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NONDECREASABLE AND WEAKLY NONDECREASABLE DILATATIONS

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Abstract

Zhou *et al.* ['On weakly non-decreasable quasiconformal mappings', *J. Math. Anal. Appl.* **386** (2012), 842–847] proved that, in a Teichmüller equivalence class, there exists an extremal quasiconformal mapping with a weakly nondecreasable dilatation. They asked whether a weakly nondecreasable dilatation is a nondecreasable dilatation. The aim of this paper is to give a negative answer to their problem. We also construct a Teichmüller class such that it contains an infinite number of weakly nondecreasable extremal representatives, only one of which is nondecreasable.

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1. Introduction

Let S be a plane domain with at least two boundary points. The Teichmüller space $T(S)$ is the space of equivalence classes of quasiconformal maps f from S to a variable domain $f(S)$. Two quasiconformal maps f from S to $f(S)$ and g from S to $g(S)$ are said to be equivalent, denoted by $f \sim g$, if there is a conformal map c from $f(S)$ onto $g(S)$ and a homotopy through quasiconformal maps h_t mapping S onto $g(S)$ such that $h_0 = c \circ f$, $h_1 = g$ and $h_t(p) = c \circ f(p) = g(p)$ for every $t \in [0, 1]$ and every p in the boundary of S. Denote by $[f]$ the Teichmüller equivalence class of f; sometimes it is more convenient to use $[\mu]$ to express the equivalence class, where μ is the Beltrami differential (or the complex dilatation) of f.

Denote by Bel(S) the Banach space of Beltrami differentials $\mu = \mu(z)\, d\bar{z}/dz$ on S with finite L^∞-norm and by $M(S)$ the open unit ball in Bel(S).

For $\mu \in M(S)$, define

$$k_0([\mu]) = \inf\{\|\nu\|_\infty : \nu \in [\mu]\}.$$

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We say that μ is extremal in $[\mu]$ if $\|\mu\|_\infty = k_0([\mu])$ (the corresponding quasiconformal map f is said to be extremal for its boundary values as well) and uniquely extremal if $\|\nu\|_\infty > k_0(\mu)$ for any other $\nu \in [\mu]$.

For any μ, define $h^*(\mu)$ to be the infimum over all compact subsets E contained in S of the essential supremum norm of the Beltrami differential $\mu(z)$ as z varies over $S\setminus E$. Define $h([\mu])$ to be the infimum of $h^*(\nu)$ taken over all representatives ν of the class $[\mu]$. It is obvious that $h([\mu]) \le k_0([\mu])$. Following [1], $[\mu]$ is called a Strebel point if $h([\mu]) < k_0([\mu])$; otherwise, $[\mu]$ is called a non-Strebel point. It is well known that the set of Strebel points is open and dense in $T(S)$ [4].

The cotangent space to $T(S)$ at the basepoint is the Banach space $Q(S)$ of integrable holomorphic quadratic differentials on S with L^1-norm

$$\|\varphi\| = \iint_S |\varphi(z)|\,dx\,dy < \infty.$$

A Beltrami differential μ (not necessarily extremal) is called to be *nondecreasable* in its class $[\mu]$ if for $\nu \in [\mu]$,

$$|\nu(z)| \le |\mu(z)| \quad \text{almost everywhere in } S$$

implies that $\mu = \nu$; otherwise, μ is called to be *decreasable*.

The notion of nondecreasable dilatation was first introduced by Reich in [7] when he studied the unique extremality of quasiconformal mappings. A uniquely extremal Beltrami differential is obviously nondecreasable. Shen and Chen [11] proved the following theorem.

Theorem A. *Let Δ denote the unit disk $\{z \in \mathbb{C} : |z| < 1\}$. Let f be a quasiconformal mapping f from Δ onto itself. Unless $[f]$ contains a conformal mapping, there exist an infinite number of quasiconformal mapping g in the Teichmüller equivalence class $[f]$ (in the universal Teichmüller space $T(\Delta)$), each of which has a nondecreasable dilatation.*

Zhou and Chen [18] studied some special nondecreasable dilatations. The author [14] proved that a Teichmüller class may contain an infinite number of nondecreasable extremal dilatations. The existence of a nondecreasable extremal in a class is generally unknown.

In [19], Zhou *et al.* defined *weakly nondecreasable dilatation* as follows. Let $\mu \in M(S)$. μ is called a *strongly decreasable dilatation* if there exists $\nu \in [\mu]$ satisfying the following conditions:

(1) $|\nu(z)| \le |\mu(z)|$ for almost all $z \in S$; and
(2) there exists a domain $G \subset S$ and a positive number $\delta > 0$ such that

$$|\nu(z)| \le |\mu(z)| - \delta \quad \text{for almost all } z \in G.$$

Otherwise, μ is called *weakly nondecreasable*. In other words, a Beltrami differential μ is called weakly nondecreasable if either μ is nondecreasable or if μ is decreasable but is not strongly decreasable. In this paper, for the sake of mathematical precision, we call a Beltrami differential μ a *pseudo nondecreasable dilatation* if it is a weakly nondecreasable dilatation but not a nondecreasable dilatation. Hence, if μ is pseudo nondecreasable, then there is another $\nu \in [\mu]$ and $|\nu(z)| \leq |\mu(z)|$ for almost all $z \in S$ such that $|\nu(z) - \mu(z)| > 0$ on a subset $E \subset S$ of positive measure.

The main result in [19] is the following theorem.

THEOREM B. *For every extremal quasiconformal mapping f from Δ onto itself, there exists an extremal quasiconformal mapping g in the Teichmüller equivalence class $[f]$ with a weakly nondecreasable dilatation.*

At the end of the paper [19], they posed the following problem.

PROBLEM. Suppose that μ is a weakly nondecreasable dilatation. Is μ is necessarily a nondecreasable dilatation?

One might expect that the answer is positive for which the existence of a nondecreasable extremal dilatation in a Teichmüller class would follow from Theorem B.

The motivation of this paper is to show that the expectation is wrong. A lot of pseudo nondecreasable (even extremal) dilatations are constructed in some kinds of Teichmüller classes and hence the original problem has a negative answer, in general.

Without any loss of generality, we restrict the consideration on the universal Teichmüller space $T(\Delta)$. Our main results are the following theorems.

THEOREM 1.1. *For any given $\lambda > 0$, the basepoint $[0]$ contains an infinite number of pseudo nondecreasable dilatations ν such that $\|\nu\|_\infty = \lambda$ and the support set of each ν in Δ has empty interior. However, 0 is the unique nondecreasable dilatation in $[0]$.*

THEOREM 1.2. *There exists an extremal Beltrami differential $\mu \in M(\Delta)$ such that μ is the unique nondecreasable extremal dilatation in $[\mu]$, while $[\mu]$ contains an infinite number of pseudo nondecreasable extremal dilatations.*

THEOREM 1.3. *There exists a Beltrami differential $\mu \in M(\Delta)$ such that $[\mu]$ contains an infinite number of nondecreasable extremal dilatations and $[\mu]$ contains an infinite number of pseudo nondecreasable extremal dilatations.*

THEOREM 1.4. *There exists a uniquely extremal Beltrami differential $\mu \in M(\Delta)$ such that $[\mu]$ is a Strebel point in $T(\Delta)$ and contains an infinite number of pseudo nondecreasable dilatations.*

THEOREM 1.5. *There exists a uniquely extremal Beltrami differential $\mu \in M(\Delta)$ such that $[\mu]$ is a non-Strebel point in $T(\Delta)\backslash\{[0]\}$ and contains an infinite number of pseudo nondecreasable dilatations.*

By Theorem A, $[\mu]$ in Theorem 1.4 or Theorem 1.5 also contains an infinite number of nondecreasable dilatations. These theorems indicate that there is some distinction between nondecreasable dilatations and pseudo nondecreasable dilatations. The author believes that, in every Teichmüller class $[\mu]$, there are an infinite number of pseudo nondecreasable dilatations; moreover, if the extremal in $[\mu]$ is not unique, then $[\mu]$ contains an infinite number of pseudo nondecreasable extremal dilatations.

After some preparation in Section 2, we prove Theorems 1.1–1.4, in Section 3. The proof of Theorem 1.5 is relatively complicated and is given in Section 4 alone.

2. Some preparations

Given $\mu \in M(\Delta)$, let f^{μ} be the uniquely determined quasiconformal mapping of Δ onto itself with Beltrami coefficient μ and normalized to fix $1, -1$ and i.

Lemma 2.1. *Let $A \subset \Delta$ be a compact set such that $\Delta \backslash A$ is connected. Let μ and ν be two equivalent Beltrami coefficients in $M(\Delta)$. Let $\widetilde{\mu}$ and $\widetilde{\nu}$ be the Beltrami coefficients of the quasiconformal mappings $(f^{\mu})^{-1}$ and $(f^{\nu})^{-1}$, respectively. In addition, suppose that $\mu(z) = \nu(z)$ for almost every $z \in \Delta \backslash A$. Then $f^{\mu}(z) = f^{\nu}(z)$ for all z in $\Delta \backslash A$ and hence $\widetilde{\mu}(w) = \widetilde{\nu}(w)$ for almost all w in $f(\Delta \backslash A)$.*

Proof. For brevity, let $f = f^{\mu}$ and $g = f^{\nu}$. Let $\mu_{g \circ f^{-1}}(w)$ denote the Beltrami coefficient of $g \circ f^{-1}$. By a simple computation,

$$\mu_{g \circ f^{-1}} \circ f(z) = \frac{1}{\tau} \frac{\mu(z) - \nu(z)}{1 - \overline{\mu(z)}\nu(z)},$$

where $\tau = \overline{f_z}/f_z$.

Thus, $\mu_{g \circ f^{-1}}(w) = 0$ for almost all $w \in f(\Delta \backslash A)$ and hence $\Psi = g \circ f^{-1}$ is conformal on $\Delta \backslash A$. Since $\Psi|_{S^1} = g \circ f^{-1}|_{S^1} = \mathrm{id}$, we conclude that $\Psi = \mathrm{id}$ in $f(\Delta \backslash A)$. Thus, $g|_{\Delta \backslash A} = f|_{\Delta \backslash A}$. In addition, it is evident that $\widetilde{\mu}(w) = \widetilde{\nu}(w)$ for almost all w in $f(\Delta \backslash A)$. □

For $\mu \in L^{\infty}(\Delta)$, $\varphi \in Q(\Delta)$, let

$$\lambda_{\mu}[\varphi] = \mathrm{Re} \iint_{\Delta} \mu(z)\varphi(z)\, dx\, dy.$$

The following Construction theorem is essentially due to Reich [8] and is very useful for the study of (unique) extremality of quasiconformal mappings (see [8, 13–15]).

Construction Theorem. *Let A be a compact subset of Δ consisting of m ($m \in \mathbb{N}$) connected components and such that $\Delta \backslash A$ is connected and each connected component of A contains at least two points. There exists a function $\mathcal{A} \in L^{\infty}(\Delta)$ and a sequence $\varphi_n \in Q(\Delta)$ ($n = 1, 2, \ldots$) satisfying the conditions (2.1)–(2.4):*

$$|\mathcal{A}(z)| = \begin{cases} 0 & \text{if } z \in A, \\ 1 & \text{for almost all } z \in \Delta \backslash A, \end{cases} \tag{2.1}$$

$$\lim_{n\to\infty} \{\|\varphi_n\| - \lambda_{\mathcal{A}}[\varphi_n]\} = 0, \tag{2.2}$$

$$\lim_{n\to\infty} |\varphi_n(z)| = \infty \quad \text{almost everywhere in } \Delta \backslash A, \tag{2.3}$$

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and, as $n \to \infty$,

$$\varphi_n(z) \to 0 \quad \textit{uniformly on } A. \tag{2.4}$$

Proof. See the proof of the Construction theorem in [15]. □

From the Construction theorem and Lemma 2.1, we get the following lemma.

Lemma 2.2. *Let* A *be as in the Construction theorem and let* $\mathcal{A}(z)$ *be constructed by the Construction theorem. Let*

$$\nu(z) = \begin{cases} k\mathcal{A}(z) & \text{if } z \in \Delta \backslash A, \\ \mathcal{B}(z) & \text{if } z \in A, \end{cases}$$

where $k < 1$ *is a positive constant and* $\mathcal{B}(z) \in L^\infty(A)$ *with* $\|\mathcal{B}\|_\infty \le k$. *Then* $\nu(z)$ *is extremal in* $[\nu]$ *and, for any* $\chi(z)$ *extremal in* $[\nu]$, $\chi(z) = \nu(z)$ *for almost all* z *in* $\Delta \backslash A$.

Proof. See the proof of [15, Lemma 5]. □

Lemma 2.3. *Let* $J_i \subset \Delta$ $(i = 1, 2, \ldots, m)$ *be* m *Jordan domains such that* $\overline{J_i} \subset \Delta$, $\overline{J_i}$ $(i = 1, 2, \ldots, m)$ *are mutually disjoint and* $\Delta \backslash \bigcup_1^m \overline{J_i}$ *is connected. Put* $A = \bigcup_1^m \overline{J_i}$. *Let* $\mathcal{A}(z)$ *be constructed by the Construction theorem. Let*

$$\nu(z) = \begin{cases} k\mathcal{A}(z) & \text{if } z \in \Delta \backslash A, \\ \mathcal{B}(z) & \text{if } z \in A, \end{cases}$$

where $k < 1$ *is a positive constant and* $\mathcal{B}(z) \in L^\infty(A)$ *with* $\|\mathcal{B}\|_\infty \le k$. *We regard* $[\nu|_{J_i}]$ *as a point in the Teichmüller space* $T(J_i)$, $i = 1, 2, \ldots, m$. *Then:*

(a) ν *is a weakly nondecreasable dilatation in* $[\nu]$ *if and only if every* $\nu|_{J_i}$ *is a weakly nondecreasable dilatation in* $[\nu|_{J_i}]$, $i = 1, 2, \ldots, m$*; and*

(b) ν *is a nondecreasable dilatation in* $[\nu]$ *if and only if every* $\nu|_{J_i}$ *is a nondecreasable dilatation in* $[\nu|_{J_i}]$, $i = 1, 2, \ldots, m$.

Proof. It is evident that ν is extremal in $[\nu]$, by Lemma 2.2.

(a) Again, the ‘only if’ part is obvious. Now, assume that every $\nu|_{J_i}$ is a weakly nondecreasable dilatation in $[\nu|_{J_i}]$, $i = 1, 2, \ldots, m$. We show that ν is a weakly nondecreasable dilatation in $[\nu]$. Suppose to the contrary. Then $[\nu]$ is a strongly decreasable dilatation in $[\nu]$. That is, there exists a Beltrami differential $\eta \in [\nu]$ such that:

(1) $|\eta(z)| \le |\nu(z)|$ for almost all $z \in \Delta$; and

(2) there exists a domain $G \subset \Delta$ and a positive number $\delta > 0$ such that

$$|\eta(z)| \le |\nu(z)| - \delta \quad \text{for almost all } z \in G.$$

Observe that η is extremal in $[\nu]$ and hence $\eta(z) = \nu(z)$ almost everywhere on $\Delta \backslash A$, by Lemma 2.2. G is forced to be contained in some J_i. Furthermore, by Lemma 2.1, $\eta|_{J_i} \in [\nu|_{J_i}]$. Thus $\nu|_{J_i}$ is a strongly decreasable dilatation in $[\nu|_{J_i}]$, which is a contradiction.

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(b) The 'only if' part is also obvious. Assume that every $\nu|_{J_i}$ is a nondecreasable dilatation in $[\nu|_{J_i}]$, $i = 1, 2, \ldots, m$. We show that ν is a nondecreasable dilatation in $[\nu]$. Suppose to the contrary. Then $[\nu]$ is a decreasable dilatation in $[\mu]$. There exists a Beltrami differential $\eta \in [\nu]$ such that $|\eta(z)| \le |\nu(z)|$ for almost all $z \in \Delta$, but $\eta(z) \ne \nu(z)$ on a subset $E \subset \Delta$ with positive measure. It causes no harm in assuming that $E \cap J_1$ has positive measure. Since $\eta(z) = \nu(z)$ almost everywhere on $\Delta \backslash A$, it follows, from Lemma 2.1, that $\eta|_{J_1} \in [\nu|_{J_1}]$. Thus, $\nu|_{J_1}$ is decreasable in $[\nu|_{J_1}]$, which is a contradiction. □

Lemma 2.4. *Let $J_i \subset \Delta$ ($i = 1, 2, \ldots, m$) be m Jordan domains such that $\overline{J_i} \subset \Delta$, $\overline{J_i}$ ($i = 1, 2, \ldots, m$) are mutually disjoint and $\Delta \backslash \bigcup_1^m \overline{J_i}$ is connected. Put $A = \bigcup_1^m \overline{J_i}$. Let $\nu \in M(\Delta)$ whose support set is contained in A. We regard $[\nu|_{J_i}]$ as a point in the Teichmüller space $T(J_i)$, $i = 1, 2, \ldots, m$. Then:*

(a) *ν is a weakly nondecreasable dilatation in $[\nu]$ if and only if every $\nu|_{J_i}$ is a weakly nondecreasable dilatation in $[\nu|_{J_i}]$, $i = 1, 2, \ldots, m$; and*
(b) *ν is a nondecreasable dilatation in $[\nu]$ if and only if every $\nu|_{J_i}$ is a nondecreasable dilatation in $[\nu|_{J_i}]$, $i = 1, 2, \ldots, m$.*

Proof. Since $\nu(z) = 0$ on $\Delta \backslash A$, by Lemma 2.1, the proof follows almost the same argument as that of Lemma 2.3. □

The following is [14, Lemma 5].

Lemma 2.5. *Set $\Delta_s = \{z : |z| < s\}$ for $s \in (0, 1)$. Let $\chi(z)$ be defined as*

$$\chi(z) = \begin{cases} 0 & \text{if } z \in \Delta - \Delta_s, \\ \widetilde{k} & \text{if } z \in \Delta_s, \end{cases}$$

where $\widetilde{k} < 1$ is a positive constant. Then $[\chi]$ contains an infinite number of nondecreasable Beltrami differentials η with $\|\eta\|_\infty < \widetilde{k}$.

3. Proofs of main results

Proof of Theorem 1.1. Let $\mathscr{C} \subset [0, 1]$ be a compact set with empty interior and positive measure. Denote by $F[\mathscr{C}]$ the collection of nonnegative measurable functions γ in $L^\infty[0, 1]$ satisfying the following conditions:

(a) $\gamma(x) \equiv 1$ for $x \in [0, 1] \backslash \mathscr{C}$;
(b) $\operatorname{essinf}_{x \in [0,1]} \gamma(x) = \rho > 0$; and
(c) $\int_0^1 \gamma(x)\, dx = 1$.

For any given $\gamma \in F[\mathscr{C}]$, define

$$\Gamma(x) = \int_0^x \gamma(t)\, dt, \qquad x \in [0, 1].$$

Then $\Gamma(x)$ is differentiable at almost every $x \in [0, 1]$. In fact, $\Gamma'(x) = \gamma(x)$ for almost all $x \in [0, 1]$.

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Let

$$\mathscr{S} = \{z = x + iy \in \mathbb{C} : x \in [0, 1], y \in \mathbb{R}\},$$
$$\mathscr{D} = \{z = x + iy \in \mathbb{C} : x \in [0, 1]\backslash\mathscr{C}, y \in \mathbb{R}\},$$
$$\mathscr{D}^c = \{z = x + iy \in \mathbb{C} : x \in \mathscr{C}, y \in \mathbb{R}\},$$

and define a mapping σ from the strip $\mathscr{S}$ onto itself by

$$\sigma(z) = \Gamma(x) + iy, \quad z = x + iy \in \overline{\mathscr{S}}.$$

Observe that $\Gamma(x)$ is a strictly increasing function with respect to $x \in [0, 1]$, $\Gamma(0) = 0$ and $\Gamma(1) = 1$. It is clear that σ is a self-homeomorphism of $\overline{\mathscr{S}}$ and keeps the boundary points fixed. Moreover, $\sigma(z)$ is differentiable at almost every $z = x + iy \in \mathscr{S}$. Precisely, for almost every $x \in (0, 1)$,

$$\begin{aligned} \partial_z \sigma(x + iy) &= \frac{\gamma(x) + 1}{2}, \\ \partial_{\bar{z}} \sigma(x + iy) &= \frac{\gamma(x) - 1}{2}. \end{aligned} \tag{3.1}$$

Let μ_σ denote the Beltrami differential of σ. Then

$$\mu_\sigma(z) = \frac{\partial_{\bar{z}}\sigma}{\partial_z\sigma} = \frac{\gamma(x) - 1}{\gamma(x) + 1}, \quad z = x + iy \in \mathscr{S}.$$

It is evident that both $\partial_z\sigma$ and $\partial_{\bar{z}}\sigma$ are locally L^2-integrable on $\mathscr{S}$. On the other hand, the conditions (a)–(c) imply that $\|\gamma\|_\infty \geq 1$ and $\rho \leq 1$. It is easy to verify that

$$\|\mu_\sigma\|_\infty = \max\left\{\frac{\|\gamma\|_\infty - 1}{\|\gamma\|_\infty + 1}, \frac{1 - \rho}{1 + \rho}\right\}.$$

Therefore, the homeomorphism σ is a generalized L^2-solution of the Beltrami equation

$$\partial_{\bar{z}} w = \mu_\sigma(z)\partial_z w.$$

By the classical characterization of quasiconformal mappings in the plane [5], we see that σ is a quasiconformal mapping.

Let $\Phi : \Delta \to \mathscr{S}$ be a conformal mapping from Δ onto $\mathscr{S}$. Then $f_\sigma = \Phi^{-1} \circ \sigma \circ \Phi$ is a quasiconformal mapping from Δ onto itself. By the analysis above, f_σ keeps the boundary points fixed. Let μ_{f_σ} and μ_σ denote the Beltrami differential of f_σ and σ, respectively. Now $\mu_{f_\sigma} \in [0]$. A simple computation shows that

$$\mu_{f_\sigma}(z) = \mu_\sigma(\Phi(z))\frac{\overline{\Phi'^2(z)}}{|\Phi'^2(z)|}. \tag{3.2}$$

By (3.1) and the assumption that $\gamma(x) = 1$ on $[0, 1]\backslash\mathscr{C}$,

$$\mu_\sigma(z) = 0, \quad z \in \mathscr{D}.$$

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Thus, when $z \in \Phi^{-1}(\mathscr{D})$, it follows, from (3.2), that $\mu_{f_\sigma}(z) = 0$. Since $\mathscr{D}^c$ has empty interior, so does $\Phi^{-1}(\mathscr{D}^c)$.

Claim. μ_{f_σ} is a weakly nondecreasable dilatation in [0].

Suppose to the contrary. Then μ_{f_σ} is a strongly decreasable dilatation. By the definition, there exists a Beltrami differential $\nu \in [0]$ such that:

(1) $|\nu(z)| \le |\mu(z)|$ for almost all $z \in \Delta$; and

(2) there exists a domain $G \subset \Delta$ and a positive number $\delta > 0$ such that

$$|\nu(z)| \le |\mu(z)| - \delta \quad \text{for almost all } z \in G.$$

Observe that the support set of μ_{f_σ} in Δ is contained in $\Phi^{-1}(\mathscr{D}^c)$, which has empty interior. We find that the condition (2) can not be satisfied. The claim is proved.

When σ varies over $F[\mathscr{C}]$, μ_{f_σ} will be mutually different in [0]. So, if we let γ vary over $F[\mathscr{C}]$ such that

$$\|\mu_\sigma\|_\infty = \max\left\{\frac{\|\gamma\|_\infty - 1}{\|\gamma\|_\infty + 1}, \frac{1-\rho}{1+\rho}\right\} = \lambda,$$

then we get an infinite number of weakly nondecreasable dilatations ν in $[\mu]$ with $\|\nu\|_\infty = \lambda$. It is clear that the support set of each ν in Δ has empty interior.

By the definition, it is obvious that 0 is the unique nondecreasable dilatation in [0] and hence any other μ_{f_σ} is a pseudo nondecreasable dilatation. This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. Choose $J = \{z \in \Delta : |z| < \frac{1}{2}\}$ and $A = \overline{J}$. Let $\mathcal{A}(z)$ be constructed by the Construction theorem and let $\mu(z) = k\mathcal{A}(z)$, where $k < 1$ is a positive constant. We now show that μ is the unique nondecreasable extremal dilatation in $[\mu]$ while $[\mu]$ contains an infinite number of pseudo nondecreasable extremal dilatations.

Note that $\mu|_J \equiv 0$ on J. $\mu|_J$ is obviously the unique nondecreasable dilatation in $[0|_J]$ (in $T(J)$). Hence, by Lemma 2.3, μ is the unique nondecreasable extremal dilatation in $[\mu]$. Also, by Lemma 2.3, if $\nu \in [\mu]$ is extremal, then it is a weakly nondecreasable dilatation if and only if $\nu|_J$ is a weakly nondecreasable dilatation in $[0|_J]$ (in $T(J)$). If we get an infinite number of pseudo nondecreasable dilatations in $[0|_J]$ with L^∞-norm of at most k, then we can get an infinite number of pseudo nondecreasable extremal dilatations in $[\mu]$. For this, we transfer Theorem 1.1 from Δ to J.

Let σ and f_σ be as in the proof of Theorem 1.1. Define the conformal mapping $\Psi : J \to \Delta$ by $w = 2z$, $z \in J$. Then $\widetilde{f_\sigma} = \Psi^{-1} \circ f_\sigma \circ \Psi$ is a quasiconformal mapping from J onto itself. By the assumption, $\widetilde{f_\sigma}$ keeps the boundary points of J fixed. Let $\widetilde{\mu}_\sigma$ be the Beltrami differential of $\widetilde{f_\sigma}$. Then $\widetilde{\mu}_\sigma \in [0|_J]$. A simple computation shows that

$$\widetilde{\mu}_\sigma(z) = \mu_\sigma(2z)\frac{\overline{\Psi'^2(z)}}{|\Psi'^2(z)|} = \mu_\sigma(2z).$$

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It is clear that $\widetilde{\mu}_\sigma$ is a weakly nondecreasable dilatation in $[0|_J]$. Since $\|\widetilde{\mu}_\sigma\|_\infty = \|\mu_\sigma\|_\infty$, by Lemma 2.3, the following Beltrami differential

$$\nu(z) = \begin{cases} \mu(z) & \text{if } z \in \Delta \backslash \overline{J}, \\ \widetilde{\mu}_\sigma(z) & \text{if } z \in J, \end{cases} \tag{3.3}$$

is a weakly nondecreasable extremal dilatation in $[\mu]$ if $\|\widetilde{\mu}_\sigma\|_\infty \le k$. By Theorem 1.1, we can find an infinite number of pseudo nondecreasable extremal dilatations $\widetilde{\mu}_\sigma$ with $\|\widetilde{\mu}_\sigma\|_\infty \le k$. Thus, we get an infinite number of pseudo nondecreasable extremal dilatations ν, defined by (3.3), in $[\mu]$. The concludes the proof of Theorem 1.2.

Proof of Theorem 1.3. Choose $J_1 = \{z \in \Delta : |z| < \frac{1}{4}\}$ and $J_2 = \{z \in \Delta : |z - \frac{1}{2}| < \frac{1}{8}\}$. Let $A = \overline{J_1} \cup \overline{J_2}$. Let $\mathcal{A}(z)$ be constructed by the Construction theorem and let $\mu(z) = k\mathcal{A}(z)$, where $k < 1$ is a positive constant. Let $\Delta_s = \{z \in \Delta : |z| < s\}$, where $s \in (0, \frac{1}{4})$ and let $\widetilde{k} \in (0, k]$ be a constant. Set

$$\mu(z) = \begin{cases} k\mathcal{A}(z) & \text{if } z \in \Delta \backslash A, \\ \widetilde{k} & \text{if } z \in \Delta_s, \\ 0 & \text{if } z \in J_1 - \Delta_s, \\ 0 & \text{if } z \in J_2. \end{cases}$$

By Theorem 1.1, $[0|_{J_2}]$ contains an infinite number of pseudo nondecreasable dilatations and $0|_{J_2}$ is the unique nondecreasable dilatation in $[0|_{J_2}]$. Applying Lemma 2.5 to J_1, we see that $[0|_{J_1}]$ contains an infinite number of nondecreasable dilatations with L^∞-norm of at most k. Thus, by Lemma 2.3, $[\mu]$ is the desired Teichmüller class. The gives Theorem 1.3.

Proof of Theorem 1.4. Let $J = \{z \in \Delta : |z| < \frac{1}{2}\}$ and $A = \overline{J}$. Choose $\chi \in M(\Delta)$ such that the support set of χ is contained in A. Assume that $\chi|_J$ is a weakly nondecreasable dilatation in $[\chi|_J]$ (in $T(J)$) and $[\chi|_J] \ne [0|_J]$, where we do not prescribe $\chi|_J$ to be nondecreasable or pseudo nondecreasable. By Lemma 2.1, $[\chi] \ne [0]$ in $T(\Delta)$ and then $k_0([\chi]) > 0$. Since the boundary dilatation $h([\chi]) = 0$, $[\chi]$ is a Strebel point in $T(\Delta)$. By Strebel's frame mapping theorem [12], $[\chi]$ can be represented by the uniquely extremal Beltrami differential μ of the form $\mu = k_0([\chi])(\overline{\varphi}/|\varphi|)$, where $\varphi \in Q(\Delta)$, $\varphi \ne 0$.

Claim. The Strebel point $[\mu]$ $(= [\chi])$ satisfies the theorem.

Applying Lemma 2.4, we see that χ is a weakly nondecreasable dilatation in $[\mu]$.

Let $D = \{z \in \Delta : |z - \frac{3}{4}| < \frac{1}{8}\}$. Regard $[0|_D]$ as the basepoint in the Teichmüller space $T(D)$. Let η be a weakly nondecreasable dilatation in $[0|_D]$. Put

$$\nu(z) = \begin{cases} 0 & \text{if } z \in \Delta \backslash (\overline{J} \cup \overline{D}), \\ \chi(z) & \text{if } z \in J, \\ \eta(z) & \text{if } z \in D. \end{cases}$$

Then $\nu \in [\mu]$ and, by Lemma 2.4, ν is a weakly nondecreasable dilatation in $[\mu]$. If either $\chi|_J$ or $\eta|_D$ is pseudo nondecreasable in their classes $[\chi|_J]$ or $[0|_D]$, then ν is pseudo nondecreasable in $[\mu]$. Because $[0|_D]$ contains an infinite number of pseudo nondecreasable dilatations, due to Theorem 1.1, $[\mu]$ contains an infinite number of pseudo nondecreasable dilatations. Now the proof of Theorem 1.4 is complete.

4. Proof of Theorem 1.5

The proof of Theorem 1.5 is somewhat lengthy and complicated. We need some new lemmas before proving it.

The Reich–Strebel inequality, also called main inequality (see [3, 9, 10]), plays an important role in the study of Teichmüller theory. To introduce the inequality, we need some notation. Suppose that f and g are two quasiconformal mappings of Δ onto itself with the Beltrami differentials μ, ν, respectively. Let $F = f^{-1}, G = g^{-1}$ and $\widetilde{\mu}, \widetilde{\nu}$ denote the Beltrami differentials of F, G, respectively. Put $\alpha = \widetilde{\mu} \circ f, \beta = \widetilde{\nu} \circ f$. Then

Main inequality. If $[\mu] = [\nu]$, that is, if f and g are equivalent, then, for any $\varphi \in Q(\Delta)$,

$$\iint_\Delta \varphi \, dx\, dy \le \iint_\Delta |\varphi(z)| \frac{|1 - \mu(z)((\varphi(z))/|\varphi(z)|)|^2}{1 - |\mu(z)|^2} \times \frac{|1 + \beta(\mu/\alpha)((1 - \overline{\mu}((\overline{\varphi})/|\varphi|))/(1 - \mu((\varphi)/|\varphi|)))|^2}{1 - |\beta|^2} \, dx\, dy,$$

or, equivalently (see [7]),

$$\mathrm{Re} \iint_\Delta \frac{(\beta - \alpha)(1 - \alpha\overline{\beta})\tau}{(1 - |\alpha|^2)(1 - |\beta|^2)} \varphi \, dx\, dy \le \iint_\Delta \frac{|\alpha - \beta|^2}{(1 - |\alpha|^2)(1 - |\beta|^2)} |\varphi| \, dx\, dy,$$

where $\tau = \overline{\partial_z f}/\partial_z f = -\mu/\alpha$.

LEMMA 4.1. *Let A be as in the Construction theorem and let $\mathcal{A}(z)$ be constructed by the Construction theorem. Let*

$$\mu(z) = \begin{cases} k\mathcal{A}(z) & \text{if } z \in \Delta \backslash A, \\ \mathcal{B}(z) & \text{if } z \in A, \end{cases}$$

where $k \in (0, 1)$ is a constant and $\mathcal{B}(z) \in L^\infty(A)$ with $\|\mathcal{B}\|_\infty < 1$. Suppose that $\nu \in [\mu]$ and that $|\nu(z)| \le k$ almost everywhere on $\Delta \backslash A$. Then $\nu(z) = \mu(z)$ for almost all z in $\Delta \backslash A$.

PROOF. Let $\{\varphi_n\}$ be the sequence associated with $\mathcal{A}(z)$ obtained by the Construction theorem. Using the above notation and applying the main inequality,

$$-\iint_\Delta \frac{|\alpha - \beta|^2}{(1 - |\alpha|^2)(1 - |\beta|^2)} |\varphi_n| \, dx\, dy \le -\mathrm{Re} \iint_\Delta \frac{(\beta - \alpha)(1 - \alpha\overline{\beta})\tau}{(1 - |\alpha|^2)(1 - |\beta|^2)} \varphi_n \, dx\, dy.$$

Let $\Lambda = \Delta \backslash A$. By the condition (2.4), we see that

$$-\iint_\Lambda \frac{|\alpha - \beta|^2}{(1 - |\alpha|^2)(1 - |\beta|^2)} |\varphi_n| \, dx\, dy \le -\mathrm{Re} \iint_\Lambda \frac{(\beta - \alpha)(1 - \alpha\overline{\beta})\tau}{(1 - |\alpha|^2)(1 - |\beta|^2)} \varphi_n \, dx\, dy + \epsilon_n, \tag{4.1}$$

where $\lim_{n\to\infty} \epsilon_n = 0$.

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We add

$$\operatorname{Re}\iint_\Lambda \frac{(\alpha-\beta)(1-\alpha\overline{\beta})}{(1-|\alpha|^2)(1-|\beta|^2)}\frac{|\alpha|}{\alpha}|\varphi_n|\,dx\,dy$$

to both sides of (4.1) and get

$$\operatorname{Re}\iint_\Lambda \frac{(\alpha-\beta)(1-\alpha\overline{\beta})}{(1-|\alpha|^2)(1-|\beta|^2)}\frac{|\alpha|}{\alpha}|\varphi_n|\,dx\,dy-\iint_\Lambda \frac{|\alpha-\beta|^2}{(1-|\alpha|^2)(1-|\beta|^2)}|\varphi_n|\,dx\,dy$$
$$\leq \operatorname{Re}\iint_\Lambda \frac{(\alpha-\beta)(1-\alpha\overline{\beta})}{(1-|\alpha|^2)(1-|\beta|^2)}\frac{1}{\alpha}(|\mu||\varphi_n|-\mu\varphi_n)\,dx\,dy+\epsilon_n.$$

By a deformation,

$$\iint_\Lambda \frac{(1-|\alpha|)|\alpha-\beta|^2+(1+|\alpha|)(|\alpha|^2-|\beta|^2)}{2|\alpha|(1+|\alpha|)(1-|\beta|^2)}|\varphi_n|\,dx\,dy$$
$$\leq \operatorname{Re}\iint_\Lambda \frac{(\alpha-\beta)(1-\alpha\overline{\beta})}{(1-|\alpha|^2)(1-|\beta|^2)}\frac{1}{\alpha}(|\mu||\varphi_n|-\mu\varphi_n)\,dx\,dy+\epsilon_n. \tag{4.2}$$

Then

$$\iint_\Lambda \frac{(1-|\alpha|)|\alpha-\beta|^2}{2|\alpha|(1+|\alpha|)(1-|\beta|^2)}|\varphi_n|\,dx\,dy$$
$$\leq \operatorname{Re}\iint_\Lambda \frac{(\alpha-\beta)(1-\alpha\overline{\beta})}{(1-|\alpha|^2)(1-|\beta|^2)}\frac{1}{\alpha}(|\mu||\varphi_n|-\mu\varphi_n)\,dx\,dy+\epsilon_n.$$

Since $|\beta(z)|\leq k=|\alpha(z)|$ almost everywhere on Λ, one finds that a lower bound on the coefficient of $|\varphi|$ on the left-hand side of (4.2) is

$$\frac{(1-|\alpha|)|\alpha-\beta|^2}{2|\alpha|(1+|\alpha|)(1-|\beta|^2)}\geq\frac{1-k}{2k(1+k)}|\alpha-\beta|^2.$$

An upper bound for the integrand on the right-hand side of (4.2) is

$$|\alpha-\beta|\frac{1+|\alpha|^2}{(1-|\alpha|^2)(1-|\beta|^2)}\frac{1}{|\alpha|}\big||\mu|\cdot|\varphi_n|-\mu\varphi_n\big|$$
$$\leq\frac{1+k^2}{k(1-k^2)^2}|\alpha-\beta|\cdot\big||\mu|\cdot|\varphi_n|-\mu\varphi_n\big|.$$

Therefore, by the identity

$$\big||w|-w\big|^2=2|w|(|w|-\operatorname{Re}w),$$

$$\iint_\Lambda |\alpha-\beta|^2|\varphi|\,dx\,dy\leq\frac{2(1+k^2)}{(1+k)(1-k)^3}\iint_\Lambda |\alpha-\beta|\cdot\big||\mu|\cdot|\varphi_n|-\mu\varphi_n\big|\,dx\,dy+\epsilon_n$$
$$=C'\iint_\Lambda |\alpha-\beta||\varphi|^{1/2}[|\mu|\cdot|\varphi_n|-\operatorname{Re}(\mu\varphi_n)]^{1/2}\,dx\,dy+\epsilon_n,$$

where $C'=C'(k)=2\sqrt{2k}(1+k^2)/[(1+k)(1-k)^3]$.

Applying Schwarz's inequality, we get

$$\iint_\Lambda |\alpha-\beta|^2|\varphi_n|\,dx\,dy \leq C'\sqrt{\iint_\Lambda |\alpha-\beta|^2|\varphi_n|\,dx\,dy}\cdot\sqrt{\iint_\Lambda [|\mu|\cdot|\varphi_n| - \mathrm{Re}\,(\mu\varphi_n)]\,dx\,dy + \epsilon_n}.$$

The condition (2.2) implies that

$$\lim_{n\to\infty}\iint_\Lambda [|\mu|\cdot|\varphi_n| - \mathrm{Re}\,(\mu\varphi_n)]\,dx\,dy = 0.$$

Hence

$$\lim_{n\to\infty}\iint_\Lambda |\alpha-\beta|^2|\varphi_n|\,dx\,dy = 0.$$

Furthermore, by the condition (2.3) and Fatou's Lemma, we obtain $\alpha(z)=\beta(z)$ almost everywhere on Λ, that is, $\widetilde{\mu}(w)=\widetilde{\nu}(w)$ almost everywhere on $f(\Lambda)$. Applying Lemma 2.1 in an opposite direction, we then get $\mu(z)=\nu(z)$ almost everywhere on Λ. The gives the lemma. □

Lemma 4.2. *Let $J_i\subset\Delta$ $(i=1,2,\ldots,m)$ be m Jordan domains such that $\overline{J_i}\subset\Delta$, $\overline{J_i}$ $(i=1,2,\ldots,m)$ are mutually disjoint and $\Delta\backslash\bigcup_1^m\overline{J_i}$ is connected. Put $A=\bigcup_1^m\overline{J_i}$. Let $\mathcal{A}(z)$ be constructed by the Construction theorem. Let*

$$\nu(z)=\begin{cases} k\mathcal{A}(z) & \text{if } z\in\Delta\backslash A,\\ \mathcal{B}(z) & \text{if } z\in A,\end{cases}$$

where $k<1$ is a positive constant and $\mathcal{B}(z)\in L^\infty(A)$ with $\|\mathcal{B}\|_\infty<1$. We regard $[\nu|_{J_i}]$ as a point in the Teichmüller space $T(J_i)$, $i=1,2,\ldots,m$. Then:

(a) *ν is a weakly nondecreasable dilatation in $[\nu]$ if and only if every $\nu|_{J_i}$ is a weakly nondecreasable dilatation in $[\nu|_{J_i}]$, $i=1,2,\ldots,m$; and*
(b) *ν is a nondecreasable dilatation in $[\nu]$ if and only if every $\nu|_{J_i}$ is a nondecreasable dilatation in $[\nu|_{J_i}]$, $i=1,2,\ldots,m$.*

Proof. (a) The 'only if' part is obvious. Now assume that every $\nu|_{J_i}$ is a weakly nondecreasable dilatation in $[\nu|_{J_i}]$, $i=1,2,\ldots,m$. We show that ν is a weakly nondecreasable dilatation in $[\nu]$. Suppose to the contrary. Then $[\nu]$ is a strongly decreasable dilatation in $[\nu]$. That is, there exists a Beltrami differential $\eta\in[\nu]$ such that:

(1) $|\eta(z)|\leq|\nu(z)|$ for almost all $z\in\Delta$; and
(2) there exists a domain $G\subset\Delta$ and a positive number $\delta>0$ such that

$$|\eta(z)|\leq|\nu(z)|-\delta \quad \text{for almost all } z\in G.$$

Observe that $|\eta(z)| \leq |\nu(z)| = k$ almost everywhere on $\Delta\backslash A$. It holds that $\eta(z) = \nu(z)$ almost everywhere on $\Delta\backslash A$, by Lemma 4.1. So G is contained in some J_i. Furthermore, by Lemma 2.1, $\eta|_{J_i} \in [\nu|_{J_i}]$. Thus $\nu|_{J_i}$ is a strongly decreasable dilatation in $[\nu|_{J_i}]$, which is a contradiction.

(b) With the help of Lemma 4.1, the proof is essentially the same as that of Case (b) of Lemma 2.3. □

Proof of Theorem 1.5. Let $\Delta_s = \{z \in \Delta : |z| < s\}$, $s \in (0, 1)$ and $A = \overline{\Delta_s}$. Let $\mathcal{A}(z)$ be constructed by the Construction theorem. By Theorem 1.4, there is a Strebel point $[\mathcal{B}]$ in $T(\Delta_s)$ such that $[\mathcal{B}]$ contains an infinite number of pseudo nondecreasable dilatations. Let $k = k_0([\mathcal{B}])$. It is convenient to assume that $\mathcal{B} = k(\overline{\varphi}/|\varphi|)$ is the uniquely determined extremal in $[\mathcal{B}]$, where $\varphi \in Q(\Delta_s)$. Put

$$\mu(z) = \begin{cases} k\mathcal{A}(z) & \text{if } z \in \Delta\backslash A, \\ k\dfrac{\overline{\varphi(z)}}{|\varphi(z)|} & \text{if } z \in A. \end{cases}$$

By [13, Lemma 4], we see that μ is uniquely extremal in $[\mu]$. Since $[\mathcal{B}]$ contains an infinite number of pseudo nondecreasable dilatations on Δ_s, by Lemma 4.2, we conclude that $[\mu]$ contains an infinite number of pseudo nondecreasable dilatations on Δ. The proof of Theorem 1.5 is complete.

5. Concluding remarks

A Beltrami differential μ (not necessarily extremal) in $\mathrm{Bel}(S)$ is said to be of *landslide type* if there exists a nonempty open subset $G \subset S$ such that

$$\operatorname*{esssup}_{z\in G} |\mu(z)| < \|\mu\|_\infty.$$

Otherwise, μ is said to be of *nonlandslide type*.

The concept of nonlandslide was first introduced by Li in [6] for extremal Beltrami differentials. Here, we generalize the definition for general Beltrami differentials. It was proved by Fan [2] and the author [16] independently that if μ contains more than one extremal, then it contains an infinite number of extremals of nonlandslide type.

The proofs of Theorems 1.2–1.5 depend extremely on Theorem 1.1. Taking a look at the pseudo nondecreasable dilatations constructed for Theorems 1.1–1.5. We find that they vanish on certain subdomains in Δ and hence all of them are of landslide type. Recently, the author constructed certain nondecreasable nonuniquely extremal dilatation of nonlandslide type by use of the main inequality in [17]. Naturally, we ask the following question.

Question 1. Is there a pseudo nondecreasable extremal dilatation of nonlandslide type?

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It is an open problem whether there exists $[\mu]$ such that the extremal in $[\mu]$ is not unique and each extremal in $[\mu]$ is nonlandslide. If such a class $[\mu]$ exists, then each extremal in $[\mu]$ is also weakly nondecreasable.

Note that, by definition, a pseudo nondecreasable extremal dilatation is necessarily nonuniquely extremal. It should also be pointed out that a nondecreasable dilatation is certainly uniquely extremal if it has a constant modulus, and a pseudo nondecreasable dilatation is certainly extremal if it has a constant modulus. However, it cannot be inferred that a pseudo nondecreasable extremal dilatation cannot have a constant modulus. The author even expects to discover a pseudo nondecreasable extremal dilatation μ of constant modulus; of course, this will lead to a strongly positive answer to Question 1. Moreover, if so, every extremal in $[\mu]$ will be nonlandslide.

It is surprising that the 'extremal' condition in Question 1 is not essential for the difficulty. With the help of the Construction theorem and Lemma 2.3, one can show that Question 1 is equivalent to a seemingly weaker question.

QUESTION 2. Is there a pseudo nondecreasable dilatation of nonlandslide type?

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