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ON LAGRANGE INTERPOLATION WITH EQUIDISTANT NODES

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A quantitative version of a classical result of S.N. Bernstein concerning the divergence of Lagrange interpolation polynomials based on equidistant nodes is presented. The proof is motivated by the results of numerical computations.

1. INTRODUCTION

In 1918 Bernstein [2] published a result concerning the divergence of Lagrange interpolation based on equidistant nodes. This result, which now has a prominent place in the study of the appoximation of functions by interpolation polynomials, may be described as follows. Throughout this paper let $f(x) = |x|$ $(-1 \leqslant x \leqslant 1)$ and $x_{k,n} = -1 + 2(k-1)/(n-1)$ $(k = 1, 2, \dots, n;\ n = 1, 2, 3, \dots)$. Define the Lagrange interpolation polynomial of degree $n-1$ to be the unique polynomial $L_{n-1}(f, x)$ of degree $n-1$ or less which satisfies the n conditions

$$L_{n-1}(f, x_{k,n}) = f(x_{k,n}) \qquad (k = 1, 2, \dots, n).$$

We can now state Bernstein's result.

THEOREM 1. (S.N. Bernstein) *If* $0 < |x| < 1$, *then the sequence* $\{L_{n-1}(f, x) \colon n = 1, 2, 3, \dots\}$ *diverges (and a fortiori does not converge to* $f(x)$*).*

Bernstein's result shows that Lagrange interpolation polynomials which are based on equidistant nodes may have very poor approximation properties.

Another source for the proof of Theorem 1 is Natanson ([4], pp.30–35) who reports that D.L. Berman proved that $\{L_{n-1}(f, 0) : n = 1, 2, 3, \dots\}$ converges to $f(0) = 0$. Clearly, since $x_{1,n} = -1$ and $x_{n,n} = +1$ for all n we have $L_{n-1}(f, +1) = f(+1)$ and $L_{n-1}(f, -1) = f(-1)$. Thus the question of convergence of $\{L_{n-1}(f, x) : n = 1, 2, 3, \dots\}$ is settled for all $x \in [-1, 1]$.

One would expect from Theorem 1 that if $0 < |x| < 1$ then $L_{n-1}(f, x)$ and $f(x)$ would differ markedly. However, the graphs of $f(x)$ and $L_{12}(f, x)$ in Figure 1 show that this is not so. Indeed in the centre of the interval $[-1, 1]$, the error $|\ L_{12}(f, x) - f(x)\ |$ appears to be quite small. These, and similar, computations

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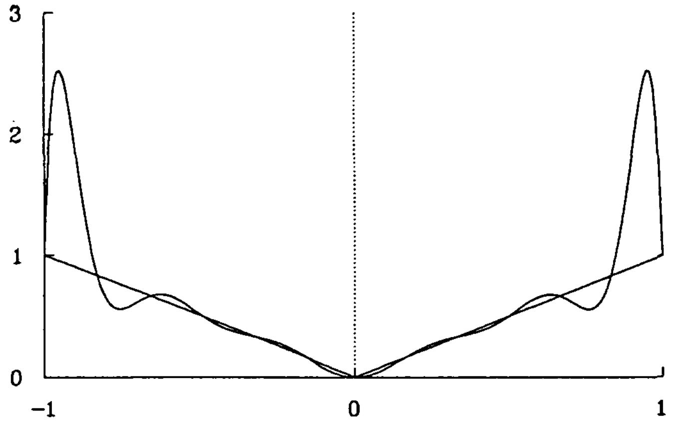

Figure 1. Graphs of $f(x) = |x|$ and $L_{12}(f, x)$

suggest that the rate of divergence of $\{L_{n-1}(f, x) : n = 1, 2, 3, \ldots\}$ depends on x. In particular, the sequence should diverge rapidly near the ends of the interval and not so rapidly near the centre of the interval. Thus the first aim of this paper is to prove a quantitative version of Theorem 1 which reflects the pointwise behaviour suggested by Figure 1. We note that similar ideas motivated the work of Runck [5, 6].

Bernstein's proof of Theorem 1 raises another problem. It is well known that there are two formulae for expressing Lagrange interpolation polynomials, namely Lagrange's formula and Newton's formula. (See, for example, Natanson ([4], Chapter I).) In studying approximation properties of interpolation polynomials, almost always one uses Lagrange's formula in preference to Newton's formula. However, in proving Theorem 1, Bernstein uses Newton's formula. This suggests the problem of proving Bernstein's result by using Lagrange's formula. This problem is important in understanding methods for the systematic study of interpolation polynomials based on equidistant nodes. Thus the second aim of this paper is to establish Bernstein's result using Lagrange's formula. We will achieve both aims by proving the following quantitative version of Theorem 1 and using Lagrange's interpolation formula.

Theorem 2. *If* $0 < |x| < 1$ *then*

$$\limsup_{n\to\infty} n^{-1} \log\, |L_{n-1}(f, x) - f(x)\,| = \frac{1}{2}[(1+x)\log(1+x) + (1-x)\log(1-x)]. \tag{1}$$

2. Preliminaries

Lagrange's formula for $L_{n-1}(f, x)$ is

$$L_{n-1}(f, x) = \sum_{k=1}^{n} f(x_k)\ell_k(x),$$

where

$$
\begin{aligned}
&x_k = x_{k,n} \qquad (k = 1, 2, \dots, n),\\
(2)\qquad &\ell_k(x) = \omega(x)/(\omega'(x_k)(x - x_k)) \quad (k = 1, 2, \dots, n),\\
&\omega(x) = (x - x_1)(x - x_2)\dots(x - x_n).
\end{aligned}
$$

Theorem 2 assumes that $0 < |x| < 1$. We shall assume henceforth that x is a fixed number in the interval $(-1, 0)$ as similar arguments can be developed for $(0, 1)$. For each integer $n \geqslant 2$, define $j = j(n)$ and $\theta = \theta(n)$ by

$$
(3)\qquad x = x_j + 2\theta/(n-1), \quad 0 \leqslant \theta < 1.
$$

We now define $(a)_k$ by

$$
\begin{aligned}
&(a)_0 = 1\\
&(a)_k = a(a+1)\dots(a+k-1) \qquad (k = 1, 2, 3, \dots).
\end{aligned}
$$

We will denote the gamma function by $\Gamma(.)$ The proof of Theorem 2 will require the following result.

Lemma. *For $-1 < x < 0$ and $n = 2m+1$,*

(4)

$$
\sum_{k=m+2}^{2m+1} x_k \ell_k(x) = (-1)^{j+m+1} \frac{\sin(\pi\theta)\Gamma(j+\theta)\Gamma(2m+2-j-\theta)}{\pi m (2m)!} \times \left[\frac{1}{2}\binom{2m}{m} - (m+1-j-\theta)\binom{2m}{m-1}\sum_{k=0}^{\infty}\frac{(j+\theta)_k}{(2m+1+k)(m+2)_k}\right].
$$

For $-1 < x \leqslant 0$ and $n = 2m$,

(5)

$$
\sum_{k=m+1}^{2m} x_k \ell_k(x) = (-1)^{j+m} \frac{\sin(\pi\theta)\Gamma(j+\theta)\Gamma(2m+1-j-\theta)}{\pi(2m-1)(2m-1)!} \times \left[2\binom{2m-2}{m-1} - (2m+1-2j-2\theta)\binom{2m-1}{m-1}\sum_{k=0}^{\infty}\frac{(j+\theta)_k}{(2m+k)(m+1)_k}\right].
$$

Proof: We establish (4), since the proof of (5) is very similar. From (2), (3) we can deduce that, for $k = m+2, m+3, \dots, 2m+1$,

$$
\ell_k(x) = (-1)^{j+1}\,\frac{\sin(\pi\theta)}{\pi}\,\frac{\Gamma(j+\theta)\Gamma(2m+2-j-\theta)}{(2m)!}\,\frac{(-1)^k}{(k-j-\theta)}\binom{2m}{k-1}.
$$

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Now consider $\sum_{k=m+2}^{2m+1} x_k \ell_k(x)$. Since $x_k = (k-m-1)/m$ it follows that

$$\sum_{k=m+2}^{2m+1} x_k\, \ell_k(x)$$
$$= (-1)^{j+1} \frac{\sin \pi\theta}{\pi} \frac{\Gamma(j+\theta)\Gamma(2m+2-j-\theta)}{m(2m)!} \sum_{k=m+2}^{2m+1} (-1)^k \frac{k-m-1}{k-j-\theta} \binom{2m}{k-1}.$$

Upon changing the index of summation from k to $r = 2m+1-k$ we obtain

$$\sum_{k=m+2}^{2m+1} x_k \ell_k(x)$$
$$= (-1)^j \frac{\sin \pi\theta}{\pi} \frac{\Gamma(j+\theta)\Gamma(2m+2-j-\theta)}{m(2m)!} \sum_{r=0}^{m-1} (-1)^r \frac{m-r}{2m+1-j-r-\theta} \binom{2m}{r}$$
$$= (-1)^j \frac{\sin \pi\theta}{\pi} \frac{\Gamma(j+\theta)\Gamma(2m+2-j-\theta)}{m(2m)!}$$
$$\times \left[\sum_{r=0}^{m-1} (-1)^r \binom{2m}{r} - (m+1-j-\theta) \sum_{r=0}^{m-1} \frac{(-1)^r}{2m+1-j-r-\theta} \binom{2m}{r} \right].$$

Since

$$\sum_{r=0}^{m-1} (-1)^r \binom{2m}{r} = \sum_{r=m+1}^{2m} (-1)^r \binom{2m}{r} = \frac{(-1)^{m+1}}{2} \binom{2m}{m},$$

the lemma will be established if we can show that

$$(6) \quad \sum_{r=0}^{m-1} \frac{(-1)^r}{2m+1-j-r-\theta} \binom{2m}{r} = (-1)^{m+1} \binom{2m}{m-1} \sum_{k=0}^{\infty} \frac{(j+\theta)_k}{(2m+1+k)(m+2)_k}.$$

Now, reversing the order of summation in the left-hand side of (6) gives

$$(7) \quad \begin{aligned} \sum_{r=0}^{m-1} \frac{(-1)^r}{2m+1-j-r-\theta} \binom{2m}{r} &= (-1)^{m+1} \sum_{k=0}^{m-1} \frac{(-1)^k}{m+2+k-j-\theta} \binom{2m}{m-1-k} \\ &= (-1)^{m+1} (2m)! \sum_{k=0}^{m-1} \frac{(-1)^k}{(m+2+k-j-\theta)(m-1-k)!(m+1+k)!}. \end{aligned}$$

We next use the identities

$$\frac{1}{m+2+k-j-\theta} = \frac{1}{m+2-j-\theta} \frac{(m+2-j-\theta)_k}{(m+3-j-\theta)_k},$$

$$(m-1-k)! = \frac{(-1)^k (m-1)!}{(1-m)_k},$$

and

$$(m+1+k)! = (m+1)!\,(m+2)_k,$$

together with the observation that

$$(1-m)_k = 0, \qquad k = m, m+1, \dots,$$

to rewrite (7) as

$$\begin{aligned}(8)\qquad &\sum_{r=0}^{m-1} \frac{(-1)^r}{2m+1-j-r-\theta}\binom{2m}{r}\\ &= \frac{(-1)^{m+1}}{m+2-j-\theta}\binom{2m}{m-1}\sum_{k=0}^{\infty}\frac{(m+2-j-\theta)_k(1-m)_k(1)_k}{(m+3-j-\theta)_k(m+2)_k\;k!}.\end{aligned}$$

If we introduce the notation ${}_pF_q\left(\begin{matrix}\alpha_1,\alpha_2\dots,\alpha_p\\ \beta_1,\beta_2\dots,\beta_q\end{matrix}\middle| z\right)$ for the generalised hypergeometric function

$${}_pF_q\left(\begin{matrix}\alpha_1,\alpha_2\dots,\alpha_p\\ \beta_1,\beta_2\dots,\beta_q\end{matrix}\middle| z\right) = \sum_{k=0}^{\infty}\frac{(\alpha_1)_k(\alpha_2)_k\dots(\alpha_p)_k}{(\beta_1)_k(\beta_2)_k\dots(\beta_q)_k}\frac{z^k}{k!},$$

then (8) can be written more concisely as

$$\begin{aligned}(9)\qquad &\sum_{r=0}^{m-1} \frac{(-1)^r}{2m+1-j-r-\theta}\binom{2m}{r}\\ &= \frac{(-1)^{m+1}}{m+2-j-\theta}\binom{2m}{m-1}{}_3F_2\left(\begin{matrix}m+2-j-\theta, & 1-m, & 1\\ m+3-j-\theta, & m+2 & \end{matrix}\middle| 1\right).\end{aligned}$$

In order to replace the (finite) alternating series on the right-hand side of (9) by a series of positive terms, we will employ the result (see p.104 of [**3**])

$$(10)\qquad {}_3F_2\left(\begin{matrix}a,b,c\\ e,f\end{matrix}\middle| 1\right) = \frac{\Gamma(e)\Gamma(f)\Gamma(s)}{\Gamma(a)\Gamma(s+b)\Gamma(s+c)}\,{}_3F_2\left(\begin{matrix}e-a, & f-a, & s\\ s+b, & s+c & \end{matrix}\middle| 1\right),$$

where $s = e+f-a-b-c$, and $s \neq 0$. Upon applying (10) with $a = m+2-j-\theta, b = 1-m, c = 1, e = m+3-j-\theta$ and $f = m+2$ (so $s = 2m+1$), (9) becomes

$$\begin{aligned}\sum_{r=0}^{m-1} \frac{(-1)^r}{2m+1-j-r-\theta}\binom{2m}{r} &= \frac{(-1)^{m+1}}{2m+1}\binom{2m}{m-1}{}_3F_2\left(\begin{matrix}1, j+\theta, & 2m+1\\ m+2, & 2m+2\end{matrix}\middle| 1\right)\\ &= (-1)^{m+1}\binom{2m}{m-1}\sum_{k=0}^{\infty}\frac{(j+\theta)_k}{(2m+1+k)(m+2)_k}.\end{aligned}$$

This proves (6), and so the lemma is established. ∎

3. Proof of Theorem 2

To prove Theorem 2 we assume $-1 < x < 0$, and suppose initially that $n = 2m+1$. We note that

$$f(x) = |x| = -x,$$
$$L_{n-1}(f,x) = \sum_{k=1}^{m}(-x_k)\ell_k(x) + \sum_{k=m+2}^{2m+1} x_k\ell_k(x).$$

Since $\varphi(t) \equiv t$ is a polynomial, we have

$$x = \sum_{k=1}^{2m+1} x_k\, \ell_k(x).$$

Hence we have the following representation of the error:

$$L_{n-1}(f,x) - f(x) = 2\sum_{k=m+2}^{2m+1} x_k\ell_k(x), \tag{11}$$

and therefore we must estimate the right-hand side of (4).

We begin by obtaining bounds for the term in the square brackets on the right-hand side of (4). We have

$$\begin{aligned} \sum_{k=0}^{\infty}\frac{(j+\theta)_k}{(2m+1+k)(m+2)_k} &< \frac{1}{2m+1}\sum_{k=0}^{\infty}\frac{(j+\theta)_k}{(m+2)_k} \\ &= \frac{1}{2m+1}\,{}_2F_1\left(\begin{matrix} j+\theta,\ 1 \\ m+2\end{matrix}\,\middle|\,1\right). \end{aligned} \tag{12}$$

The hypergeometric series can be summed using the well-known result (see, for example, p.99 of [3])

$${}_2F_1\left(\begin{matrix} a,b \\ c\end{matrix}\,\middle|\,1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{13}$$

which is valid if c is not a negative integer or zero, and if $c-a-b>0$. Applying (13) to (12) gives

$$\begin{aligned} \sum_{k=0}^{\infty}\frac{(j+\theta)_k}{(2m+1+k)(m+2)_k} &< \frac{1}{2m+1}\,\frac{\Gamma(m+2)\Gamma(m+1-j-\theta)}{\Gamma(m+2-j-\theta)\Gamma(m+1)} \\ &= \frac{m+1}{(2m+1)(m+1-j-\theta)}, \end{aligned}$$

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and so

$$
\begin{aligned}
\frac{1}{2}\binom{2m}{m} &> \frac{1}{2}\binom{2m}{m} - (m+1-j-\theta)\binom{2m}{m-1}\sum_{k=0}^{\infty}\frac{(j+\theta)_k}{(2m+1+k)(m+2)_k} \\
&> \frac{1}{2}\binom{2m}{m} - \frac{m+1}{2m+1}\binom{2m}{m-1} \\
&= \frac{1}{2(2m+1)}\binom{2m}{m}.
\end{aligned}
\tag{14}
$$

From (4) and (14) it then follows that

$$
\begin{aligned}
\frac{\sin\pi\theta}{\pi}\,\frac{\Gamma(j+\theta)\Gamma(2m+2-j-\theta)}{2m(2m+1)(\Gamma(m+1))^2} &\leqslant \left|\sum_{k=m+2}^{2m+1} x_k\ell_k(x)\right| \\
&\leqslant \frac{\sin\pi\theta}{\pi}\,\frac{\Gamma(j+\theta)\Gamma(2m+2-j-\theta)}{2m(\Gamma(m+1))^2}.
\end{aligned}
\tag{15}
$$

We next work with the right-hand inequality of (15), which can be rewritten in the form

$$
\begin{aligned}
\frac{1}{m}\log\left|\sum_{k=m+2}^{2m+1} x_k\ell_k(x)\right| \leqslant{}& \frac{1}{m}\log\Gamma(j+\theta) + \frac{1}{m}\log\Gamma(2m+2-j-\theta) \\
&- \frac{2}{m}\log\Gamma(m+1) + O\left(\frac{\log m}{m}\right), \quad \text{as } m\to\infty.
\end{aligned}
$$

Now, as $m\to\infty$ then $j\to\infty$ and $m-j\to\infty$, and so we can employ the asymptotic expansion of $\log\,\Gamma(x)$ (see pp.252–3 of [**7**])

$$
\log\Gamma(x) = \left(x-\frac{1}{2}\right)\log x - x + \frac{1}{2}\log(2\pi) + O\left(\frac{1}{x}\right), \quad \text{as } x\to\infty,
$$

to obtain

$$
\begin{aligned}
\frac{1}{m}\log\left|\sum_{k=m+2}^{2m+1} x_k\ell_k(x)\right| \leqslant{}& \frac{j+\theta-\frac{1}{2}}{m}\log(j+\theta) + \frac{2m+\frac{3}{2}-j-\theta}{m}\log(2m+2-j-\theta) \\
&- \frac{2m+1}{m}\log(m+1) + O\left(\frac{\log m}{m}\right) \\
={}& \left(\frac{j+\theta-\frac{1}{2}}{m}\right)\log\left(\frac{j+\theta}{m+1}\right) \\
&+ \left(\frac{2m+\frac{3}{2}-j-\theta}{m}\right)\log\left(\frac{2m+2-j-\theta}{m+1}\right) + O\left(\frac{\log m}{m}\right).
\end{aligned}
\tag{16}
$$

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From the definition (3) of $j = j(n)$, it follows that

$$\lim_{m \to \infty} \frac{j}{m} = 1 + x.$$

Thus, on letting $m \to \infty$ in (16), we obtain

$$\limsup_{m \to \infty} \frac{1}{m} \log \left| \sum_{k=m+2}^{2m+1} x_k \ell_k(x) \right| \leqslant (1+x)\log(1+x) + (1-x)\log(1-x). \tag{17}$$

Finally, we consider the left-hand inequality of (15). Choose c_1 and c_2 so that $0 < c_1 < c_2 < 1$. Then, by Berman ([1], Lemma 1), for each x there exists an increasing sequence $\{r_m\}_{m=1}^{\infty}$ of positive integers so that if we write

$$x = x_j + \frac{\theta}{r_m}, \qquad 0 \leqslant \theta < 1,$$

where $j = j(n)$ and $\theta = \theta(n)$, then the inequalities

$$c_1 \leqslant \theta \leqslant c_2$$

hold for all m. Because $(\sin \pi\theta)/\pi$ has a positive lower bound for $c_1 \leqslant \theta \leqslant c_2$, (15) yields

$$\begin{aligned} \frac{1}{r_m} \log \left| \sum_{k=r_m+2}^{2r_m+1} x_k \ell_k(x) \right| \geqslant{}& \frac{1}{r_m} \log \Gamma(j+\theta) + \frac{1}{r_m} \log \Gamma(2r_m + 2 - j - \theta) \\ &- \frac{2}{r_m} \log \Gamma(r_m + 1) + O\left(\frac{\log r_m}{r_m}\right), \text{ as } m \to \infty. \end{aligned}$$

As before, the right-hand side of this inequality approaches $(1+x)\log(1+x) + (1-x)\log(1-x)$ as $m \to \infty$, and so we deduce that

$$\limsup_{m \to \infty} \frac{1}{m} \log \left| \sum_{k=m+2}^{2m+1} x_k \ell_k(x) \right| \geqslant (1+x)\log(1+x) + (1-x)\log(1-x). \tag{18}$$

Thus (11), (17), (18) imply that

(19)

$$\limsup_{n=2m+1 \to \infty} n^{-1} \log |L_{n-1}(f,x) - f(x)| = \frac{1}{2}[(1+x)\log(1+x) + (1-x)\log(1-x)].$$

To complete the proof, suppose that $n = 2m$. From (5) it follows that

$$\begin{aligned} |L_{n-1}(f,x) - f(x)| \leqslant{}& \frac{2\sin \pi\theta}{\pi} \frac{\Gamma(j+\theta)\Gamma(2m+1-j-\theta)}{(2m-1)(2m-1)!} \binom{2m-1}{m-1} \\ &\times \left[\frac{2m}{2m-1} + (2m+1-2j-2\theta) \sum_{k=0}^{\infty} \frac{(j+\theta)_k}{(2m+k)(m+1)_k} \right], \end{aligned}$$

and then similar arguments to those employed above show that

(20) $$\limsup_{n=2m\to\infty} n^{-1}\log|L_{n-1}(f,x)-f(x)| \leqslant \frac{1}{2}[(1+x)\log(1+x)+(1-x)\log(1-x)].$$

(1) then follows from (19) and (20).

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