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A CHARACTERIZATION OF WEAK PEAK SETS FOR FUNCTION ALGEBRAS

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Let $A \subset C(S)$ be a function algebra. In this paper we prove that $S_o = \overline{S}_o \subset S$ is a weak peak set for A if and only if for any open neighbourhood U of S_o there is an f in A such that $\|f\| \leqslant 2$, $|f(x) - 1| \leqslant 1/3$ on S_o and $|f(x)| \leqslant 1/3$ on $S \setminus U$.

Let S be a compact Hausdorff space. We shall denote by $C(S)$ $(C_R(S))$ the Banach algebra of all complex (real) valued continuous functions on S, provided with a usual supremum norm. Let A be a function algebra on S, i.e. a closed subalgebra of $C(S)$ which contains the constant functions and which separates points of S. A closed subset S_o of S is called a weak peak set for A if for any open neighbourhood U of S_o there is an f in A such that $1 = \|f\| = f(s)$ for $s \in S_o$ and $|f(s)| < 1$ for $s \in S \setminus U$. A one-point weak peak set is called a weak peak point. A classical Bishop's the so-called "1/4 - 3/4" criterion (see e.g. [4] p. 263) gives the following characterization of weak peak points:

THEOREM [Bishop]. *Let A be a function algebra on S. A point $s_o \in S$ is a weak peak point for A if and only if there are constants $K \geqslant 1$ and $c < 1$ such that for any open neighbourhood U of s_o there is an f in A such that $\|f\| \leqslant K$, $f(s_o) = 1$ and $|f(s)| \leqslant c$ for $s \in S \setminus U$.*

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The aim of this note is to give the following generalization of the above criterion:

THEOREM. *Let* A *be a function algebra on a compact Hausdorff space* S *and let* S_o *be a closed subset of* S. *Then* S_o *is a weak peak set for* A *if and only if there are constants* $K \geq 1$, $c_1 \geq 0$, $c_2 \geq 0$ *with* $c_1 + c_2 < 1$ *such that for any open neighbourhood* U *of* S_o *there is an* f *in* A *such that* $\|f\| \leq K$, $|f(s) - 1| \leq c_2$ *for* $s \in S_o$ *and* $|f(s)| \leq c_1$ *for* $s \in S \setminus U$.

Please note that the definition of a weak peak set is evidently equivalent to the following one ([4]):

Let A be a function algebra on S and let S_o be a closed subset of S then S_o is a weak peak set for A if and only if for any $\varepsilon > 0$ and any open neighbourhood U of S_o there is an f in A such that $1 = \|f\| = f(s)$ for $s \in S_o$ and $|f(s)| < \varepsilon$ for $s \in S \setminus U$.

Hence the "only if" part of our theorem is trivial: that is, if $S_o \subset S$ is a weak peak set for A then we can take c_1, c_2 any positive numbers and put $K = 1$.

We divide the proof of our theorem into four steps; the proof of the second step is based on Bishop's proof of his criterion.

Assume that the assumption of Theorem is fulfilled.

STEP 1. For any $\varepsilon > 0$ there is a positive constant $K(\varepsilon)$ such that for any open neighbourhood U of S_o there is an f in A such that $|f(s)| \leq \varepsilon$ for $s \in S \setminus U$; $|f(s)| \leq 1$ and $|f(s) - 1| \leq \varepsilon$ for $s \in S_o$ and $\|f\| \leq K(\varepsilon)$.

Proof. Let $f_1 \in A$ be such that $|f_1(s)| \leq c_1$ for $s \in S \setminus U$, $|f_1(s) - 1| \leq c_2$ for $s \in S_o$ and $\|f_1\| \leq K$. Since the discs $D_1 = \{z \in \mathbb{C} : |z| \leq c_1\}$ and $D_2 = \{z \in \mathbb{C} : |z - 1| \leq c_2\}$ are disjoint then, by Runge Theorem, there is a polynomial p such that $p(z) \leq \varepsilon$ for $z \in D_1$ and $|p(z) - (1 - \frac{\varepsilon}{2})| \leq \frac{\varepsilon}{2}$ for $z \in D_2$.

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Put $K(\varepsilon) = \sup\{|p(z)| : |z| \leqslant K\}$ and $f = p \circ f_1 \in A$.

STEP 2. Assume that there are constants K_1 and $\varepsilon < 1$ such that for any open neighbourhood U of S_o there is an f in A such that $|f(s)| \leqslant \varepsilon$ for $s \in S \setminus U$; $|f(s)| \leqslant 1$ and $|f(s) - 1| \leqslant \varepsilon$ for $s \in S_o$ and $\|f\| \leqslant K_1$, then for any open neighbourhood U of S_o there is a g in A such that $|g(s)| \leqslant \varepsilon$ for $s \in S \setminus U$; $|g(s) - 1| \leqslant \varepsilon$ for $s \in S_o$ and $\|g\| \leqslant 1$.

Proof. Fix any $x < 1$ with

$$a = (K_1 - 1) - x(K_1 - \varepsilon) < 0$$

and a decreasing sequence of positive numbers ε_n such that

$$\varepsilon_n(1 - x^n) + x^n a < 0 \quad \text{for} \quad n \geqslant 1.$$

We define by induction a sequence of functions $(h_n)_{n=1}^{\infty}$ from A. Let $h_1 \in A$ be any function from A such that $|h_1(s)| \leqslant \varepsilon$ for $s \in S \setminus U$; $|h_1(s)| \leqslant 1$ and $|h_1(s) - 1| \leqslant \varepsilon$ for $s \in S_o$ and $\|h_1\| \leqslant K_1$.

Assume we have defined $h_1, \ldots, h_n$ then put

$$W_n = \{s : \max_{1 \leqslant j \leqslant n} |h_j(s)| \geqslant 1 + \varepsilon_n\} .$$

The set W_n is a closed subset of $S \setminus S_o$ so there is an $h_{n+1} \in A$ such that

$$|h_{n+1}(s)| \leqslant \varepsilon \quad \text{for} \quad s \in (S \setminus U) \cup W_n ;$$

$$|h_{n+1}(s)| \leqslant 1 \quad \text{and} \quad |h_{n+1}(s) - 1| \leqslant \varepsilon \quad \text{for} \quad s \in S_o$$

and

$$\|h_{n+1}\| \leqslant K_1 .$$

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Let

$$g = (1 - x) \sum_{j=1}^{\infty} x^{j-1} \cdot h_j \quad .$$

We have evidently $|g(s)| \leqslant 1$ and $|g(s) - 1| \leqslant \varepsilon$ for $s \in S_o$, $|g(s)| \leqslant \varepsilon$ for $s \in S \setminus U$, and $|g(s)| \leqslant 1$ for $s \in S \setminus \bigcup_{n=1}^{\infty} W_n$. It remains to show that if $s \in \bigcup_{n=1}^{\infty} W_n$ then $|g(s)| \leqslant 1$.

The sequence $(W_n)_{n=1}^{\infty}$ is an increasing sequence of compact sets so if $s \in \bigcup_{n=1}^{\infty} W_n$ then there is a positive integer m such that $s \in W_{m+1}$ but $s \notin W_m$ for $m \geqslant 0$ (we put $W_o = \emptyset$).

We have

$$|h_j(s)| \leqslant 1 + \varepsilon_m \quad \text{for} \quad j \leqslant m ,$$

$$|h_{m+1}(s)| \leqslant K_1 ,$$

$$|h_j(s)| \leqslant \varepsilon \quad \text{for} \quad j \geqslant m + 2 ,$$

hence

$$|g(s)| \leqslant (1 - x) \ ((1 + \varepsilon_m) \sum_{j=1}^{m} x^{j-1} + K_1 x^m + \varepsilon \sum_{j=m+2}^{\infty} x^{j-1})$$

$$= 1 + \varepsilon_m (1 - x^m) + x^m (K_1 - 1 - x(K_1 - \varepsilon)) < 1 .$$

STEP 3. Let M be a closed subspace of $C(S)$, let π be a canonical map from M onto $M_{|S_o} = \{f_{|S_o} : f \in M\} \subset C(S_o)$ $\pi(f) = f_{|S_o}$, and let $\tilde{\pi} : M_{/\ker\pi} \to M_{|S_o} : \tilde{\pi}(f + \ker\pi) = f_{|S_o}$. Then if $\tilde{\pi}$ is not an isometry then there are measures μ on S_o and ν on $S \setminus S_o$ such that $\mu - \nu \perp M$ but the measure μ represents a non zero functional on M.

Proof. Assume $\tilde{\pi}$ is not an isometry, there is a functional F_o on

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$M|_{S_o}$ represented by a measure μ_o on S_o such that $\|F_o\| = \text{var}(\mu_o) = 1$ but $\|\pi^*(F_o)\| = t < 1$. Let ν_o be any measure on S which represents the functional $\pi^*(F_o)$ with $\text{var}(\nu_o) = t$. Since $\text{var}(\nu_o|_{S_o}) \leqslant t < 1$ then the measure $\mu_o - \nu_o|_{S_o}$ is not orthogonal to M and we can put $\mu = \mu_o - \nu_o|_{S_o}$, $\nu = \nu_o|_{S \setminus S_o}$.

To end the proof of Theorem now fix an open neighbourhood U of S_o and let $q \in C_R(S)$ be such that

$$q|_{S_o} \equiv 1 \quad , \quad q|_{S \setminus U} \equiv 3 \quad , \quad 1 \leqslant q \leqslant 3$$

and put

$$M = \{q \cdot g \in C(S) \; : \; g \in A\} \quad .$$

We shall prove that $\tilde{\pi} : M/_{\ker\pi} \to M|_{S_o}$ is an isometry. Assume the contrary and let μ, ν be as in Step 3. We can assume that the norm of μ on $M|_{S_o}$ is equal to 2 and let $f \in M$ be such that $\int f d\mu = 1$ and $\|f\|_{S_o} \leqslant 1$. Fix $0 < \delta < \frac{1}{2}(\|f\| (1 + \text{var}(\nu)))^{-1}$. The regularity of the measure ν provides there is an open neighbourhood V of S_o such that $|\nu|(V) < \delta$ and, by Steps 1 and 2, there is an f_o in A such that $\|f_o\|_{S \setminus V} \leqslant \delta$, $\|f_o - 1\|_{S_o} \leqslant \delta$ and $\|f_o\| \leqslant 1$. Since $\nu - \mu \perp M$ and $ff_o \in M$ we have

$$\tfrac{1}{2} < 1 - 2\delta \leqslant \left|\int_{S_o} f d\mu\right| - \left|\int_{S_o} f(1 - f_o) d\mu\right|$$

$$\leqslant \left|\int_{S_o} ff_o d\mu\right| = \left|\int_{S \setminus S_o} ff_o d\nu\right| \leqslant \left|\int_V ff_o d\nu\right| + \left|\int_{S \setminus V} ff_o d\nu\right|$$

$$\leqslant \delta\|f\| + \text{var}(\nu)\delta\|f\| < \tfrac{1}{2} \quad .$$

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So we have proved that $\tilde{\pi}$ is an isometry and the same time we have established that:

for any open neighbourhood U of S_o there is an f_U in A such $\|f_U\| \leqslant 2$, $f_{U|S_o} \equiv 1$ and $|f_U(s)| \leqslant \frac{1}{2}$ for $s \in S \setminus U$.

Now the Theorem follows from the Bishop criterion applied to the algebra $\tilde{A} = \{f \in A : f_{|S_o} = \text{const}\}$. //

REMARKS. The theorem we have just proved is applied to the theory of small perturbations of multiplication in function algebras (see the author's paper "Perturbations of Banach algebras" to appear and also [2], [3], [5]) but one can also get another application; for example, the following theorem due to Badé and Curtis is an immediate consequence of our theorem:

THEOREM. [Badé and Curtis]. *Let A be a function algebra on a compact Hausdorff space S and assume A is boundedly ε-normal for some $\varepsilon < \frac{1}{2}$ then $A = C(S)$.*

Where A is called boundedly ε-normal if there is a constant K such that for any closed disjoint subsets F, G of S there is an f in A with $\|f\| \leqslant K$, $|f(s)| \leqslant \varepsilon$ for $s \in F$ and $|f(s) - 1| \leqslant \varepsilon$ for $s \in G$.

Proof. Assume $A \not\subset C(S)$ then there is a measure μ on S orthogonal to A; let $F = \overline{F} \subset S$ be such that $\mu(F) \neq 0$, we can assume $\mu(F) = 1$, and let U be an open neighbourhood of F with $|\mu|(U \setminus F) \leqslant \frac{1}{4}$. By our Theorem and the assumptions F is a weak peak set so there is an f in A such that $\|f\| \leqslant 1$, $f|_F \equiv 1$ and $|f(s)| < \frac{1}{2}\text{var}(\mu)$ for $s \in S \setminus U$. We have

$$0 = \left|\int_S f d\mu\right| = \left|\int_F f d\mu + \int_{U\setminus F} f d\mu + \int_{S\setminus U} f d\mu\right|$$

$$\geqslant 1 - \frac{1}{2} - \frac{|\mu|(S\setminus U)}{2\text{var}(\mu)} > 0$$

and this ends the proof.

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