overline{\big(\beta*(g-1)\big)(x)}\Big) = \sum_{x\in G}\sum_{g\in S}(a_{xg^{-1}} - a_x)(\overline{b_{xg^{-1}} - b_x})$, where $\alpha = \sum_{x\in G} a_x x \in D^p(G)/\mathbb{C}$. The sum is finite since $\alpha * (g-1) \in L^p(G)$ and $\beta * (g-1) \in L^q(G)$ for each $g \in S$. For $y \in G$, define δ_y by $\delta_y(x) = 0$ if $x \neq y$ and $\delta_y(y) = 1$.

Lemma 5.1 *Let* $\alpha \in \mathcal{F}(G)$. *Then* α *is a harmonic function if and only if* $\langle \delta_y, \alpha\rangle = 0$ *for all* $y \in G$.

Proof Represent α by $\sum_{x\in G} a_x x$ and let $y \in G$. Now

$$\langle \delta_y, \alpha\rangle = -2\sum_{g\in S}(\overline{a_{yg^{-1}} - a_y}).$$

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If α is harmonic, then $\langle \delta_y, \alpha \rangle = 0$. Conversely, if $\langle \delta_y, \alpha \rangle = 0$ for all $y \in G$, then α is harmonic since $\sum_{g \in S}(a_{yg^{-1}} - a_y) = 0$ for all $y \in G$. ■

For $X \subseteq D^p(G)/\mathbb{C}$, let $(\bar{X})_{D(p)}$ denote the closure of X in $D^p(G)/\mathbb{C}$.

Proposition 5.2 *If $\alpha \in (\overline{\mathbb{C}G})_{D(p)}$ and $\beta \in \mathrm{LHD}^q(G)$, then $\langle \alpha, \beta \rangle = 0$.*

Proof Let $\{\alpha_n\}$ be a sequence in $\mathbb{C}G$ which converges to α in $D^p(G)/\mathbb{C}$. It follows from Lemma 5.1 that $\langle \alpha_n, \beta \rangle = 0$ for each n. We now obtain,

$$\begin{aligned}
0 &\le \Big| \sum_{x \in G} \sum_{g \in S} \Big(\big(\alpha * (g-1)\big)(x) \Big) \Big(\overline{\big(\beta * (g-1)\big)(x)} \Big) \Big| \\
&= \Big| \sum_{x \in G} \sum_{g \in S} \Big(\big((\alpha - \alpha_n) * (g-1)\big)(x) \Big) \Big(\overline{\big(\beta * (g-1)\big)(x)} \Big) \Big| \\
&\le \sum_{x \in G} \sum_{g \in S} \Big| \Big(\big((\alpha - \alpha_n) * (g-1)\big)(x) \Big) \Big(\overline{\big(\beta * (g-1)\big)(x)} \Big) \Big| \\
&\le \|\alpha - \alpha_n\|_{D(p)} \|\beta\|_{D(q)} \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

The last inequality follows from Hölder's inequality. ■

We are now ready to prove

Theorem 5.3 *If $1 < p \in \mathbb{R}$ and G is a finitely generated group with a central element of infinite order, then $\bar{H}^1_{(p)}(G) = 0$.*

Proof The space of continuous linear functionals on $D^p(G)/\mathbb{C}$ is $D^q(G)/\mathbb{C}$. Let $\big(\overline{L^p(G)}\big)^{\perp}_{D(p)} = \big\{ \beta \in D^q(G)/\mathbb{C} \mid \langle \alpha, \beta \rangle = 0 \text{ for all } \alpha \in \big(\overline{L^p(G)}\big)_{D(p)} \big\}$. Since $(\overline{\mathbb{C}G})_{D(p)} = \big(\overline{L^p(G)}\big)_{D(p)}$ it follows from Proposition 5.2 that $\mathrm{LHD}^q(G)/\mathbb{C}$ is contained in $\big(\overline{L^p(G)}\big)^{\perp}_{D(p)}$.

Let $\beta \in D^q(G)/\mathbb{C}$ and represent β by $\sum_{x \in G} b_x x$. Suppose that β is not harmonic on G. Then there exists an $x \in G$ such that $\sum_{g \in S}(b_{xg^{-1}} - b_x) \neq 0$. If α is supported only on x, then by Lemma 5.1 we have that $\langle \alpha, \beta \rangle \neq 0$. Thus the space of continuous linear functionals on $D^p(G)/\big(\overline{L^p(G) \bigoplus \mathbb{C}}\big)$ is $\mathrm{LHD}^q(G)/\mathbb{C}$. The theorem now follows from Proposition 4.3. ■

Remark 5.4 If G is a group for which $L^2(G)$ does not satisfy the hypothesis of Theorem 3.2, then using the proof of the above theorem we can obtain the well known result $D^2(G) = (\overline{\mathbb{C}G})_{D^2(G)} \bigoplus \mathrm{LHD}^2(G)$.

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6 A Description of $H^1\big(G, L^2(G)\big)$

Let $d > 1$. We shall say that G satisfies condition S_d if there exists a constant $C > 0$ such that $\|\alpha\|_{\frac{d}{d-1}} \le C\|\alpha\|_{D(1)}$ for all $\alpha \in \mathbb{C}G$. In this section we will describe the nonzero elements of $H^1\big(G, L^2(G)\big)$ for groups that satisfy property S_d and have a central element of infinite order. If $\alpha \in \mathcal{F}(G)$ and $t \ge 1$, then α^t will denote the function $\alpha^t(x) = \big(\alpha(x)\big)^t$. Let us start with

Lemma 6.1 *Let G be a finitely generated group and let t be a real number greater than or equal to 2. If α is a non-negative, real function in $\mathcal{F}(G)$, then*

$$\|\alpha^t\|_{D(1)} \le 2t\sum_{x\in G}\alpha^{t-1}(x)\Big(\sum_{g\in S}\big|\big(\alpha*(g-1)\big)(x)\big|\Big).$$

Proof Represent α by $\sum_{x\in G} a_x x$. Let $x \in G$ and let $g \in S$. It follows from the Mean Value Theorem applied to x^t that $(r^t - s^t) \le t(r^{t-1} + s^{t-1})(r - s)$ where r and s are real numbers with $0 \le s \le r$. Thus $|a^t_{xg^{-1}} - a^t_x| \le t(a^{t-1}_x + a^{t-1}_{xg^{-1}})|a_{xg^{-1}} - a_x|$. Now $\|\alpha^t\|_{D(1)} = \sum_{x\in G}\sum_{g\in S}\big|\big(\alpha^t*(g-1)\big)(x)\big| = \sum_{x\in G}\sum_{g\in S}|a^t_{xg^{-1}} - a^t_x| \le t\sum_{x\in G}\sum_{g\in S}(a^{t-1}_x + a^{t-1}_{xg^{-1}})|a_{xg^{-1}} - a_x| = 2t\sum_{x\in G}\sum_{g\in S}a^{t-1}_x|a_{xg^{-1}} - a_x|$. ■

We will now use this lemma to prove the following:

Proposition 6.2 *Let $d > 2$. If G satisfies condition S_d, then there is a constant $C' > 0$ such that $\|\alpha\|_{\frac{2d}{d-2}} \le C'\|\alpha\|_{D(2)}$ for all $\alpha \in \mathbb{C}G$.*

Proof Put $t = \frac{2d-2}{d-2}$ and represent α by $\sum_{x\in G} a_x x$. By property S_d, Lemma 6.1 and Schwartz's inequality we have (assuming without loss of generality that α is non-negative).

$$\begin{aligned}\|\alpha^{\frac{2d-2}{d-2}}\|_{\frac{d}{d-1}} &\le C\|\alpha^{\frac{2d-2}{d-2}}\|_{D(1)}\\ &\le 2C\Big(\frac{2d-2}{d-2}\Big)\sum_{x\in G}\alpha^{\frac{d}{d-2}}(x)\Big(\sum_{g\in S}\big|\big(\alpha*(g-1)\big)(x)\big|\Big)\\ &= 2C\Big(\frac{2d-2}{d-2}\Big)\sum_{x\in G}\sum_{g\in S}a_x^{\frac{d}{d-2}}|a_{xg^{-1}} - a_x|\\ &\le 2C\Big(\frac{2d-2}{d-2}\Big)\|\alpha^{\frac{d}{d-2}}\|_2\|\alpha\|_{D(2)}.\end{aligned}$$

Observe $\|\alpha^{\frac{2d-2}{d-2}}\|_{\frac{d}{d-1}} = \|\alpha^{\frac{2d}{d-2}}\|_1^{\frac{d-1}{d}}$ and $\|\alpha^{\frac{d}{d-2}}\|_2 = \|\alpha^{\frac{2d}{d-2}}\|_1^{\frac{1}{2}}$. Substituting we obtain $\|\alpha^{\frac{2d}{d-2}}\|_1^{\frac{d-1}{d}} \le C'\|\alpha^{\frac{2d}{d-2}}\|_1^{\frac{1}{2}}\|\alpha\|_{D(2)}$. The proposition follows by dividing both sides by $\|\alpha^{\frac{2d}{d-2}}\|_1^{\frac{1}{2}}$ and observing that $\|\alpha^{\frac{2d}{d-2}}\|_1^{\frac{d-2}{2d}} = (\|\alpha\|^{\frac{2d}{d-2}}_{\frac{2d}{d-2}})^{\frac{d-2}{2d}}$. ■

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If G is finitely generated, then $L^p(G) \subseteq L^{p'}(G)$ for $1 \le p \le p'$. If $\alpha \in L^p(G)$, then α is in the zero class of $H^1\big(G, L^p(G)\big)$. Our next result will show, for the case G a group that has a central element of infinite order and satisfies property S_d, that each nonzero class in $H^1\big(G, L^2(G)\big)$ can be represented by a function in $L^{p'}(G)$ for some fixed real number $p' > 2$.

Theorem 6.3 *If G is a finitely generated group that has a central element of infinite order and satisfies condition S_d for $d > 2$, then each nonzero class in $H^1\big(G, L^2(G)\big)$ can be represented by a function from $L^{\frac{2d}{d-2}}(G)$.*

Proof Let 1_G denote the constant function one on G. If $1_G \in (\overline{\mathbb{C}G})_{D^2(G)}$, then there exists a sequence $\{\alpha_n\}$ in $\mathbb{C}G$ such that $\|1_G - \alpha_n\|_{D^2(G)} \to 0$ but $\|\alpha_n\|_{\frac{2d}{d-2}} \not\to 0$ contradicting Proposition 6.2. Hence $(\overline{\mathbb{C}G})_{D^2(G)} \neq D^2(G)$. By Remark 5.4 we have the decomposition $D^2(G) = \overline{L^2(G)} \bigoplus \mathrm{LHD}^2(G)$. By Proposition 4.3, $\mathrm{LHD}^2(G) = \mathbb{C}$. Thus nonzero classes in $H^1\big(G, L^2(G)\big)$ can be represented by functions in $(\overline{\mathbb{C}G})_{D^2(G)} \setminus L^2(G)$. Let $\alpha \in (\overline{\mathbb{C}G})_{D^2(G)}$, so there exists a sequence $\{\alpha_n\}$ in $\mathbb{C}G$ such that $\alpha_n \to \alpha$ in the Banach space $D^2(G)$. Thus $\{\alpha_n\}$ is a Cauchy sequence in $D^2(G)$. By Proposition 6.2 $\{\alpha_n\}$ forms a Cauchy sequence in $L^{\frac{2d}{d-2}}(G)$. Now $\|\bar{\alpha} - \alpha_n\|_{\frac{2d}{d-2}} \to 0$ for some $\bar{\alpha} \in L^{\frac{2d}{d-2}}(G)$. Since L^p-convergence implies pointwise convergence $\|(\bar{\alpha} - \alpha_n) * (g-1)\|_2 \to 0$ as $n \to \infty$ for each $g \in S$. Hence $\|(\bar{\alpha} - \alpha_n)\|_{D^2(G)} \to 0$ as $n \to \infty$. Therefore $\bar{\alpha} = \alpha$ ■

Let A be a finite subset of G and define

$$\partial A := \{x \in A \mid \text{ there exists } g \in S \text{ with } xg \notin A\}.$$

We shall say G satisfies condition $(IS)_d$ if $|A|^{d-1} < |\partial A|^d$ for all finite subsets A of G and some positive constant C. Varopoulous proves the following proposition on page 224 of [5].

Proposition 6.4 *A finitely generated group G satisfies the condition $(IS)_d$ for some $d \ge 1$ if and only if it satisfies condition S_d.*

Now $\mathbb{Z}^d$ satisfies condition $(IS)_d$ but does not satisfy $(IS)_{d+\epsilon}$ for any $\epsilon > 0$. We now have the following:

Corollary 6.5 *Let $d \ge 3$. Each nonzero class in $H^1\big(\mathbb{Z}^d, L^2(\mathbb{Z}^d)\big)$ can be represented by a function from $L^{\frac{2d}{d-2}}(\mathbb{Z}^d)$.*

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