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# ON TOPOLOGIES IN A TOPOS

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Johnstone and Paré have each given a way of constructing the largest topology allowing a given object to be a sheaf. In this paper we use the notion of a partial map to construct such a topology in a simple way.

Throughout this paper, $E$ denotes an elementary topos. The preliminaries and other notation can be found in [**1**].

## PRELIMINARIES

A *topology (Lawvere-Tierney)* in $E$ is a morphism $j\colon \Omega \to \Omega$ such that

$$
\begin{array}{ccc}
 & & \Omega \\
 & \nearrow & \big\downarrow j \\
1 & & \\
 & \searrow & \\
 & & \Omega
\end{array}
\qquad
\begin{array}{ccc}
 & & \Omega \\
 & \overset{j}{\nearrow} & \big\downarrow j \\
\Omega & & \\
 & \underset{j}{\searrow} & \\
 & & \Omega
\end{array}
\qquad
\begin{array}{ccc}
\Omega\times\Omega & \xrightarrow{\ \wedge\ } & \Omega \\
{\scriptstyle j\times j}\big\downarrow & & \big\downarrow{\scriptstyle j} \\
\Omega\times\Omega & \xrightarrow{\ \wedge\ } & \Omega
\end{array}
$$

commute. We write $J \rightarrowtail \Omega$ for the subobject classified by $j$. Since every topology in $E$ is a closure operator on $E$, we say $X_0 \rightarrowtail X$ is *$j$-dense* (*$j$-closed*) if $jX_0 = \overline{X}_0 = X\left(jX_0 = \overline{X}_0 = X_0\right)$. An object $F \in E$ is a *$j$-sheaf* if for every $j$-dense $X_0 \overset{\sigma}{\rightarrowtail} X$ and every $X_0 \xrightarrow{f} F$ there exists a unique $X \xrightarrow{\overline{f}} F$ such that $\overline{f}\sigma = f$. If $F \in E$, then all partial maps with codomain $F$ are representable, that is, there is a mono $F \overset{\eta}{\rightarrowtail} \widetilde{F}$ such that for every $Y_0 \rightarrowtail Y$ and $Y_0 \xrightarrow{f} F$, there exists a unique $Y \xrightarrow{\widetilde{f}} \widetilde{F}$ making

$$
\begin{array}{ccc}
Y_0 & \xrightarrow{\ f\ } & F \\
\big\downarrow & & \big\downarrow{\scriptstyle \eta} \\
Y & \xrightarrow{\ \widetilde{f}\ } & \widetilde{F}
\end{array}
$$

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a pullback diagram.

Let $I$ be an object in $E$ and let $X \xrightarrow{f} Y$ be in $E/I$. We define (see [3]) $Iso(f)$ to be a subobject of $I$ with the following property: $\forall J \in E$, $\forall J \xrightarrow{\alpha} I$, $\alpha$ factors through $Iso(f) \Longleftrightarrow f$ is an isomorphism in $E/J$.

LEMMA 1. *Let $F$ be an object and $\Omega \xrightarrow{j} \Omega$ a topology in $E$. Then $F$ is a $j$-sheaf if and only if*

$$\begin{array}{ccc} F & \xrightarrow[(*)]{\eta} & \widetilde{F} \\ \downarrow & & \downarrow \varphi \\ J & \xrightarrow{\tau} & \Omega \end{array}$$

*is a pullback, where $\varphi$ and $j$ are the characteristic morphisms of $F$ and $J$, respectively.*

PROOF: Let $F$ be a $j$-sheaf and $T$ an object of $E$. Let $\alpha \in \widetilde{F}$, $\beta \in J$ be two $T$-elements, such that $\varphi\alpha = \tau\beta$, then there exists a $j$-dense $T_0 \overset{\sigma}{\rightarrowtail} T$ such that

$$\begin{array}{ccc} T_0 & \xrightarrow{\sigma} & T \\ \downarrow & & \downarrow \alpha \\ F & \xrightarrow{\eta} & \widetilde{F} \end{array}$$

is a pullback diagram. However, $F$ is a sheaf, therefore there exists a unique $T$-element $\gamma \in A$ such that $\eta\gamma = \alpha$. Hence, the required diagram is a pullback.

Conversely, assume that $(*)$ is a pullback and $X_0 \overset{\sigma}{\rightarrowtail} X$ is $j$-dense. If $X_0 \xrightarrow{f} F$, then there exists a unique $X \xrightarrow{\widetilde{f}} \widetilde{F}$ making

$$\begin{array}{ccc} X_0 & \xrightarrow{\sigma} & X \\ \downarrow & & \downarrow \widetilde{f} \\ F & \xrightarrow{\eta} & \widetilde{F} \end{array}$$

a pullback diagram. However $X_0 \overset{\sigma}{\rightarrowtail} X$ is $j$-dense, therefore the characteristic morphism of $\sigma$ factors through $J$, that is we have $X \xrightarrow{g} J$ such that $\varphi\widetilde{f} = \tau g$. Therefore there exists a unique $X \xrightarrow{h} F$ such that $h\sigma = f$. Hence $F$ is a $j$-sheaf. ∎

Now, for the identity morphism $1_F$ and $F \overset{(t,f)}{\rightarrowtail} \Omega \times F$, there exists a unique $\Omega \times F \xrightarrow{\gamma} \widetilde{F}$ such that

$$\begin{array}{ccc} F & \xrightarrow[P.B.]{1_F} & F \\ {\scriptstyle (t,F)}\downarrow & & \downarrow \eta \\ \Omega \times F & \xrightarrow{\gamma} & \widetilde{F} \end{array}$$

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is a pullback diagram. If $\varphi$ is the characteristic morphism for $\eta$, then $\varphi\gamma\colon \Omega \times F \longrightarrow \Omega$ is the projection. Hence $\Omega^* F \xrightarrow{\gamma} \varphi$ is a morphism in $E/\Omega$ and therefore $Iso(\gamma)$ is a subobject of $\Omega$. We characterise the elements of $J = Iso(\gamma)$ as follows.

LEMMA 2. *Let $X$ be in $E$ and let $X \xrightarrow{\alpha} \Omega$ represent a subobject of $X$. Then $\alpha$ factors through $J$ if and only if*

$$\begin{array}{ccccc} X \times F & \xrightarrow{\alpha \times F} & \Omega \times F & \xrightarrow{\gamma} & \tilde{F} \\ {\scriptstyle \pi}\downarrow & & & & \downarrow{\scriptstyle \varphi} \\ X & & \xrightarrow{\alpha} & & \Omega \end{array}$$

*is a pullback.*

PROOF: By the definition of $J$, $\alpha$ factors through $J$ if and only if

$$X^* F \cong \alpha^* \Omega^* F \xrightarrow[\sim]{\alpha^* \gamma} \alpha^* \varphi$$

which is equivalent to the condition that the required diagram is a pullback. ∎

LEMMA 3. *Let $J = Iso(\gamma)$ be the above-mentioned object. Then $F \overset{\eta}{\rightarrowtail} \tilde{F} \xrightarrow{\varphi} \Omega$ factors through $J$ and the diagram*

$$\begin{array}{ccc} F & \xrightarrow[(*)]{\eta} & \tilde{F} \\ \downarrow & & \downarrow{\scriptstyle \varphi} \\ J & \longrightarrow & \Omega \end{array}$$

*is a pullback.*

PROOF: The following diagram is a pullback.

$$\begin{array}{ccccc} F \times F & \xrightarrow{\pi} & F & \xrightarrow{\eta} & \tilde{F} \\ {\scriptstyle \pi}\downarrow & P.B. & \downarrow & P.B. & \downarrow{\scriptstyle \varphi} \\ F & \xrightarrow{\tau} & 1 & \xrightarrow{t} & \Omega \end{array}$$

However, $\eta\pi = \gamma(\varphi\eta \times F)$ and $t\tau = \varphi\eta$. Therefore, by Lemma 2 $\varphi\eta$ factors through $J$, that is $(*)$ commutes.

Let $X \xrightarrow{\tilde{f}} \tilde{F}$, $X \xrightarrow{\alpha} J$ be two morphisms, such that $\varphi\tilde{f} = \alpha$. Since $\alpha$ factors through $J$ then by Lemma 2,

$$\begin{array}{ccc} X \times F & \xrightarrow{\gamma(\alpha \times F)} & \tilde{F} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ X & \xrightarrow{\alpha} & \Omega \end{array}$$

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is a pullback. Therefore there exists a unique $X \xrightarrow{f} F$ such that $\gamma(\alpha, f) = \tilde{f}$. On the other hand, if $X_0 \overset{\sigma}{\rightarrowtail} X$ is classified by $\alpha$, then we have,

$$\begin{aligned}\tilde{f}\sigma &= \gamma(\alpha, f)\sigma = \gamma(\alpha\sigma,\, f\sigma) = \gamma(t,\, f\sigma)\\ &= \eta f\sigma.\end{aligned}$$

Therefore, by the uniqueness of $\tilde{f}$, $\tilde{f} = \eta g$ and hence $(*)$ is a pullback. ☐

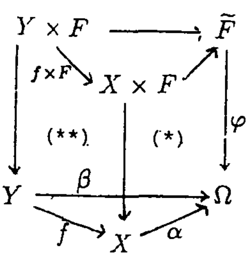
**Theorem 4.** *Let $F$ be an object in $E$ and let $Iso(\gamma) = J \rightarrowtail \Omega$ be the same subobject as above. If $\Omega \xrightarrow{j} \Omega$ is the characteristic morphism of $J$, then $j$ is a topology in $E$.*

Proof: Since

$$\begin{array}{ccc} F & \xrightarrow{\eta} & \tilde{F} \\ \downarrow & & \downarrow{\scriptstyle\varphi} \\ 1 & \xrightarrow[t]{} & \Omega \end{array}$$

is a pullback then, by Lemma 3, the truth value factors through $J$, that is $jt = t$. Suppose $X \xrightarrow{\alpha} \Omega$ factors through $J$. If $\alpha$ classifies the subobject $X_0 \overset{\sigma}{\rightarrowtail} X$, and $Y \xrightarrow{f} X$ is a morphism, then we show that $Y \xrightarrow{\beta} \Omega$, the characteristic morphism of $f^*X_0 \overset{\sigma'}{\rightarrowtail} Y$, factors through $J$.

Consider




In this diagram, by assumption, $(*)$ is a pullback. However, $\alpha f = \beta$, so the two triangles are commutative. Also $(**)$ is a pullback therefore

$$\begin{array}{ccc} Y \times F & \xrightarrow{\gamma(\beta \times F)} & F \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ Y & \xrightarrow{\beta} & \Omega \end{array}$$

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is a pullback, that is $\beta$ factors through $J$. Hence $J$ is stable under pullbacks. It remains to show that $j^2 = j$. Consider the diagram

$$\begin{array}{ccccc} K & \xrightarrow{i} & J & \longrightarrow & 1 \\ \big\downarrow \scriptstyle P.B. & & \big\downarrow \scriptstyle P.B. & & \big\downarrow \scriptstyle t \\ \Omega & \xrightarrow[j]{} & \Omega & \xrightarrow[j]{} & \Omega. \end{array}$$

We will show that $K \cong J$. Let $X \xrightarrow{\alpha} J$ be an $X$-element of $J$. Then $j\alpha = t$ so $jj\alpha = jt = t$. Therefore, $\alpha$ is an $X$-element of $K$. Conversely, suppose $X \xrightarrow{\alpha} K$ is an $X$-element of $K$, then $i\alpha = j\alpha \in J$. Therefore, by Lemma 2,

$$\begin{array}{ccccc} X \times F & & \xrightarrow{\gamma(j\alpha \times F)} & & \tilde{F} \\ {\scriptstyle \pi}\big\downarrow & & & & \big\downarrow{\scriptstyle \varphi} \\ X & \xrightarrow{\alpha} & \Omega & \xrightarrow{j} & \Omega \end{array}$$

is a pullback. On the other hand $j\varphi = \varphi$, because

$$\begin{array}{ccccc} F & \longrightarrow & J & \longrightarrow & 1 \\ {\scriptstyle \eta}\big\downarrow \scriptstyle P.B. & & \big\downarrow \scriptstyle P.B. & & \big\downarrow \scriptstyle t \\ \tilde{F} & \xrightarrow{\varphi} & \Omega & \xrightarrow{j} & \Omega \end{array}$$

is a pullback. Also, by Lemma 3, $J \times F \cong F$ hence $\gamma(j \times F) = \gamma$ and $\gamma(j\alpha \times F) = \gamma(\alpha \times F)$. Therefore,

$$\begin{array}{ccc} X \times F & \xrightarrow{\gamma(\alpha \times F)} & \tilde{F} \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \varphi} \\ X & \xrightarrow{\alpha} & \Omega \end{array}$$

is a pullback diagram, that is $\alpha \in J$. Hence $K \cong J$, that is $j^2 = j$, and $j$ is a topology. ∎

THEOREM 5. *Let $j$ be the topology of Theorem 4. Then $F$ is a $j$-sheaf and $j$ is the largest topology allowing $F$ to be a sheaf.*

PROOF: By Lemmas 1 and 3, $F$ is a $j$-sheaf. Now suppose that $X_0 \overset{\sigma}{\rightarrowtail} X$ is a subobject of $X$ with characteristic morphism $X \xrightarrow{\alpha} \Omega$. Let $X_0 \xrightarrow{f} F$ be a morphism.

Then $f$ can be extended uniquely to $X$ if and only if

$$\begin{array}{ccc} X \times F & \xrightarrow{\gamma(\alpha \times F)} & \tilde{F} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ X & \xrightarrow{\alpha} & \Omega \end{array}$$

is a pullback, that is if and only if $\alpha$ factors through $J$. Or equivalently if and only if $\sigma$ is $j$-dense. Therefore $j$ is the largest topology. $\square$

## References

[1] P. Johnstone, *Topos Theory* (LMS Mathematical Monographs No.10, Academic Press, New York, 1977).

[2] R. Paré, 'Indexed categories and generated topologies', *J. Pure Appl. Algebra* **19** (1980), 385–400.

[3] R. Paré and D. Schumacher, *Abstract Families and the Adjoint Functor Theorems*, Lecture Notes in Mathematics, 661 (Springer-Verlag, Berlin, Heidelberg, New York, 1978).

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