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# COMPETITIVE ANALYSIS OF INTERRELATED PRICE ONLINE INVENTORY PROBLEMS WITH DEMANDS

SHUGUANG HAN ✉[1], JUELIANG HU[1] and DIWEI ZHOU[2]

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**Abstract**

This paper investigates interrelated price online inventory problems, in which decisions as to when and how much of a product to replenish must be made in an online fashion to meet some demand even without a concrete knowledge of future prices. The objective of the decision maker is to minimize the total cost while meeting the demands. Two different types of demand are considered carefully, that is, demands which are linearly and exponentially related to price. In this paper, the prices are online, with only the price range variation known in advance, and are interrelated with the preceding price. Two models of price correlation are investigated, namely, an exponential model and a logarithmic model. The corresponding algorithms of the problems are developed, and the competitive ratios of the algorithms are derived as the solutions by use of linear programming.

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## 1. Introduction

In recent years, online problem and competitive algorithm theory have received an increasing amount of attention. After the economic order quantity (EOQ) model was proposed by Wilson [18] in 1934, inventory theory has gradually developed. In classical inventory problems, prices are generally assumed to be constant, or they follow a probability distribution. Serel [13] studied the optimal ordering and pricing problem based on interrelated demand and price in the rapid response system. Banerjee and Sharma [3] investigated the inventory model with seasonal demand in two potentially replaceable markets. Sana [11] generalized the EOQ model in the case of perishable products with sensitive demand to price. Lin and Ho [9] studied

[1]Department of Mathematics, Zhejiang Sci-Tech University, Hangzhou 310018, China;
e-mail: dawn1024@zstu.edu.cn, hujlhz@163.com.
[2]Department of Mathematical Sciences, Loughborough University, Loughborough LE11 3TU, UK;
e-mail: d.zhou2@lboro.ac.uk.

the optimal ordering and pricing problem of the joint inventory model with sensitive demand to price based on the quantity discount. Kalymon [7] studied the problem with price dependency on previous prices, where demand was also uncertain. Webster and Weng [17] studied the ordering and pricing problem of the fashion product's supply chain which consisted of producer and seller, and where random demand is sensitive to price in the supply chain. Ali and Masinga [1] presented a nonlinear programming model that could handle random demands and incorporate price changes for optimal order quantities. Shu et al. [14] considered an inventory control problem with stochastic demand in which the demand mean and variance were assumed to be known for each market. Liu et al. [10] investigated a single-period inventory problem with discrete stochastic demand. Yang et al. [20] studied the product pricing and material replenishment strategy with price-sensitive demand. Sicilia et al. [15] studied an inventory model for deteriorating items with shortages and time-varying demand. Drezner and Scott [5] derived approximate formulas for the optimal solution in the particular case of an exponential demand distribution for the stochastic inventory model.

The price online inventory problem [8] is challenging because the decision maker, or a retailer, must decide when and how much to purchase without knowing future prices. The price online inventory problem can be seen as an extension of the time-series search problem and the financial one-way trading problem [4, 6, 12, 19, 21], in which a decision maker wants to purchase $L$ units of product through a sequence of $n$ sellers $v_1, v_2, \dots, v_n$ arriving online, and he needs to decide the fraction to purchase from each $v_i$ at the then-prevailing market price $p_i$. His objective is to minimize the cost. It is easy to solve the off-line version of the problem; if the decision maker knows all the future prices, he can simply wait for the lowest price and then purchase all his products at that price.

Specifically, in our price online inventory problem, there is a buyer who has $L$ units of product to be purchased, and there is a sequence of sellers $v_1, v_2, \dots, v_n$ arriving. When a seller $v_i$ arrives, the unit price $p_i$ is revealed and the buyer needs to decide the amount $x_i$ of product to be purchased from $v_i$ at price $p_i$, and the objective is to minimize $\sum_i p_i x_i$ subject to $\sum_i x_i \le L$. This optimization problem is challenging because: (1) the buyer has no control on the prices, which fluctuate with time; (2) the future prices are uninformative, that is, when $v_i$ arrives, any price $p_j$, where $j > i$ is unknown; and (3) he needs to decide the amount of product to be purchased from a seller $v_i$ as soon as $v_i$ arrives. Larsen and Wohlk [8] considered a real-time version of the inventory problem with continuous deterministic demand involving the fixed order cost and the inventory cost, and they obtained algorithmic upper and lower bounds of the competitive ratio, where the gap grows with the complexity of the modes. The inventory problem considered is a demand online inventory problem where the decision maker only knows the upper bound and lower bound of the daily demand and decides how many products should be prepared every day.

We apply the competitive ratio to evaluate the performance of the algorithm. An arbitrary online algorithm ALG is referred to as *r-competitive*, if an arbitrary input price instance $I$ satisfies $\mathrm{ALG}(I) \le r \cdot \mathrm{OPT}(I)$, where ALG(I) denotes the cost of the online algorithm ALG and OPT(I) is the cost of the optimal off-line algorithm OPT. The competitive ratio of ALG algorithm is defined as the minimum $r$ that satisfies the above inequality.

To improve the work of Larsen and Wohlk [8] on a real-time inventory problem, we focus on two main facts of one inventory system: the price and the demand. The impacts of price, the price-related patterns and the relevant algorithms are discussed. In the Chinese stock market, the stock prices of today are known to be bounded in the interval from 90% to 110% of yesterday's closing price. So we modify the assumptions and assume that the variation range of each price is interrelated with its preceding price. In the inventory system, there are certain demands for the items, since there are some retailers and customers. The demand is negatively correlated to the price, because customers are more willing to purchase cheaper products. The problems considered in this paper become more practical than the problems of Larsen and Wohlk [8]. Two types of relationship and price of demand are considered, linear relationships and exponential relationships [16], and, for each of the two types, the exponential and logarithmic price interrelations, respectively, are considered [21].

## 2. Problem statement

We consider an inventory problem in which the decision maker, a retailer, decides when and how much to purchase every day without knowing future prices during the purchasing process. Let $U$ be the storage capacity, which must be reached when the purchasing process is over. Additionally, the initial inventory is zero. The objective of the decision maker is to minimize the total cost with the demands met. In order to generalize the model, we consider different price variation ranges. That is, the price has its own variation range, and the range is variable.

Let $n$ denote the number of purchasing days. Denote by $D_i$ and $p_i$ the demand and the price on the $i$th day, respectively. Let $\theta_1$ and $\theta_2$ denote the parameters of price variation ranges. We make some basic observation on the values of $\theta_1$ and $\theta_2$. If $1 \le \theta_1$ or $\theta_2 \le 1$, implying that the price sequence is monotonously increasing or decreasing, respectively, the optimal solution can be obtained by selecting the first or last price for the problems, respectively. So we just focus on the case where $0 < \theta_1 \le 1 \le \theta_2$. Let $p$ denote the initial price, where $p_1 \in [\theta_1 p, \theta_2 p]$. As in the Chinese stock market, the stock prices follow the exponential model with $\theta_1 = 0.9$ and $\theta_2 = 1.1$. The following two price interrelation models are considered:

(i) the exponential model with $p_i \in [\theta_1 p_{i-1}, \theta_2 p_{i-1}], 2 \le i \le n$; and
(ii) the logarithmic model with $p_i \in [\theta_1 p_1 \ln i, \theta_2 p_1 \ln i], 2 \le i \le n$.

## 3. The competitive analysis for linearly related demand

For the inventory problem, the demand is assumed to have a negative linear relationship with price [16]. Without loss of generality, we assume that $D_i = a_i - b_i p_i$.

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**3.1. Competitive analysis of the exponential model** A linear programming problem with variables $\{r, s_1, s_2, \dots, s_n\}$ is investigated as follows. The second and third constraint conditions are transformed by the range of the total purchase quantity at the end of the $j$th day ($j = 1, 2, \dots, n$) and the relationship between the demand, price and the price correlation in the exponential model. The assumption of the exponential model is that $p_i \in [\theta_1 p_{i-1}, \theta_2 p_{i-1}]$ for every $2 \le i \le n$, and there exists one positive $p \in [\theta_1 p, \theta_2 p]$ with $\theta_1 \le 1 \le \theta_2$. The linear programming problem is given below.

$$\begin{aligned} \text{minimize} \quad & r && \text{(LP1)} \\ \text{such that} \quad & H_i(s_1, s_2, \dots, s_n) \le r \\ & U + \sum_{i=1}^{n}(a_i - b_i\theta_2^i p) \le \sum_{i=1}^{n} s_i \le U + \sum_{i=1}^{n}(a_i - b_i\theta_1^i p) \\ & \sum_{i=1}^{j}(a_i - b_i\theta_2^i p) \le \sum_{i=1}^{j} s_i \le U + \sum_{i=1}^{j}(a_i - b_i\theta_1^i p), \quad j = 1, 2, \dots, n-1 \\ & s_i \ge 0 \quad \text{for } i = 1, 2, \dots, n, \end{aligned}$$

where

$$H_i(s_1, s_2, \dots, s_n) = \frac{s_1/\theta_1^{i-1} + s_2/\theta_1^{i-2} + \cdots + s_{i-1}/\theta_1 + s_i + s_{i+1}\theta_2 + \cdots + s_n\theta_2^{n-i}}{U + \sum_{j=1}^{n}(a_j - b_j\theta_2^j p)}.$$

THEOREM 3.1. *The solution to the above linear programming problem (LP1) exists.*

PROOF. We only need to prove that there exists $\{r', s_1', s_2', \dots, s_n'\}$ such that

$$H_i(s_1', s_2', \dots, s_n') \le r', \quad i = 1, 2, \dots, n, \tag{3.1}$$

$$U + \sum_{i=1}^{n}(a_i - b_i\theta_2^i p) \le \sum_{i=1}^{n} s_i' \le U + \sum_{i=1}^{n}(a_i - b_i\theta_1^i p), \tag{3.2}$$

$$\sum_{i=1}^{j}(a_i - b_i\theta_2^i p) \le \sum_{i=1}^{j} s_i' \le U + \sum_{i=1}^{j}(a_i - b_i\theta_1^i p) \quad j = 1, 2, \dots, n-1, \tag{3.3}$$

$$s_i' \ge 0, i = 1, 2, \dots, n. \tag{3.4}$$

We construct the set as follows. Let $s_1' = U + a_1 - b_1\theta_1 p$ and $s_i' = a_i - b_i\theta_1^i p$ for every $i$, $2 \le i \le n$. It is obvious that $s_i' \ge 0$ for $1 \le i \le n$ and $\sum_{i=1}^{n} s_i' = U + \sum_{i=1}^{n}(a_i - b_i\theta_1^i p)$, and $\sum_{i=1}^{j} s_i' = U + \sum_{i=1}^{j}(a_i - b_i\theta_1^i p)$ holds for $1 \le j \le n-1$. With the assumption $\theta_1 \le \theta_2$, $\{s_1', s_2', \dots, s_n'\} = \{U + a_1 - b_1\theta_1 p, a_2 - b_2\theta_1^2 p, \dots, a_n - b_n\theta_1^n p\}$ satisfies the inequalities (3.2)–(3.4). In addition, for any $i = 1, 2, \dots, n$,

$$\begin{aligned} H_i(s_1', s_2', \dots, s_n') = {} & \frac{1}{U + \sum_{j=1}^{n}(a_j - b_j\theta_2^j p)} \\ & \times \bigg\{ \frac{U + a_1 - b_1\theta_1 p}{\theta_1^{i-1}} + \cdots + \frac{a_{i-1} - b_{i-1}\theta_1^{i-1} p}{\theta_1} \\ & + a_i - b_i\theta_1^i p + (a_{i+1} - b_{i+1}\theta_1^{i+1} p)\theta_2 + \cdots + (a_n - b_n\theta_1^n p)\theta_2^{n-i} \bigg\}. \end{aligned}$$

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Since $\theta_1, \theta_2, U, n, p, a_i(1 \le i \le n)$ and $b_i(1 \le i \le n)$ are all known parameters, the values of $H_i(s'_1, s'_2, \ldots, s'_n) = H_i(U + a_1 - b_1\theta_1 p, a_2 - b_2\theta_1^2 p, \ldots, a_n - b_n\theta_1^n p)$ can be calculated for all $1 \le i \le n$. Let $r' = \max_{1\le i\le n} H_i(U + a_1 - b_1\theta_1 p, a_2 - b_2\theta_1^2 p, \ldots, a_n - b_n\theta_1^n p)$. Note that $H_i(U + a_1 - b_1\theta_1 p, a_2 - b_2\theta_1^2 p, \ldots, a_n - b_n\theta_1^n p) \le r'$ for all $1 \le i \le n$. From the above analysis, there exists

$$\{r', s'_1, s'_2, \ldots, s'_n\} = \max_{1\le i\le n}\{H_i(U + a_1 - b_1\theta_1 p, a_2 - b_2\theta_1^2 p, \ldots, a_n - b_n\theta_1^n p),\\ U + a_1 - b_1\theta_1 p, a_2 - b_2\theta_1^2 p, \ldots, a_n - b_n\theta_1^n p\}$$

satisfying the inequalities (3.1)–(3.4). Thus, the solution of the linear programming problem (LP1) exists. □

The online algorithm for this model is designed according to the solution of the linear programming problem (LP1), and is denoted by $\mathrm{SLP}_1$.

**Algorithm 1**: $\mathrm{SLP}_1$

1: Solve the linear programming problem (LP1), and let $\{r^*, s_1^*, s_2^*, \ldots, s_n^*\}$ be the solution.
2: Define $s_i^*$ to be the quantity of units for purchasing at period $i$ for every $1 \le i \le n$.

THEOREM 3.2. *The competitive ratio of* $\mathrm{SLP}_1$ *algorithm is* $r^*$.

PROOF. Let $\sigma = p_1, p_2, \ldots, p_n$ be an arbitrary price sequence. Without loss of generality, we assume that the lowest price in $\sigma$ is $p_i$; then the optimal solution $\mathrm{OPT}(\sigma) \ge (U + \sum_{j=1}^n D_j)p_i$ and $\mathrm{SLP}_1(\sigma) = \sum_{j=1}^n s_j^* p_j$.

If $p_j \in [\theta_1 p_{j-1}, \theta_2 p_{j-1}]$ for $2 \le j \le n$, then $p_j \le p_i/\theta_1^{i-j}$ for $j = 1, 2, \ldots, i$ and $p_j \le \theta_2^{j-i} p_i$ for $j = i+1, i+2, \ldots, n$. From

$$\frac{\mathrm{SLP}_1(\sigma)}{\mathrm{OPT}(\sigma)} \le \sum_{j=1}^n \frac{s_j^* p_j}{(U + \sum_{j=1}^n D_j)p_i},$$

we obtain

$$\begin{aligned}\frac{\mathrm{SLP}_1(\sigma)}{\mathrm{OPT}(\sigma)} &\le \frac{s_1^* p_i/\theta_1^{i-1} + s_2^* p_i/\theta_1^{i-2} + \cdots + s_{i-1}^* p_i/\theta_1 + s_i^* p_i + s_{i+1}^*\theta_2 p_i + \cdots + s_n^*\theta_2^{n-i} p_i}{(U + \sum_{j=1}^n D_j)p_i}\\ &\le \frac{s_1^*/\theta_1^{i-1} + s_2^*/\theta_1^{i-2} + \cdots + s_{i-1}^*/\theta_1 + s_i^* + s_{i+1}^*\theta_2 + \cdots + s_n^*\theta_2^{n-i}}{U + \sum_{j=1}^n (a_j - b_j\theta_2^j p)}\\ &= H_i(s_1^*, s_2^*, \ldots, s_n^*).\end{aligned}$$

Combined with the optimal solution to the linear programming problem (LP1), the above inequality can be rewritten as $\mathrm{SLP}_1(\sigma)/\mathrm{OPT}(\sigma) \le H_i(s_1^*, s_2^*, \ldots, s_n^*) \le r^*$ for $i = 1, 2, \ldots, n$, where $r^*$ is the minimum one satisfying the above inequality. Hence, $r^*$ is the competitive ratio of the algorithm $\mathrm{SLP}_1$. □

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**3.2. Competitive analysis of the logarithmic model** The assumption of logarithmic model is that $p_i \in [\theta_1 p_1 \ln i, \theta_2 p_1 \ln i]$ for $2 \le i \le n$, and there exists one positive $p$ satisfying $p_1 \in [\theta_1 p, \theta_2 p]$ with $\theta_1 \le \theta_2$. Let

$$K_1(s_1, s_2, \dots, s_n) = \frac{s_1 + s_2\theta_2 \ln 2 + \cdots + s_n\theta_2 \ln n}{U + a_1 - b_1\theta_2 p + \sum_{j=2}^{n}(a_j - b_j\theta_2^2 p \ln j)},$$

$$K_i(s_1, s_2, \dots, s_n) = \frac{s_1 + s_2\theta_2 \ln 2 + \cdots + s_n\theta_2 \ln n}{\{U + a_1 - b_1\theta_2 p + \sum_{j=2}^{n}(a_j - b_j\theta_2^2 p \ln j)\}\theta_1 \ln i}, \quad i = 2, 3, \dots, n.$$

Before giving the competitive ratio, the following linear programming problem with variables $\{r, s_1, s_2, \dots, s_n\}$ is considered, in which the constraint conditions (3.2)–(3.4) are transformed by the range of the total purchase quantity at the end of the $j$th day $(j = 1, 2, \dots, n)$ in the logarithmic model.

$$\begin{aligned}
\text{minimize} \quad & r \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(LP2)}\\
\text{such that} \quad & K_i(s_1, s_2, \dots, s_n) \le r,\\
& a_1 - b_1\theta_2 p \le s_1 \le U + a_1 - b_1\theta_1 p,\\
& a_1 - b_1\theta_2 p + \sum_{i=2}^{j}(a_i - b_i\theta_2^2 p \ln i) \le \sum_{i=1}^{j} s_i, \quad j = 2, \dots, n-1,\\
& \sum_{i=1}^{j} s_i \le U + a_1 - b_1\theta_1 p + \sum_{i=2}^{j}(a_i - b_i\theta_1^2 p \ln i), \quad j = 2, \dots, n-1,\\
& U + a_1 - b_1\theta_2 p + \sum_{i=2}^{n}(a_i - b_i\theta_2^2 p \ln i) \le \sum_{i=1}^{n} s_i,\\
& \sum_{i=1}^{n} s_i \le U + a_1 - b_1\theta_1 p + \sum_{i=2}^{n}(a_i - b_i\theta_1^2 p \ln i),\\
& s_i \ge 0, \quad i = 1, 2, \dots, n.
\end{aligned}$$

THEOREM 3.3. *The solution to the linear programming problem (LP2) exists.*

The online algorithm of this model is designed according to the solution of the linear programming problem (LP2), and is denoted by $\text{SLP}_2$.

**Algorithm 2**: $\text{SLP}_2$

1: Solve the linear programming problem (LP2) and let $\{\hat{r}, \hat{s}_1, \hat{s}_2, \dots, \hat{s}_n\}$ be the solution.
2: Define $\hat{s}_i$ to be the quantity of units for purchasing at period $i$ for every $1 \le i \le n$.

THEOREM 3.4. *The competitive ratio of* $\text{SLP}_2$ *algorithm is* $\hat{r}$.

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PROOF. Let $\sigma = p_1, p_2, \ldots, p_n$ denote an arbitrary price sequence. Without loss of generality, we assume that the lowest price in $\sigma$ is $p_i$. For $i = 1$, $\mathrm{OPT}(\sigma) \geq \{U + a_1 - b_1\theta_2 p + \sum_{j=2}^{n}(a_j - b_j\theta_2^2 p \ln j)\}p_1$ and $\mathrm{SLP}_2(\sigma) = \sum_{j=1}^{n} \hat{s}_j p_j$ with the assumptions that $p_j \in [\theta_1 p_1 \ln j, \theta_2 p_1 \ln j]$ for $2 \leq j \leq n$ and $p_j \leq \theta_2 p_1 \ln j$ for every $j = 2, 3, \ldots, n$. Then

$$\begin{aligned}\frac{\mathrm{SLP}_2(\sigma)}{\mathrm{OPT}(\sigma)} &\leq \frac{\sum_{j=1}^{n} \hat{s}_j p_j}{(U + a_1 - b_1\theta_2 p + \sum_{j=2}^{n}(a_j - b_j\theta_2^2 p \ln j))p_1} \\ &\leq \frac{\hat{s}_1 p_1 + \hat{s}_2\theta_2 \ln 2 p_1 + \cdots + \hat{s}_n\theta_2 \ln n p_1}{(U + a_1 - b_1\theta_2 p + \sum_{j=2}^{n}(a_j - b_j\theta_2^2 p \ln j))p_1} \\ &= \frac{\hat{s}_1 + \hat{s}_2\theta_2 \ln 2 + \cdots + \hat{s}_n\theta_2 \ln n}{U + a_1 - b_1\theta_2 p + \sum_{j=2}^{n}(a_j - b_j\theta_2^2 p \ln j)} \\ &= K_1(\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_n).\end{aligned}$$

For $2 \leq i \leq n$, $\mathrm{OPT}(\sigma) \geq \{U + a_1 - b_1\theta_2 p + \sum_{j=2}^{n}(a_j - b_j\theta_2^2 p \ln j)\}p_i$ and $\mathrm{SLP}_2(\sigma) = \sum_{j=1}^{n} \hat{s}_j p_j$. By the assumptions of this model, $p_i \geq \theta_1 p_1 \ln i$ holds for $i = 2, 3, \ldots, n$ and $p_j \leq \theta_2 p_1 \ln j$ holds for $j = 2, 3, \ldots, n$. Then

$$\begin{aligned}\frac{\mathrm{SLP}_2(\sigma)}{\mathrm{OPT}(\sigma)} &\leq \frac{\sum_{j=1}^{n} \hat{s}_j p_j}{(U + a_1 - b_1\theta_2 p + \sum_{j=2}^{n}(a_j - b_j\theta_2^2 p \ln j))p_i} \\ &\leq \frac{\hat{s}_1 p_1 + \hat{s}_2\theta_2 \ln 2 p_1 + \cdots + \hat{s}_n\theta_2 \ln n p_1}{(U + a_1 - b_1\theta_2 p + \sum_{j=2}^{n}(a_j - b_j\theta_2^2 p \ln j))\theta_1 p_1 \ln i} \\ &= \frac{\hat{s}_1 + \hat{s}_2\theta_2 \ln 2 + \cdots + \hat{s}_n\theta_2 \ln n}{(U + a_1 - b_1\theta_2 p + \sum_{j=2}^{n}(a_j - b_j\theta_2^2 p \ln j))\theta_1 \ln i} \\ &= K_i(\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_n).\end{aligned}$$

Combining the above two cases with $i = 1$ and $2 \leq i \leq n$,

$$\frac{\mathrm{SLP}_2(\sigma)}{\mathrm{OPT}(\sigma)} \leq K_i(\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_n) \leq \hat{r}, \quad i = 1, 2, \ldots, n,$$

where $\hat{r}$ is the minimum one satisfying the above inequality. Hence, $\hat{r}$ is the competitive ratio of the algorithm $\mathrm{SLP}_2$. □

## 4. The competitive analysis for exponentially related demand

In this inventory problem, the demand is assumed to have a negative exponential relationship with price [2, 16]. Without loss of generality, we assume that $D_i = a_i \exp(-b_i p_i)$.

**4.1. Competitive analysis of the exponential model** First, a linear programming problem with variables $\{r, s_1, s_2, \ldots, s_n\}$, as follows, is investigated, in which the second and third constraint conditions are transformed by the range of the total

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purchase quantity at the end of the $j$th day ($j = 1, 2, \dots, n$).

$$\begin{aligned} &\text{minimize} && r && \text{(LP3)}\\ &\text{such that} && M_i(s_1, s_2, \dots, s_n) \le r \\ & && U + \sum_{i=1}^{n} a_i \exp(-b_i\theta_2^i p) \le \sum_{i=1}^{n} s_i \le U + \sum_{i=1}^{n} a_i \exp(-b_i\theta_1^i p) \\ & && \sum_{i=1}^{j} a_i \exp(-b_i\theta_2^i p) \le \sum_{i=1}^{j} s_i \le U + \sum_{i=1}^{j} a_i \exp(-b_i\theta_1^i p), \quad j = 1, 2, \dots, n-1 \\ & && s_i \ge 0, \quad i = 1, 2, \dots, n, \end{aligned}$$

where

$$M_i(s_1, s_2, \dots, s_n) = \frac{s_1/\theta_1^{i-1} + s_2/\theta_1^{i-2} + \cdots + s_{i-1}/\theta_1 + s_i + s_{i+1}\theta_2 + \cdots + s_n\theta_2^{n-i}}{U + \sum_{j=1}^{n} a_j \exp(-b_j\theta_2^j p)}.$$

Theorem 4.1. *The solution to the linear programming problem* (LP3) *exists.*

The online algorithm of this model can be designed according to the solution of the linear programming problem (LP3), and is denoted by $\text{SLP}_3$.

**Algorithm 3**: $\text{SLP}_3$

1: Solve the linear programming problem (LP3) and let $\{r^*, s_1^*, s_2^*, \dots, s_n^*\}$ be the solution.
2: Define $s_i^*$ to be the quantity of units for purchasing at period $i$ for every $1 \le i \le n$.

Theorem 4.2. *The competitive ratio of* $\text{SLP}_3$ *algorithm is* $r^*$.

Proof. Let $\sigma = p_1, p_2, \dots, p_n$ be an arbitrary price sequence. Without loss of generality, we assume that the lowest price in $\sigma$ is $p_i$. Note that $\text{OPT}(\sigma) \ge (U + \sum_{j=1}^{n} D_j)p_i$ and $\text{SLP}_3(\sigma) = \sum_{j=1}^{n} s_j^* p_j$. For $2 \le j \le n$, $p_j \in [\theta_1 p_{j-1}, \theta_2 p_{j-1}]$ holds, and then $p_j \le p_i/\theta_1^{i-j}$ for $j = 1, 2, \dots, i$ and $p_j \le \theta_2^{j-i} p_i$ for $j = i+1, i+2, \dots, n$. From $\text{SLP}_3(\sigma)/\text{OPT}(\sigma) \le \sum_{j=1}^{n} s_j^* p_j/(U + \sum_{j=1}^{n} D_j)p_i$,

$$\begin{aligned} \frac{\text{SLP}_3(\sigma)}{\text{OPT}(\sigma)} &\le \frac{s_1^* p_i/\theta_1^{i-1} + s_2^* p_i/\theta_1^{i-2} + \cdots + s_{i-1}^* p_i/\theta_1 + s_i^* p_i + s_{i+1}^*\theta_2 p_i + \cdots + s_n^*\theta_2^{n-i} p_i}{(U + \sum_{j=1}^{n} D_j)p_i} \\ &\le \frac{s_1^*/\theta_1^{i-1} + s_2^*/\theta_1^{i-2} + \cdots + s_{i-1}^*/\theta_1 + s_i^* + s_{i+1}^*\theta_2 + \cdots + s_n^*\theta_2^{n-i}}{U + \sum_{j=1}^{n} a_j \exp(-b_j\theta_2^j p)} \\ &= M_i(s_1^*, s_2^*, \dots, s_n^*). \end{aligned}$$

Combined with the optimal solution to the linear programming problem (LP3), the above inequality can be rewritten as

$$\frac{\text{SLP}_3(\sigma)}{\text{OPT}(\sigma)} \le M_i(s_1^*, s_2^*, \dots, s_n^*) \le r^*, \quad i = 1, 2, \dots, n,$$

where $r^*$ is the minimum one satisfying the above inequality. Hence, $r^*$ is the competitive ratio of the algorithm $\mathrm{SLP}_3$. □

**4.2. Competitive analysis of the logarithmic model** Let

$$Q_1(s_1, s_2, \ldots, s_n) = \frac{s_1 + s_2\theta_2 \ln 2 + \cdots + s_n\theta_2 \ln n}{U + a_1 \exp(-b_1\theta_2 p) + \sum_{j=2}^{n} a_j \exp(-b_j\theta_2^2 p \ln j)},$$

$$Q_i(s_1, s_2, \ldots, s_n) = \frac{s_1 + s_2\theta_2 \ln 2 + \cdots + s_n\theta_2 \ln n}{(U + a_1 \exp(-b_1\theta_2 p) + \sum_{j=2}^{n} a_j \exp(-b_j\theta_2^2 p \ln j))\theta_1 \ln i}$$

for $i = 2, 3, \ldots, n$. Before giving the competitive ratio, the following linear programming problem with variables $\{r, s_1, s_2, \ldots, s_n\}$ is considered, in which the constraint conditions are transformed by the range of the total purchase quantity at the end of the $j$th day ($j = 1, 2, \ldots, n$).

$$\begin{aligned}
\text{minimize}\quad & r \qquad\qquad \text{(LP4)}\\
\text{such that}\quad & Q_i(s_1, s_2, \ldots, s_n) \le r,\\
& a_1 \exp(-b_1\theta_2 p) \le s_1 \le U + a_1 \exp(-b_1\theta_1 p),\\
& a_1 \exp(-b_1\theta_2 p) + \sum_{i=2}^{j} a_i \exp(-b_i\theta_2^2 p \ln i) \le \sum_{i=1}^{j} s_i, \quad j = 2, \ldots, n-1,\\
& \sum_{i=1}^{j} s_i \le U + a_1 \exp(-b_1\theta_1 p) + \sum_{i=2}^{j} a_i \exp(-b_i\theta_1^2 p \ln i), \quad j = 2, \ldots, n-1,\\
& U + a_1 \exp(-b_1\theta_2 p) + \sum_{i=2}^{n} a_i \exp(-b_i\theta_2^2 p \ln i) \le \sum_{i=1}^{n} s_i,\\
& \sum_{i=1}^{n} s_i \le U + a_1 \exp(-b_1\theta_1 p) + \sum_{i=2}^{n} a_i \exp(-b_i\theta_1^2 p \ln i),\\
& s_i \ge 0, \quad i = 1, 2, \ldots, n.
\end{aligned}$$

THEOREM 4.3. *The solution to the linear programming problem* (LP4) *exists.*

The online algorithm of this model can be designed according to the solution of the linear programming problem (LP4), and is denoted by $\mathrm{SLP}_4$.

**Algorithm 4**: $\mathrm{SLP}_4$

1: Solve the linear programming problem (LP4) and let $\{\bar{r}, \bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n\}$ be the solution.
2: Define $\bar{s}_i$ to be the quantity of units for purchasing at period $i$ for every $1 \le i \le n$.

THEOREM 4.4. *The competitive ratio of* $\mathrm{SLP}_4$ *algorithm is* $\bar{r}$.

PROOF. Let $\sigma$ denote an arbitrary price sequence. Without loss of generality, we assume that the lowest price in $\sigma$ is $p_i$.

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For $i = 1$, $\mathrm{OPT}(\sigma) \geq \{U + a_1 \exp(-b_1\theta_2 p) + \sum_{j=2}^{n} a_j \exp(-b_j\theta_2^2 p \ln j)\}p_1$, and $\mathrm{SLP}_4(\sigma) = \sum_{j=1}^{n} \bar{s}_j p_j$. For $2 \leq j \leq n$, $p_j \in [\theta_1 p_1 \ln j, \theta_2 p_1 \ln j]$, the inequality $p_j \leq \theta_2 p_1 \ln j$ holds for every $j = 2, 3, \ldots, n$.

$$\begin{aligned}
\frac{\mathrm{SLP}_4(\sigma)}{\mathrm{OPT}(\sigma)} &\leq \frac{\sum_{j=1}^{n} \bar{s}_j p_j}{\{U + a_1 \exp(-b_1\theta_2 p) + \sum_{j=2}^{n} a_j \exp(-b_j\theta_2^2 p \ln j)\}p_1} \\
&\leq \frac{\bar{s}_1 p_1 + \bar{s}_2\theta_2 \ln 2p_1 + \cdots + \bar{s}_n\theta_2 \ln np_1}{\{U + a_1 \exp(-b_1\theta_2 p) + \sum_{j=2}^{n} a_j \exp(-b_j\theta_2^2 p \ln j)\}p_1} \\
&= \frac{\bar{s}_1 + \bar{s}_2\theta_2 \ln 2 + \cdots + \bar{s}_n\theta_2 \ln n}{U + a_1 \exp(-b_1\theta_2 p) + \sum_{j=2}^{n} a_j \exp(-b_j\theta_2^2 p \ln j)} \\
&= Q_1(\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n).
\end{aligned}$$

For $2 \leq i \leq n$, $\mathrm{OPT}(\sigma) \geq \{U + a_1 \exp(-b_1\theta_2 p) + \sum_{j=2}^{n} a_j \exp(-b_j\theta_2^2 p \ln j)\}p_i$ and $\mathrm{SLP}_4(\sigma) = \sum_{j=1}^{n} \bar{s}_j p_j$ hold. By the assumptions of this model, $p_i \geq \theta_1 p_1 \ln i$ and $p_j \leq \theta_2 p_1 \ln j$ for $i = 2, 3, \ldots, n$ and $j = 2, 3, \ldots, n$. Then

$$\begin{aligned}
\frac{\mathrm{SLP}_4(\sigma)}{\mathrm{OPT}(\sigma)} &\leq \frac{\sum_{j=1}^{n} \bar{s}_j p_j}{(U + a_1 \exp(-b_1\theta_2 p) + \sum_{j=2}^{n} a_j \exp(-b_j\theta_2^2 p \ln j))p_i} \\
&\leq \frac{\bar{s}_1 p_1 + \bar{s}_2\theta_2 \ln 2p_1 + \cdots + \bar{s}_n\theta_2 \ln np_1}{(U + a_1 \exp(-b_1\theta_2 p) + \sum_{j=2}^{n} a_j \exp(-b_j\theta_2^2 p \ln j))\theta_1 p_1 \ln i} \\
&= \frac{\bar{s}_1 + \bar{s}_2\theta_2 \ln 2 + \cdots + \bar{s}_n\theta_2 \ln n}{(U + a_1 \exp(-b_1\theta_2 p) + \sum_{j=2}^{n} a_j \exp(-b_j\theta_2^2 p \ln j))\theta_1 \ln i} \\
&= Q_i(\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n).
\end{aligned}$$

Combining the above two cases,

$$\frac{\mathrm{SLP}_4(\sigma)}{\mathrm{OPT}(\sigma)} \leq Q_i(\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n) \leq \bar{r}, \quad i = 1, 2, \ldots, n,$$

where $\bar{r}$ is the minimum one satisfying the above inequality. Hence, $\bar{r}$ is the competitive ratio of the algorithm $\mathrm{SLP}_4$. □

## 5. Conclusions

We investigate two models for the interrelated price online inventory problem with two types of demand. The corresponding algorithms are designed, and the competitive ratios are derived for the exponential and the logarithmic model, respectively, with the daily demand. In future, it would be interesting to consider a problem where both the price and demand are online. It would also be challenging to investigate the online inventory problem, where the price information is updated in real-time.

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