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ACTIONS OF FINITE GROUPS ON $\mathbf{R}^{n+k}$ WITH FIXED SET $\mathbf{R}^k$

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In this paper we study the existence problem for topological actions of finite groups on euclidean spaces $\mathbf{R}^{n+k}$ which are free outside a fixed point set $\mathbf{R}^k$ (embedded as a vector subspace). We refer to such an action as a *semi-free action on* $(\mathbf{R}^{n+k}, \mathbf{R}^k)$ and note that all our actions will be assumed orientation-preserving.

Suppose the finite group π acts semi-freely on $(\mathbf{R}^{n+k}, \mathbf{R}^k)$, then it acts freely on $\mathbf{R}^{n+k} - \mathbf{R}^k = S^{n-1} \times \mathbf{R}^{k+1}$. Since this space is homotopy equivalent to S^{n-1}, π will have periodic integral cohomology and n will be a multiple of the period. In fact the orbit space

$$W = (\mathbf{R}^{n+k} - \mathbf{R}^k)/\pi$$

is a finitely-dominated Poincaré complex of formal dimension $n - 1$ with $\pi_1 W = \pi$ and $\widetilde{W} \simeq S^{n-1}$ as considered by Swan [**41**]. We call such spaces *Swan complexes* for short and refer to the groups with periodic cohomology as $\mathscr{P}$-groups.

Some $\mathscr{P}$-groups admit complex representations V such that no element g has $+1$ as an eigenvalue; then π acts semi-freely on $(V, 0)$. Such representations will be called $\mathscr{F}$-representations and the actions they provide on $(V \times \mathbf{R}^k, \mathbf{R}^k)$ *linear actions*. Any semi-free action on $(\mathbf{R}^{n+k}, \mathbf{R}^k)$ which can be smoothed is conjugate to the linear action provided by the differential at 0.

Other examples can be obtained from the solution of the spherical space form problem by Madsen, Thomas and Wall [**27**, **28**]. Start with a free action of π on S^{n-1} and extend it to a semi-free action on $(\mathbf{R}^n, 0)$ by regarding $\mathbf{R}^n$ as $S^{n-1} \times \mathbf{R} \cup \{0\}$. This gives many examples of non-linear actions.

Our main results, Theorems A, B and C below, give necessary and sufficient conditions for a finite group π to have a semi-free action on $(\mathbf{R}^{n+k}, \mathbf{R}^k)$ when $n \geqq 6$. Strangely enough the conditions depend only on n and not on $k \geqq 0$. This however does not imply that every such semi-free action can be desuspended to an action on $(\mathbf{R}^n, 0)$. Indeed we give counterexamples to such a statement in Section 3.

If n is divisible by twice the cohomological period of π then it acts

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semi-freely on $(\mathbf{R}^{n+k}, \mathbf{R}^k)$ if and only if it acts freely on S^{n-1}. In contrast there are an infinite number of groups π which can act semi-freely on $(\mathbf{R}^n, 0)$ in the period dimension but which cannot act freely on S^{n-1} and so are not obtained by the coning construction above.

Semi-free actions on $(\mathbf{R}^n, 0)$ have been studied by R. J. Milgram in [**26**] by completely different methods. It would be interesting to compare the two approaches.

0. Statement of results. Before giving our results it is useful to summarize some facts about the finite groups with periodic cohomology. We refer the reader to [**27**, **28**], [**43**], and [**44**], [**47**] for more details. First recall that $\mathscr{P}$-groups are just those whose Sylow p-subgroups are cyclic or (if $p = 2$) possibly quaternion. The maximal normal subgroup $O(\pi)$ of odd order is metacyclic and the groups π are divided into six types depending on the structure of $\pi/O(\pi)$: type I groups are metacyclic and this quotient is a cyclic 2-group; type II groups have $\pi/O(\pi)$ a (generalized) quaternion 2-group. For the other types we mainly need to know that any 2-hyperelementary subgroup of a $\mathscr{P}$-group is of type I or II.

We shall need the subclassification of 2-hyperelementary type II groups given in [**24**]. Let

$$Q2^k = \langle x, y : x^{2^{k-2}} = y^2, yxy^{-1} = x^{-1}, y^4 = 1\rangle$$

be the quaternion group of order 2^k. Let A, B, C and D be odd order cyclic groups of coprime orders a, b, c, d and suppose $Q2^k$ acts on them such that

x inverts $B \times C$ and centralizes $A \times D$

y inverts $A \times C$ and centralizes $B \times D$.

The corresponding semi-direct product $(A \times B \times C \times D) \tilde{\times} Q2^k$ is denoted $Q(2^k a, b, c) \times \mathbf{Z}/d$. We normalize the notation so that $b \geqq c$, and, when $k = 3$, $a \geqq b \geqq c$ by changing the presentation of $Q2^k$ if necessary. We now introduce the following 2-hyperelementary subtypes:

type IIK if $b = 1$

type IIL if $b > 1$ and $k > 3$

type IIM if $b > 1$ and $k = 3$.

It will be convenient to call a type IIM group *special* if $c = d = 1$; in this case we write $Q(8a, b)$ instead of $Q(8a, b, 1)$.

After this summary of relevant group theory we now return to the question of semi-free actions. The first invariant of a semi-free action on $(\mathbf{R}^{n+k}, \mathbf{R}^k)$ is the homotopy type of

$$W = (\mathbf{R}^{n+k} - \mathbf{R}^k)/\pi.$$

Since this is an $(n - 1)$-dimensional Swan complex for π, its homotopy type is determined by a single element $g \in H^n(\pi; \mathbf{Z})$ called the *k-invariant of the action* and $2d(\pi)|n$ where $2d(\pi)$ is the *period* of π.

THEOREM A. *Let π be a $\mathcal{P}$-group and $n \geqq 6$. There is a semi-free action of π on $(\mathbf{R}^{n+k}, \mathbf{R}^k)$ with k-invariant g if and only if there is a semi-free action of π on $(\mathbf{R}^n, 0)$ with k-invariant g.*

The proof of Theorem A is based upon work of D. Anderson and E. K. Pedersen. They prove in **[1]** that a semi-free action on $(\mathbf{R}^{n+k}, \mathbf{R}^k)$ is equivalent to the vanishing of $\lambda^p(f, \hat{f})$ in the group

$$L_3^{\langle -k\rangle}(\mathbf{Z}\pi) \subset L_3^p(\mathbf{Z}[\pi \times \mathbf{Z}^k])$$

under a natural map

$$J: L_3^p(\mathbf{Z}\pi) \to L_3^{\langle -k\rangle}(\mathbf{Z}\pi).$$

The groups $L_*^{\langle -k\rangle}$ and $L_*^{\langle -k+1\rangle}$ are connected by an exact triangle where the third term is the Tate cohomology of $K_{-k}(\mathbf{Z}\pi)$. Since $K_{-k}(\mathbf{Z}\pi) = 0$ for $k > 1$ by results from **[6]**,

$$L_*^{\langle -1\rangle}(\mathbf{Z}\pi) \cong L_*^{\langle -k\rangle}(\mathbf{Z}\pi)$$

for finite π and there is an exact sequence

$$H^0(K_{-1}(\mathbf{Z}\pi)) \to L_3^p(\mathbf{Z}\pi) \xrightarrow{J} L_3^{\langle -k\rangle}(\mathbf{Z}\pi) \to H^1(K_{-1}(\mathbf{Z}\pi)).$$

The group $H^0(K_{-1}(\mathbf{Z}\pi)) \neq 0$ in general, even for $\pi = Q(8a, b)$ but nevertheless we prove in Section 3 that J is injective, and hence Theorem A.

Since the obstruction to a semi-free action on $(\mathbf{R}^n, 0)$ with prescribed k-invariant lies in the projective L-groups (Proposition 2.10), Dress' induction theorem **[11]** together with the induction results for normal invariants from **[22]** show that we can restrict our attention to the 2-hyperelementary groups. More precisely we get that π acts semi-freely on $(\mathbf{R}^n, 0)$ with k-invariant $g \in H^n(\pi; \mathbf{Z})$ if and only if each 2-hyperelementary subgroup $\rho \subseteq \pi$ acts semi-freely on $(\mathbf{R}^n, 0)$ with k-invariant Res $g \in H^n(\rho; \mathbf{Z})$.

Thus to prove a general existence theorem for actions with prescribed k-invariant there are two difficulties to be overcome, namely to understand which k-invariants on subgroups ρ extend to π, and to solve the 2-hyperelementary case. With the detailed knowledge we have of the groups only the second presents any real problem.

The semi-characteristic argument used by **[20]** in the case of spherical space forms generalizes to the non-compact setting (see Section 1). It shows that if π acts semi-freely on $(\mathbf{R}^n, 0)$ then every subgroup of order $2p$ (p prime) is cyclic, and if $n \equiv 4 \pmod 8$ then π has no type IIL subgroups. Note that the first condition is equivalent to assuming that every

2-hyperelementary type I subgroup of π has an $\mathscr{F}$-representation. Since the 2-hyperelementary subgroups of π are all type I or II, we can ask whether π has a generator $g \in H^n(\pi; \mathbf{Z})$ such that

$$\text{Res } g \in H^n(\rho; \mathbf{Z})$$

is the Euler class of an $\mathscr{F}$-representation for every 2-hyperelementary subgroup $\rho \subseteq \pi$. If such a generator exists for π we call it a *linear k-invariant* and then π will act on $(\mathbf{R}^n, 0)$. It follows from the structure of $\mathscr{P}$-groups (see [**47**, Sections 11, 12]) that these exist whenever

$$n \equiv 0 \pmod{4d(\pi)}$$

and π satisfies the $2p$-conditions. The basic obstacle to the existence of linear generators is the fact that the 2-hyperelementary type IIL or IIM groups have $d(\pi) = 2$ and no $\mathscr{F}$-representation of dimension 4. From [**43**], the groups of type III or V have no type IIL or IIM subgroups and a type IV or VI group with no type IIL subgroup has no type IIM subgroup either. It follows that any group satisfying Lee's conditions which is not of type IIM has a linear k-invariant in the period dimension $2d(\pi)$. To discuss this last case, it is convenient to remark that every type IIM group has an *almost linear* k-invariant in the following sense.

Definition 0.1. Let π be a $\mathscr{P}$-group of period $2d(\pi)$ and let $n \equiv 0 \pmod{2d(\pi)}$. A generator $g \in H^n(\pi; \mathbf{Z})$ is called *almost linear* if its restriction to any 2-hyperelementary type I or type IIK subgroup of π is linear.

Our next result summarizes the connection between actions of π and its subgroups. It is proved in Sections 1, 7 following the discussion above.

THEOREM B. *Let π be a $\mathscr{P}$-group of period $2d(\pi)$ and $n \geqq 6$ such that $n \equiv 0 \pmod{2d(\pi)}$. If $n \equiv 0 \pmod 8$ then π has a semi-free action on $(\mathbf{R}^n, 0)$ if and only if it satisfies the $2p$-conditions. If $n \equiv 4 \pmod 8$, and $g \in H^n(\pi; \mathbf{Z})$ is an almost linear generator, then π has a semi-free action on $(\mathbf{R}^n, 0)$ with k-invariant g if and only if*

(i) *π contains no subgroups of type* IIL, *and*

(ii) *each special type* IIM *subgroup $\rho \subseteq \pi$ acts semi-freely on $(\mathbf{R}^n, 0)$ with k-invariant* Res $g \in H^n(\rho; \mathbf{Z})$.

We have left to describe our result about actions of type IIM groups in dimensions $n = 8l + 4$ (for $l \geqq 1$). It turns out that the answer involves a number theoretic condition on the integers a, b in the order of special type IIM subgroups $Q(8a, b)$ of π.

For any integer r, let $\eta_r = \zeta_r + \zeta_r^{-1}$ where $\zeta_r = e^{2\pi i/r}$ is a primitive r'th root of unity.

Given a, b we consider the domain $A = \mathbf{Z}[\eta_a, \eta_b]$ and its fraction field $F = \mathbf{Q}[\eta_a, \eta_b]$. From $F^{(2)}$, the elements in F with even valuation at all finite primes in A, there are reduction maps $r_{\mathfrak{p}}$ defined for each prime (in A) which divides ab: for $x \in F^{(2)}$ write $x_{\mathfrak{p}}$ as a product of a unit $u_{\mathfrak{p}}$ in $A_{\mathfrak{p}}$ with an (even) power of a uniformizer at $\mathfrak{p}$, then set $r_{\mathfrak{p}}(x) = u_{\mathfrak{p}}$ in $(A/\mathfrak{p})^{\times}$ modulo squares. Of course, if $x \in A^{\times} \subset F^{(2)}$ then $r_{\mathfrak{p}}(x) \in (A/\mathfrak{p})^{\times}$ has no indeterminacy.

Definition 0.2. Let

$$\Phi_A : F^{(2)} \to (A/ab)^{\times}/\text{squares}$$

$$\varphi_A : F^{(2)} \to (A/4A)^{\times}/\text{squares} \cong A/2A$$

be the product of the $r_{\mathfrak{p}}$ for $\mathfrak{p}|ab$ and the reduction modulo 4, respectively.

For integers α and a we use the notation $\alpha \| a$ to mean that α is a full prime power divisor of a, that is, $\alpha = p^m$ for some prime p and $\alpha | a$ but $p \nmid (a/\alpha)$. We define the element $\nu_p(ab) \in A$ by

$$\nu_p(ab) = \prod \{2 - \eta_{\alpha} : \alpha \| ab \text{ and } (\alpha, p) = 1\}$$

for each (rational) prime $p|a$. Next let

$$\nu(a) = \{\nu_p(ab) : p|a\} \in \sum_{p|a}^{\oplus} (A/pA)^{\times}$$

and similarly for $\nu(b)$. We will identify

$$(A/aA)^{\times}_{(2)} = \sum_{p|a}^{\oplus} (A/pA)^{\times}_{(2)}$$

and

$$(A/ab)^{\times} = (A/aA)^{\times} \oplus (A/bA)^{\times}$$

so that elements of the left-hand side can be represented by vectors. We let

$$V(a, b) = (-1)^{r+1}(\nu(a), \nu(b)) \in (A/ab)^{\times}/\text{squares}$$

where r is the number of rational prime divisors of ab.

Definition 0.3. We say that the condition $C(a, b)$ is satisfied if $V(a, b)$ is in the image of Φ_A restricted to the kernel of φ_A.

We can now state the main result of the paper as

THEOREM C. *Let π be a type* IIM *group and let $l \geqq 1$. Then π acts semi-freely on $(\mathbf{R}^{8l+4}, 0)$ with almost linear k-invariant if and only if condition $C(a, b)$ is satisfied for each special subgroup $Q(8a, b)$ of π.*

The condition $C(a, b)$ used here is not easy to check in general. The smallest example where it is satisfied is $(a, b) = (3, 13)$. Moreover, if a and

b are prime numbers ($\not\equiv 1 \bmod 16$) then a necessary condition for $C(a, b)$ to be satisfied is that

$$\left(\frac{a}{b}\right) = \left(\frac{b}{a}\right) = 1, \quad \left(\frac{a}{b}\right)_4 \left(\frac{b}{a}\right)_4 = 1$$

where $\left(\frac{a}{b}\right)$ is the Legendre symbol and $\left(\frac{a}{b}\right)_4$ is the bi-quadratic symbol with value $+1$ if a is a fourth power mod b and value -1 if not. We refer the reader to the appendix (Section 10) for a more detailed discussion of condition $C(a, b)$ based on the work of Bentzen [**3**].

The Theorems A, B, C together give a complete answer (modulo number theory) to the question of existence of semi-free actions on $(\mathbf{R}^{n+k}, \mathbf{R}^k)$ with almost linear k-invariant. For arbitrary k-invariants we do not have complete information. From [**15**] we know that the quaternion group $Q(4p) = Q(4p, 1, 1)$ can only act freely on S^{8l+3} with linear k-invariant: for a non-linear k-invariant there is a non-zero surgery obstruction in L_3^h. In fact, the obstruction maps non-trivially to L_3^p, so $Q(4p)$ cannot act semi-freely on $(\mathbf{R}^{8l+4}, 0)$ with non-linear k-invariant either. This also implies that a type IIK can only act with linear k-invariant. It follows that the results above give the correct dimensional bounds in the sense that if π acts semi-freely on $(\mathbf{R}^{n+k}, \mathbf{R}^k)$ at all, then it can also act with an almost linear k-invariant. We shall not go further into these questions in the present paper, but we point out that for general 2-hyperelementary groups of type I (satisfying the $2p$-conditions) it is not known at present precisely which k-invariants can be realized by free or semi-free actions.

It is in order to compare the case of free actions treated in [**21**] with the semi-free case. We restrict ourselves to almost linear k-invariants.

The first difference between the two cases is that Theorem B fails for free actions: with quite a lot of work on the finiteness obstruction for groups of type I, cf. [**47**], one can reduce to 2-hyperelementary groups. However, it is not true that a type IIM group acts freely on S^{8l+3} if its special subgroups do. The L^h-surgery obstruction is detected on the special subgroups but the finiteness obstruction is not. In fact, the group $Q(8p, q) \times \mathbf{Z}/r$ with $(p, q, r) = (3, 313, 7)$ has non-zero finiteness obstruction but

$$\sigma_4(Q(24, 313)) = 0.$$

Secondly, for a special type IIM group the number theoretic conditions for a free action on S^{8l+3} are far more restrictive than the conditions $C(a, b)$. There are 4 groups $Q(8p, q)$ of order less than 16,000 which act freely on S^{8l+3} but 42 which act semi-freely on $(\mathbf{R}^{8l+4}, 0)$ where p and q are (odd) primes.

We recall the precise condition for a free action on S^{8l+3} from [**21**]. There are reduction maps

$$\hat{\Phi}:A^{\times} \to (A/ab)^{\times} \to (A/a)^{\times}_{(2)} \oplus (A/b)^{\times}_{(2)}$$
$$\hat{\varphi}:A^{\times} \to (A/4A)^{\times} \to (A/4A)^{\times}_{(2)} \cong A/2A$$

similar to the maps in Definition 0.2. Here the subscript (2) indicates the 2-primary part. In analogy with Definition 0.3 we have the closely related condition $\hat{C}(a, b)$:

$$\hat{V}(a, b) \in \mathrm{Image}(\hat{\Phi}|\mathrm{Kernel}\ \hat{\varphi})$$

where $\hat{V}(a, b) \in (A/ab)^{\times}_{(2)}$ is defined by the same formula as $V(a, b)$. Since in condition $\hat{C}(a, b)$ we do not divide out squares in the target, this is a much more difficult condition to check than condition $C(a, b)$. The main result in **[21]** is the following analogue of Theorem C:

THEOREM $\hat{\mathrm{C}}$. *The group $Q(8a, b)$ acts freely on S^{8l+3}, $l \geqq 1$, if and only if*

(i) *condition $\hat{C}(a, b)$ is satisfied, and*
(ii) $(2, 2) \in \mathrm{Image}(\hat{\Phi})$, $(1, -1) \in \mathrm{Image}(\hat{\Phi})$.

It is instructive to elaborate a little on the similarities and differences between Theorems C and $\hat{\mathrm{C}}$. In both cases one sets up a surgery problem (over a Swan complex) and then calculate the surgery obstruction on L_3^p or L_3^h. The actions exist precisely when the relevant obstruction vanishes. The actual calculations in fact take place in suitable L_3^Y-groups. The groups L_3^p, L_3^h, L_3^Y are calculated using the arithmetic square exact sequences given in Section 3. The surgery obstructions are calculated using various Reidemeister torsion invariants and restriction to subgroups. To make these calculations it is useful to extend Wall's splitting of L^Y for a 2-hyperelementary group (indexed by odd divisors of $|\pi|$) to the other L- and K-groups (see Section 6).

It is a key fact for this paper (which might have other applications) that $L_3^p(\mathbf{Z}\pi)$ is detected by three invariants σ_*, χ and δ_2^Y as follows (of Sections 4, 5):

$$\chi:L_3^p(\mathbf{Z}\pi) \to L_3^h(\hat{\mathbf{Z}}_2\pi)$$
$$\sigma_*:L_3^p(\mathbf{Z}\pi) \to H^0(\widetilde{K}_0(\mathbf{Z}\pi))$$
$$\delta_2^Y:\ker \chi \cap \ker \sigma_* \to \frac{H^0(Wh(\hat{\mathbf{Q}}_2\pi)_+)}{H^0(Wh'(\mathbf{Z}\pi)) + d^*H^1(\widetilde{K}_0(\mathbf{Z}\pi))}$$

where $Wh(\hat{\mathbf{Q}}_2\pi)_+$ indicates the part of $Wh(\hat{\mathbf{Q}}_2\pi)$ from type 0 summands. One consequence is (the map d^* is defined in (5.8)):

PROPOSITION 5.18. *Let π be a finite group. The natural map*

$$i^P:L_3^Y(\mathbf{Z}\pi) \to L_3^P(\mathbf{Z}\pi)$$

induces an isomorphism

$$\ker(L_3^P(\mathbf{Z}\pi) \to H^0(\widetilde{K}_0(\mathbf{Z}\pi))) \cong \frac{\mathrm{Im}(L_3^Y(\mathbf{Z}\pi) \to L_3^Y(\hat{\mathbf{Z}}_2\pi))}{H^0(Wh'(\mathbf{Z}\pi)) + d^*H^1(\widetilde{K}_0(\mathbf{Z}\pi))}.$$

Given a surgery problem $(f, \hat{f})$ with $f:M \to X$ where X is a Swan complex, the L^p-surgery obstruction is denoted $\lambda^p(f, \hat{f})$. As the formal dimension $n - 1 \equiv 3 \pmod 4$, $\sigma_*(\lambda^p(f, \hat{f}))$ is the cohomology class of the finiteness obstruction for X and $\chi(\lambda^p(f, \hat{f})$ can be calculated as a semi-characteristic, following [**20**]. If $\pi = Q(8a, b)$ and X has an almost linear k-invariant, the final invariant δ_2^Y can be calculated (as a difference of "weak" Reidemeister torsions given in Section 8) using the fact that the relevant part of $L_3^Y(\hat{\mathbf{Z}}_2\pi)$ injects into $L_3^Y(\hat{\mathbf{Z}}_2\tau)$ for $\tau = Q(4ab)$. This is done in (7.7), (8.16) and Section 9.

If X is homotopy equivalent to a finite complex then we have the obstruction $\lambda^h(f, \hat{f})$ in $L_3^h(\mathbf{Z}\pi)$ to a free action of π on S^{8l+3}. By picking a finite cell structure on X, the obstruction can be lifted back to $\lambda^Y(f, \hat{f})$ in $L_3^Y(\mathbf{Z}\pi)$. In [**21**] these obstructions were evaluated in three stages: the image of λ^Y in $L_3^Y(\hat{\mathbf{Z}}_2\pi)$ essentially as above and the image of λ^Y in $L_3^Y(\hat{\mathbf{Z}}_l\pi)$, l odd, by an argument involving a modular Reidemeister torsion. If these two vanish, the ordinary Reidemeister torsion and restriction to τ determines

$$\lambda^Y \in \operatorname{Im}(L_0^Y(\hat{\mathbf{Q}}\pi) \to L_3^Y(\mathbf{Z}\pi)).$$

Many difficulties in the finite case arise from the type Sp factors in $\mathbf{Q}\pi$. In particular they lead to Condition (ii) in Theorem $\hat{\mathrm{C}}$. One consequence of Proposition 5.18 is that type Sp factors play no role for L^P-obstructions. This is later confirmed by calculating (9.35) that the type Sp part of

$$\operatorname{Im}(L_3^Y(\mathbf{Z}\pi) \to L_3^Y(\hat{\mathbf{Z}}_2\pi))$$

lies in the indeterminacy $H^1(\widetilde{K}_0(\mathbf{Z}\pi))$.

1. Necessary conditions for semi-free actions. The main results of this section give the part of Theorem B in which certain necessary conditions are stated for the existence of a semi-free action on $(\mathbf{R}^{n+k}, \mathbf{R}^k)$. In order to do this, we define *proper actions* (Definition (1.1) and the following paragraph) on $(\mathbf{R}^{n+k}, \mathbf{R}^k)$ for any $k \geqq 0$ and observe that R. Lee's original semi-characteristic argument for deriving these conditions in the spherical case remains valid in this wider context. For the case of main interest here ($k = 0$) we show in (1.6) that any action is proper. Finally in (1.5) we begin the reduction of our existence question to surgery theory. This is completed in the next section.

Suppose that a finite group π acts semifreely on $\mathbf{R}^{n+k}$ with fixed-point set $\mathbf{R}^k$ (embedded as the last k co-ordinates, say). Let

$$W = (\mathbf{R}^{n+k} - \mathbf{R}^k)/\pi$$

and observe that $W \simeq X^{n-1}$, an $(n - 1)$-dimensional Swan complex for π since

$$\mathbf{R}^{n+k} - \mathbf{R}^k \approx S^{n-1} \times \mathbf{R}^{k+1}.$$

Following [**33**, Section 5] we make

Definition 1.1. A *proper* TOP-*manifold of type* (n, k) consists of (i) an open $(n + k)$-dimensional Top manifold W, (ii) a free $\mathbf{Z}^{k+1}$-action on W such that the orbit space $\bar{W}$ is compact, and (iii) a homotopy retraction $r:\bar{W} \to W$ of the projection $W \to \bar{W}$.

Under these conditions

$$r \times c:\bar{W} \to W \times T^{k+1}$$

is a homotopy equivalence where

$$c:\bar{W} \to B\mathbf{Z}^{k+1} = T^{k+1}$$

classifies the free $\mathbf{Z}^{k+1}$-action and W is a finitely-dominated Poincaré complex of formal dimension $(n - 1)$. Furthermore, the normal invariant

$$\eta(r \times c) \in \operatorname{Im} p_1^* + \operatorname{Im} p_2^* \subseteq [W \times T^{k+1}, G/\mathrm{TOP}]$$

where p_1, p_2 are projections on the two factors. A semi-free topological action of a finite group π on $(\mathbf{R}^{n+k}, \mathbf{R}^k)$ as above will be called *proper* if

$$W = (\mathbf{R}^{n+k} - \mathbf{R}^k)/\pi$$

is a proper TOP manifold of type (n, k). If $\mathcal{N}_{n-1}^{\langle k\rangle}(\pi)$ denotes the unoriented bordism group of pairs (W, ϕ) where W is a proper TOP manifold of type (n, k) and $\phi:W \to B\pi$ a continuous map, then $\mathcal{N}_{n-1}^{\langle k\rangle}(\pi)$ is a subgroup of

$$\mathcal{N}_{n+k}(\pi \times \mathbf{Z}^{k+1})$$

and so is detected by characteristic numbers.

PROPOSITION 1.2. *Suppose π is a finite group acting properly and semi-freely on $(\mathbf{R}^{n+k}, \mathbf{R}^k)$ preserving the orientation. Then π is a $\mathcal{P}$-group, $n \equiv 0 \pmod{2d(\pi)}$ and all subgroups of order $2p$ (p any prime) in π are cyclic. In addition if $n \equiv 4 \pmod 8$ then π has no type* IIL *subgroups.*

Proof. Since W is homotopy equivalent to a Swan complex for π, π must be a $\mathcal{P}$-group and n a multiple of $2d(\pi)$. For the rest observe that the semi-characteristic invariant of R. Lee gives a homomorphism:

(1.3) $\quad \mathcal{N}_{n-1}^{\langle -k\rangle}(\pi) \to \widetilde{R}_{GL,ev}(\pi, F)$

for any field F of characteristic 2, defined by the formula

(1.4) $\quad \chi_{1/2}(W, \phi) = \sum_{i=0}^{l-1} (-1)^i [H_i(\widetilde{W}; F)]$

where $n = 2l$ and $\widetilde{W}$ is the π-covering associated to ϕ. This invariant has the formal properties expressed in [**20**, 2.4, 2.7, 3.8, 4.10, 4.11] after we note that Bredon's result [**4**, 7.4] applies to our non-compact situation and proves again that $\chi_{1/2}$ is well-defined. It then follows as in [**20**, 4.14] that every subgroup of order $2p$ in π is cyclic and from [**20**, 4.15] that π has no type IIL subgroups when $n \equiv 4 \pmod 8$.

Next we point out that relationship between proper actions of a $\mathscr{P}$-group π and manifold structures on Swan complexes.

PROPOSITION 1.5. *Let X be an $(n - 1)$-dimensional Swan complex for π with k-invariant $g \in H^n(\pi; \mathbf{Z})$. Then π has a proper semi-free action on $(\mathbf{R}^{n+k}, \mathbf{R}^k)$ with k-invariant g if and only if*

(i) *X is homotopy equivalent to a proper manifold of type (n, k)*

or equivalently,

(ii) *there exists a closed manifold $\bar{U}$ of dimension $(n + k)$ and a homotopy equivalence*

$$\bar{g}:\bar{U} \to X \times T^{k+1} \quad \textit{with } \eta(\bar{g}) \in \operatorname{Im} p_1^* + \operatorname{Im} p_2^*.$$

Proof. If π has a proper action on $(\mathbf{R}^{n+k}, \mathbf{R}^k)$ with k-invariant g then

$$W = (\mathbf{R}^{n+k} - \mathbf{R}^k)/\pi$$

is a proper manifold of type (n, k) homotopy equivalent to X and $\bar{W} \simeq X \times T^{k+1}$.

Conversely suppose that

$$W \overset{h}{\underset{\simeq}{\to}} X$$

where W is a proper manifold of type (n, k). Then

$$\bar{W} \overset{r \times c}{\underset{\simeq}{\to}} W \times T^{k+1} \overset{h \times 1}{\underset{\simeq}{\to}} X \times T^{k+1}$$

is a homotopy equivalence and so (i) $\Rightarrow$ (ii). If $\hat{W}$ denotes the π-covering of $\bar{W}$ induced by this homotopy equivalence then

$$\hat{W} \simeq S^{n-1} \times T^{k+1}.$$

After replacing this by a suitable finite covering we can assume that the homotopy equivalence is homotopic to a homeomorphism (note that $\operatorname{Im} \hat{p}_1^* = 0$ since $(n - 1)$ is odd and that every fake $(k + 1)$-torus is finitely covered by T^{k+1}, [**16**]). For $k = 0$, we easily complete the argument by compactifying one end of the infinite cyclic covering $\widetilde{W}$ to get $\mathbf{R}^n$. For $k > 0$, if $\widetilde{W}$ denotes the $\mathbf{Z}^{k+1}$-covering, we obtain a free π action on $S^{n-1} \times \mathbf{R}^{k+1}$ which is bounded in the $\mathbf{R}^{k+1}$-factor since it comes from a $\pi \times \mathbf{Z}^{k+1}$ action. Consider the homomorphism

$$\gamma:\mathrm{Homeo}_b(S^{n-1} \times \mathbf{R}^{k+1}) \to \mathrm{Homeo}(S^{n-1} * S^k, S^k)$$

described by Anderson-Pedersen [**1**] where the domain denotes homeomorphisms bounded in the $\mathbf{R}^{k+1}$-factor and the range homeomorphisms on the join which are the identity on S^k. After applying γ, we obtain a semi-free π action on (S^{n+k}, S^k) and therefore a semi-free action on $(\mathbf{R}^{n+k}, \mathbf{R}^k)$ after removing one point. By construction this action is proper.

We will show later (in Section 2) that proper actions are the ones most naturally studied by the techniques of surgery theory. It is therefore necessary to ask how closely an arbitrary semi-free action on $(\mathbf{R}^{n+k}, \mathbf{R}^k)$ resembles a proper one.

PROPOSITION 1.6. *Any semi-free topological action of π on $(\mathbf{R}^n, 0)$ for $n \geqq 6$ is conjugate to a proper action.*

Proof. We will show that the given free π action on $\mathbf{R}^n - \{0\}$ is conjugate to the restriction of a free $\pi \times \mathbf{Z}$ action satisfying the conditions of (1.1) by applying the twist-gluing construction of Siebenmann [**40**] to

$$W = (\mathbf{R}^n - \{0\})/\pi.$$

According to [**40**, 5.1] we must prove that W is homeomorphic to arbitrarily small neighbourhoods of each end ϵ_+ or ϵ_- by homeomorphisms $f_\pm:U_\pm \to W$ that fix point-wise a smaller neighbourhood of $\epsilon_\pm$. (The "niceness condition" (**) is satisfied in our case since $W \cup \{\epsilon_\pm\}$ is compact and Hausdorff.) To construct such homeomorphisms we use a proper map

$$g: W \to X \times \mathbf{R}$$

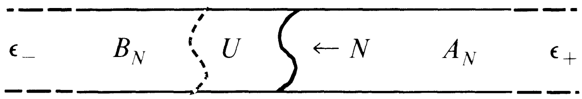
constructed from a proper map $W \to \mathbf{R}$ (average the projection $S^{n-1} \times \mathbf{R} \to \mathbf{R}$ over π) and a homotopy equivalence of W to a suitable $(n - 1)$-dimensional Swan complex X for π. Set $\dim W = n = 2l$. From [**12**, 4.1] for a fixed t_0, g is properly homotopic to a map $g_1: W \to X \times \mathbf{R}$ such that there is a codim. 1 submanifold $N \subset W$ with $g_1|N:N \to X \times t_0$ satisfying the following:

(0) the inclusion $N \subset W$ is $(l - 1)$-connected,
(i) $K_i(A_N, N) = 0$ for $i \leqq l$ and $K_i(B_N, N) = 0$ for $i < l$,
(ii) $K_l(B_N, N) = P$ is a finitely generated projective $\mathbf{Z}\pi$-module.
Here

$$A_N = g_1^{-1}(X \times [t_0, \infty)) \quad \text{and} \quad B_N = g_1^{-1}(X \times (-\infty, t_0]).$$

Next we apply [**12**, 6.2] to find an open subset U of a compact set in B_N

which is a manifold with boundary N and has a tame end ϵ whose finiteness obstruction $\sigma(\epsilon) = [P]$ in $\widetilde{K}_0(\mathbf{Z}\pi)$.



We may assume that the end ϵ_- of W is highly-connected in the sense of [**39**] so that B_N is expressed as the union of compact highly-connected cobordisms. By an engulfing argument, it follows that there exists a homeomorphism $f:W \to W$ such that $f|A_N$ is the identity and $f(U) = B_N$.

Choose now open subset $U_\pm$ of $\epsilon_\pm$ and homeomorphisms

$$f_\pm : U_\pm \to W$$

constructed as above with $U_+ \cap U_- = \phi$ and form

$$\overline{W} = W/\{x \sim f_+^{-1}f_-(x) \text{ for } x \in U_-\}.$$

From [**40**, 5.2, 5.5] it follows that $\overline{W}$ is a compact topological manifold whose homeomorphism class depends only on W (and not on the choices of $(U_\pm, f_\pm)$). Also, by the way the $U_\pm$ were constructed, there is an induced map

$$\overline{g}:\overline{W} \to X \times S^1$$

which is a homotopy equivalence.

Clearly the infinite cyclic covering of $\overline{W}$ induced by the composite of $\overline{g}$ with the projection to S^1 is a proper TOP manifold of type $(n, 0)$ as in Definition (1.1). However this infinite cyclic covering is also homeomorphic to W since it is the union $\cup\{W \times \{m\};\ m \text{ an integer}\}$ under identification of $U_+ \times \{m\}$ to $U_- \times \{m + 1\}$ by $f_-^{-1}f_+$.

2. Proper surgery theory. In order to study the existence question for semi-free proper actions, we need a version of surgery theory which will start with reductions of the Spivak normal fibre space to a Swan complex X and (in favourable circumstances) produce a homotopy equivalence between $X \times T^{k+1}$ and a closed manifold (cf. Proposition 1.5). For $k = 0$, there are two possibilities: surgery on paracompact open manifolds due to Maumary [**29**] and Taylor [**42**] or the compact version due to Pedersen-Ranicki [**33**]. The second approach seems to generalize most easily to $k > 0$ to describe the $L^{\langle -k \rangle}$ groups. Most of the properties we need of these groups are given in [**1**]. We have included them here (with slightly different proofs) for the reader's convenience.

On the algebraic side, let $L_n^p(\mathbf{Z}\pi)$ denote the projective surgery groups defined using forms and formations on finitely-generated projective $\mathbf{Z}\pi$-modules ([**34**]). Next let

$$L_n^{\langle -1\rangle}(\mathbf{Z}\pi) = \operatorname{coker}(\bar{\epsilon}:L_{n+1}^p(\mathbf{Z}\pi) \to L_{n+1}^p(\mathbf{Z}[\pi \times \mathbf{Z}]))$$

where $\bar{\epsilon}$ is the (split) injection induced by the inclusion $\pi \subset \pi \times \mathbf{Z}$ into the first factor. This process may be repeated to produce (for $k > 0$):

(2.1) $$L_n^{\langle -k\rangle}(\mathbf{Z}\pi) = \operatorname{coker}(\bar{\epsilon}:L_{n+1}^{\langle -k+1\rangle}(\mathbf{Z}\pi) \to L_{n+1}^{\langle -k+1\rangle}(\mathbf{Z}[\pi \times \mathbf{Z}])).$$

It is sometimes convenient to set

$$L_n^{\langle 0\rangle}(\mathbf{Z}\pi) = L_n^p(\mathbf{Z}\pi)$$

and

$$L_n^{\langle 1\rangle}(\mathbf{Z}\pi) = L_n^h(\mathbf{Z}\pi)$$

in order to unify the notation.

One geometric interpretation of these groups can be given as follows. For any CW complex K, let $L_n^{1,p}(K)$ denote the bordism group of normal maps from compact n-manifolds to finitely-dominated oriented Poincaré pairs (Z, Y) with Y finite, equipped with a reference map $\omega:Z \to K$. An element

(2.2) $$\varphi:(N, M) \to (Z, Y)$$

satisfying the conditions of [**33**, Section 2] represents zero in this bordism group if

$$\varphi \times 1:(N, M) \times S^1 \to (Z, Y) \times S^1$$

is normally bordant to a homotopy equivalence (respecting the reference map $\omega \times 1:Z \times S^1 \to K \times S^1$).

THEOREM 2.3. [**33**, 2.1]. *If K has a finite 2-skeleton and $n \geqq 5$ then there is a natural isomorphism*

$$L_n^{1,p}(K) \to L_n^p(\pi_1(K)).$$

Now for $k > 0$ we define

(2.4) $$L_n^{1,\langle -k\rangle}(K) = \operatorname{coker}(\bar{\epsilon}:L_{n+1}^{1,\langle -k+1\rangle}(K) \to L_{n+1}^{1,\langle -k+1\rangle}(K \times S^1))$$

where $\bar{\epsilon}$ is induced by the inclusion $K \subset K \times S^1$ and as before,

$$L_n^{1,\langle 0\rangle}(K) = L_n^{1,p}(K).$$

The groups $L_n^{1,\langle -k\rangle}(K)$ are just the bordism groups of normal maps as in (2.2) with reference map to $K \times T^k$ modulo those from $L_{n+k}^{1,p}(K \times T^{k-1})$ via the k inclusions of subgroups $T^{k-1} \subset T^k$.

Our Definitions (2.1) and (2.4) together with Theorem (2.3) imply:

PROPOSITION 2.5. *If K has a finite 2 skeleton, $k \geqq 0$ and $n \geqq 5$ then there is a natural isomorphism*

$$L_n^{1,\langle -k\rangle}(K) \to L_n^{\langle -k\rangle}(\pi_1(K)).$$

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One consequence is that a reduction of the Spivak bundle to a Swan complex X for π in dimension $(n - 1)$ gives an element in the group $L_{n-1}^{\langle -k \rangle}(\mathbf{Z}\pi)$. Indeed, let $(f, \hat{f}):M \to X$ be the resulting normal map and consider the element represented by

$$(f, \hat{f}) \times \mathrm{id}:M \times T^k \to X \times T^k \quad \text{in } L_{n-1}^{1,\langle -k \rangle}(\mathbf{Z}\pi).$$

Later we will see that the vanishing of this element in $L_{n-1}^{\langle -k \rangle}(\mathbf{Z}\pi)$ is necessary and sufficient for the existence of a proper semi-free action on $(\mathbf{R}^{n+k}, \mathbf{R}^k)$. Another consequence is the following naturality property:

PROPOSITION 2.6. *For any $k \geqq 0$ and any finitely presented group π,*
(i) *there exists a natural transformation*

$$i_k:L_n^{\langle -k \rangle}(\mathbf{Z}\pi) \to L_n^{\langle -k-1 \rangle}(\mathbf{Z}\pi)$$

and a (split) monomorphism

$$\theta_k:L_n^{\langle -k \rangle}(\mathbf{Z}\pi) \to L_{n+1}^{\langle -k+1 \rangle}(\mathbf{Z}[\pi \times \mathbf{Z}])$$

defining a natural transformation of functors, and

(ii) *the diagram*

$$\begin{tikzcd} L_n^{\langle -k \rangle}(\mathbf{Z}\pi) \arrow[r, "i_k"] \arrow[d, "\theta_k"] & L_n^{\langle -k-1 \rangle}(\mathbf{Z}\pi) \arrow[d, "\theta_{k+1}"] \\ L_{n+1}^{\langle -k+1 \rangle}(\mathbf{Z}[\pi \times \mathbf{Z}]) \arrow[r, "i_{k+1}"] & L_{n+1}^{\langle -k \rangle}(\mathbf{Z}[\pi \times \mathbf{Z}]) \end{tikzcd}$$

is commutative.

Proof. (i) From (2.4) elements of $L_n^{\langle -k \rangle}(\mathbf{Z}\pi)$ can be represented by normal maps (of dimension $(n + k)$)

$$\varphi:(N, M) \to (Z, Y)$$

together with a reference map $\omega:Z \to K \times T^k$ where (Z, Y) is a finitely dominated Poincaré pair (with Y finite) and K is a finite complex with $\pi_1 K = \pi$. Since $Z \times S^1$ has a canonical finite structure extending the given one on $Y \times S^1$, the normal map

$$\varphi \times 1:(N, M) \times S^1 \to (Z, Y) \times S^1$$

together with

$$\omega \times 1:Z \times S^1 \to (K \times S^1) \times T^k$$

represents an element of

$$L_{n+k+1}^h(\mathbf{Z}[(\pi \times \mathbf{Z}) \times \mathbf{Z}^k]).$$

This correspondence induces a well-defined natural homomorphism

$$\theta_k : L_n^{\langle -k \rangle}(\mathbf{Z}\pi) \rightarrow L_{n+1}^{\langle -k+1 \rangle}(\mathbf{Z}[\pi \times \mathbf{Z}])$$

where we identify

$$(2.7) \qquad L_n^{\langle -k \rangle}(\mathbf{Z}\pi) = \frac{L_{n+k}^p(\mathbf{Z}[\pi \times \mathbf{Z}^k])}{\Sigma\, \{L_{n+k}^p(\mathbf{Z}[\pi \times \mathbf{Z}^{k-1}]);\ \mathbf{Z}^{k-1} \subset \mathbf{Z}^k\}}$$

and similarly in the range (using L^h instead of L^p).

To define i_k it is easiest to write

$$(2.8) \qquad L_n^{\langle -k \rangle}(\mathbf{Z}\pi) = \frac{L_{n+k+1}^h(\mathbf{Z}[\pi \times \mathbf{Z}^{k+1}])}{\Sigma\, \{L_{n+k+1}^h(\mathbf{Z}[\pi \times \mathbf{Z}^k]) : \mathbf{Z}^k \subset \mathbf{Z}^{k+1}\}}$$

and then let i_k be the map induced by the "forgetful" homomorphism

$$(2.9) \qquad \gamma_* : L_{n+k+1}^h(\mathbf{Z}[\pi \times \mathbf{Z}^{k+1}]) \rightarrow L_{n+k+1}^p(\mathbf{Z}[\pi \times \mathbf{Z}^{k+1}]).$$

After dividing out in domain and range and comparing with (2.7) we obtain

$$i_k : L_n^{\langle -k \rangle}(\mathbf{Z}\pi) \rightarrow L_n^{\langle -k-1 \rangle}(\mathbf{Z}\pi).$$

From the given definitions of i_k and θ_k, the commutativity of diagram (ii) in general follows from the case $k = 0$.

Consider the diagram:

$$\begin{tikzcd}
 & L_{n+1}^h(\mathbf{Z}[\pi \times \mathbf{Z}']) \arrow[r, "\gamma_*"] \arrow[dd, "\theta_1"] \arrow[ddr, phantom, "(2)"] & L_{n+1}^p(\mathbf{Z}[\pi \times \mathbf{Z}']) \arrow[dr, "j_1"] \arrow[dd, "\theta_0"] & \\
L_n^p(\mathbf{Z}\pi) \arrow[ur, "\theta_0"] \arrow[dd, "\theta_0"] & & & L_n^{\langle -1 \rangle}(\mathbf{Z}\pi) \arrow[dd, "\theta_1"] \\
(1) & L_{n+2}^s(\mathbf{Z}[\pi \times \mathbf{Z}' \times \mathbf{Z}]) \arrow[r, "\gamma_*"] \arrow[dd, "\tau_*"] \arrow[ddr, phantom, "(3)"] & L_{n+2}^h(\mathbf{Z}[\pi \times \mathbf{Z}' \times \mathbf{Z}]) \arrow[dd, "\tau_*"] & (4) \\
L_{n+1}^h(\mathbf{Z}[\pi \times \mathbf{Z}]) \arrow[dr, "\theta_1"] & & & L_{n+1}^p(\mathbf{Z}[\pi \times \mathbf{Z}]) \\
 & L_{n+2}^s(\mathbf{Z}[\pi \times \mathbf{Z} \times \mathbf{Z}']) \arrow[r, "\gamma_*"'] & L_{n+2}^h(\mathbf{Z}[\pi \times \mathbf{Z} \times \mathbf{Z}']) \arrow[ur, "j_0"] &
\end{tikzcd}$$

where the composite $j_1\gamma_*\theta_0 = i_0$, $j_0\gamma_*\theta_1 = i_1$ in diagram (ii) and τ_* is induced by the interchange of factors $\mathbf{Z}$, $\mathbf{Z}'$. The maps j_k are the natural projections from (2.4). All the sub-diagrams (1)-(4) commute so diagram (ii) does also for $k = 0$.

A similar inductive argument starting with the fact that θ_0 is split [**35**] by the epimorphism j_0 shows that θ_k is a split monomorphism with $j_k\theta_k = \text{identity}$.

PROPOSITION 2.10. *Let X^{n-1} be a Swan complex for π and $n \geqq 6$.*

(i) *There exists a semi-free action of π on $(\mathbf{R}^n, 0)$ with*

$$(\mathbf{R}^n - \{0\})/\pi \simeq X$$

if and only if there is a normal invariant for X with vanishing obstruction in $L^p_{n-1}(\mathbf{Z}\pi)$.

(ii) *For any $k > 0$, there exists a proper semi-free action of π on $(\mathbf{R}^{n+k}, \mathbf{R}^k)$ with*

$$(\mathbf{R}^{n+k} - \mathbf{R}^k)/\pi \simeq X$$

if and only if there is a normal invariant for X with vanishing obstruction in $L^{\langle -k \rangle}_{n-1}(\mathbf{Z}\pi)$.

Proof. Let $\mathcal{N}(X)$ denote the set of normal invariants for the Swan complex X and

$$p^*:\mathcal{N}(X) \to \mathcal{N}(X \times T^{k+1})$$

the map induced by the projection

$$p:X \times T^{k+1} \to X.$$

First we claim that there is a commutative diagram (for $k \geqq 0$):

$$\begin{tikzcd}
\mathcal{N}(X) \arrow[rr, "\lambda^{\langle -k \rangle}"] \arrow[d, "p^*"] & & L^{\langle -k \rangle}_{n-1}(\mathbf{Z}\pi) \arrow[d, "\theta"] \\
\mathcal{N}(X \times T^{k+1}) \arrow[rr, "\lambda^h"] & & L^h_{n+k}(\mathbf{Z}[\pi \times \mathbf{Z}^{k+1}])
\end{tikzcd} \tag{2.11}$$

where $\theta = \theta_1 \circ \theta_2 \circ \ldots \circ \theta_k$ and $\lambda^{\langle -k \rangle}$, λ^h are the surgery obstruction maps (cf. the discussion just before (2.6) for $\lambda^{\langle -k \rangle}$). This diagram follows easily from considering the two (equivalent) descriptions of $\mathcal{N}(X)$: equivalence classes of liftings $\xi:X \to \mathrm{BTOP}$ of the Spivak normal fibre space $\nu_X:X \to BG$, or normal bordism classes of normal maps

$$\{f:M^{n-1} \to X, \hat{f}:\nu_M \to \xi \simeq \nu_X\}.$$

The map p^* in the first case is the composition

$$\xi \circ p:X \times T^{k+1} \to X \to \mathrm{BTOP}$$

and in the second case is just crossing the normal map with T^{k+1}. Choose now a reduction for ν_X and identify

$$\mathcal{N}(X) \simeq [X, G/\mathrm{TOP}].$$

According to (1.5) if π has a proper semi-free action on $(\mathbf{R}^{n+k}, \mathbf{R}^k)$ then there exists a homotopy equivalence

$$\bar{g}: \bar{W} \to X \times T^{k+1}$$

with normal invariant

$$\eta(\bar{g}) \in \operatorname{Im} p_1^* + \operatorname{Im} p_2^* \subseteq [X \times T^{k+1}, G/\mathrm{TOP}].$$

Because $\operatorname{Im} p_2^*$ can be eliminated by a finite covering of T^{k+1}, we may assume that

$$\eta(\bar{g}) \in \operatorname{Im} p_1^*.$$

Since $\lambda^h(\mathcal{N}(\bar{g})) = 0$ the result follows from (2.11).

LEMMA 2.12.

$$\operatorname{Im} p_1^* = \left\{ \begin{array}{l} \alpha:(1 \times t)^*(\alpha) = \alpha \textit{ for all finite covering} \\ \textit{maps}(1 \times t): X \times T^{k+1} \to X \times T^{k+1}. \end{array} \right\}$$

Proof. It is enough to consider the case $k = 0$, and the diagram:

$$\begin{tikzcd} 0 \arrow[r] & {[X, G/\mathrm{TOP}]} \arrow[r, "p^*"] \arrow[d, equal] & {[X \times S^1, G/\mathrm{TOP}]} \arrow[r] \arrow[d, "t^*"] & {[X_+ \wedge S^1, G/\mathrm{TOP}]} \arrow[r] \arrow[d, "t^*_+"] & 0 \\ 0 \arrow[r] & {[X, G/\mathrm{TOP}]} \arrow[r, "p_1^*"] & {[X \times S^1, G/\mathrm{TOP}]} \arrow[r] & {[X_+ \wedge S^1, G/\mathrm{TOP}]} \arrow[r] & 0 \end{tikzcd}$$

where t^*_+ is multiplication by the degree of the covering.

Remark 2.13. This suggests that a structure set $S^{\langle -k \rangle}(X)$ for proper actions can be defined as the similar fixed set in $S^h(X \times T^{k+1})$. Then we have a surgery exact sequence (for each $k \geqq 0$):

$$(2.14) \qquad L_n^{\langle -k \rangle}(\mathbf{Z}\pi) \to S^{\langle -k \rangle}(X) \to [X, G/\mathrm{TOP}] \to L_{n-1}^{\langle -k \rangle}(\mathbf{Z}\pi).$$

The various structure sets can be compared. For example, there is a diagram of exact sequences when $n - 1 \equiv 3 \pmod 4$:

$$(2.15) \qquad \begin{tikzcd} {[\Sigma X_+, G/\mathrm{TOP}]} \arrow[r] \arrow[d, equal] & L_0^p(\mathbf{Z}\pi) \arrow[r] \arrow[d] & S^p(X) \arrow[r] \arrow[d] & {[X, G/\mathrm{TOP}]} \arrow[d, equal] \\ {[\Sigma X_+, G/\mathrm{TOP}]} \arrow[r] & L_0^{\langle -1 \rangle}(\mathbf{Z}\pi) \arrow[r] & S^{\langle -1 \rangle}(X) \arrow[r] & {[X, G/\mathrm{TOP}]} \end{tikzcd}$$

which we will use in Section 3 to give an example of a proper action on $(\mathbf{R}^{n+1}, \mathbf{R}^1)$ which cannot be desuspended to $(\mathbf{R}^n, 0)$. In contrast, for $k \geqq 1$

$$\mathcal{S}^{\langle -k \rangle}(X) \cong \mathcal{S}^{\langle -1 \rangle}(X)$$

and so any proper action on $(\mathbf{R}^{n+k}, \mathbf{R}^k)$ desuspends to $(\mathbf{R}^{n+1}, \mathbf{R}^1)$.

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3. Arithmetic sequences in K- and L-theory. After (2.10) it is clear that the proofs of Theorems A, B and C will rely on detailed knowledge of the relevant surgery obstruction groups. The calculation of these groups will be based on the arithmetic sequences introduced by Wall (see [**46**] for a more detailed account and complete references). Since it is necessary in our method to compare various different versions of L-theory, we give a series of braid diagrams which will be used later for this purpose. As a first application, we give the proof of Theorem A.

Let (R, α, u) be an anti-structure consisting of a ring R (with unity) an anti-automorphism α of R, and a unit u in R such that

$$u^{\alpha} = u^{-1} \quad \text{and} \quad x^{\alpha\alpha} = uxu^{-1} \quad \text{for all } x \in R.$$

Wall defines groups $L_n^X(R, \alpha, u)$ depending functorially on the anti-structure and an α-invariant subgroup $X \subseteq K_1(R)$. We point out that in these definitions forms always have even rank, in contrast to the version of L-theory given in [**34**] based on α-invariant subgroups of

$$\widetilde{K}_1(R) = K_1(R)/\{\pm 1\}$$

but without the even rank hypothesis. The main cases of interest will be group rings $R = \Lambda\pi$ where π is a finite group and $\Lambda = \mathbf{Z}, \hat{\mathbf{Z}}, \mathbf{Q}$ or $\hat{\mathbf{Q}}$. In this paper we further assume that the involution α on $\Lambda\pi$ is induced by the map

$$x \to x^{-1} \quad \text{for all } x \in \pi.$$

If in addition $u = +1$, we will simplify the notation of the L-theory to $L_n^X(\Lambda\pi)$.

Let

$$X(\Lambda\pi) = SK_1(\Lambda\pi) \quad \text{and}$$
$$Y(\Lambda\pi) = \{\pm 1\} \oplus \pi/\pi' \oplus X(\Lambda\pi)$$

where π' is the commutator subgroup of π and

$$SK_1(\Lambda\pi) = 0$$

by definition if $\Lambda = \mathbf{Q}$ or $\hat{\mathbf{Q}}$. From now on these particular subgroups of $K_1(\Lambda\pi)$ will be abbreviated X or Y with $\Lambda\pi$ understood from the context. The L-groups corresponding to the subgroups $\{0\}$ and $K_1(R)$ are written L^S and L^K respectively. The main exact sequence:

$$(3.1) \quad \ldots \to L_{n+1}^S(\hat{\mathbf{Q}}\pi) \to L_n^X(\mathbf{Z}\pi) \to L_n^X(\hat{\mathbf{Z}}\pi) \oplus L_n^S(\mathbf{Q}\pi) \to L_n^S(\hat{\mathbf{Q}}\pi) \to \ldots$$

is the starting point for Wall's calculations of $L_n^X(\mathbf{Z}\pi)$. Since $X \subseteq Y$ the corresponding L-groups are related by means of a Rothenberg sequence:

(3.2) $\quad \ldots \to H^{n+1}(Y/X) \to L_n^X(\mathbf{Z}\pi) \to L_n^Y(\mathbf{Z}\pi) \to H^n(Y/X) \to \ldots$

where we have used the abbreviation $H^*(Y/X)$ for the Tate cohomology of the group $\mathbf{Z}/2$ with coefficients in the $\mathbf{Z}/2$-module Y/X. Another case of the same sequence links L^Y and L^K:

(3.3) $\quad \ldots \to H^{n+1}(Wh'(\mathbf{Z}\pi)) \to L_n^Y(\mathbf{Z}\pi)$
$$\to L_n^K(\mathbf{Z}\pi) \to H^n(Wh'(\mathbf{Z}\pi)) \to \ldots$$

where

$$Wh'(\Lambda\pi) = K_1'(\Lambda\pi)/\{\pm 1\} \oplus \pi/\pi' \quad \text{and}$$
$$K_1'(\Lambda\pi) = K_1(\Lambda\pi)/X(\Lambda\pi).$$

To obtain the surgery obstruction groups $L_n'(\mathbf{Z}\pi)$ and $L_n^h(\mathbf{Z}\pi)$ when n is odd, we must factor out the class of

$$\tau = \begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix}$$

from L^Y and L^K respectively. (When n is even no modification is necessary.) Since L^Y and L^K have better formal properties (such as commuting with products) it will be convenient to use them in making calculations.

There is a close connection between (3.1)-(3.3) and the Mayer-Vietoris sequence in K-theory arising from the arithmetic square:

(3.4) $\quad 0 \to Wh'(\mathbf{Z}\pi) \to Wh'(\hat{\mathbf{Z}}\pi) \oplus Wh(\mathbf{Q}\pi)$
$$\to Wh(\hat{\mathbf{Q}}\pi) \to \widetilde{K}_0(\mathbf{Z}\pi) \to 0.$$

Let

$$\bar{W}(\pi) = \ker(Wh(\hat{\mathbf{Q}}\pi) \to \widetilde{K}_0(\mathbf{Z}\pi))$$

and consider the short exact sequences

(3.5) $\quad 0 \to Wh'(\mathbf{Z}\pi) \to Wh'(\hat{\mathbf{Z}}\pi) \oplus Wh(\mathbf{Q}\pi) \to \bar{W}(\pi) \to 0$

and

(3.6) $\quad 0 \to \bar{W}(\pi) \to Wh(\hat{\mathbf{Q}}\pi) \to \widetilde{K}_0(\mathbf{Z}\pi) \to 0.$

The given involution on $\Lambda\pi$ induces a $\mathbf{Z}/2$-module structure agreeing with complex conjugation on the centres of the simple summands of $\mathbf{Q}\pi$. For (3.6) to be an exact sequence of $\mathbf{Z}/2$-modules the involution $[P] \to -[P^*]$ must be used. This convention (which is different from that of [**34**]) will be adopted for all K_0-groups occurring in the paper.

PROPOSITION 3.7. *There is a commutative braid of exact sequences*:

$$\begin{tikzcd}[column sep=small]
H^0(Wh'(\mathbf{Z}\pi)) \arrow[drr] \arrow[rr, bend left] & & L_3^Y(\mathbf{Z}\pi) \arrow[drr] \arrow[rr, bend left] & & L_3^Y(\hat{\mathbf{Z}}\pi) \oplus L_3^S(\mathbf{Q}\pi) \arrow[drr] \arrow[rr, bend left] & & L_3^K(\hat{\mathbf{Z}}\pi) \oplus L_3^K(\mathbf{Q}\pi) \\
L_0^S(\hat{\mathbf{Q}}\pi) \arrow[urr] \arrow[drr] & & H^0(Wh'(\hat{\mathbf{Z}}\pi) \oplus Wh(\mathbf{Q}\pi)) \arrow[urr] \arrow[drr] & & L_3^K(\mathbf{Z}\pi) \arrow[urr] \arrow[drr] & & L_3^S(\hat{\mathbf{Q}}\pi) \\
L_0^K(\hat{\mathbf{Z}}\pi) \oplus L_0^K(\mathbf{Q}\pi) \arrow[urr] \arrow[rr, bend right] & & L_0^{\overline{W}(\pi)}(\hat{\mathbf{Q}}\pi) \arrow[urr] \arrow[rr, bend right] & & H^0(\overline{W}(\pi)) \arrow[urr] \arrow[rr, bend right] & & H^1(Wh'(\mathbf{Z}\pi))
\end{tikzcd}$$

Remarks. 1) The proof follows from the diagrams given in [**37**, Section 6]. Note that the sequences given there all use the version of L-theory without the even rank assumption on forms so that some (easy) diagram-chasing is required to obtain the given braid.

2) The braid extends in both directions; the piece given here is the one needed later.

Next let $L_n^P(R, \alpha; u)$ denote the version of projective L-theory assuming the even rank condition. (To get a complete definition in the style of [**34**, p. 112] simply add this condition to the definition of $X_n(A)$.) These groups were introduced by Pardon and are denoted $f\,W_*$ in [**32**]. The arithmetic sequence for computing $L_n^P(\mathbf{Z}\pi)$ can be derived from [**37**] and we get

PROPOSITION 3.8. *There is a commutative braid of exact sequences*:

$$\begin{tikzcd}
L_0^K(\hat{\mathbf{Z}}\pi) \oplus L_0^K(\mathbf{Q}\pi) \arrow[dr] \arrow[rr, bend left] & & L_0^K(\hat{\mathbf{Q}}\pi) \arrow[dr] \arrow[rr, bend left] & & H^0(\tilde{K}_0(\mathbf{Z}\pi)) \arrow[dr] & \\
& L_0^{\overline{W}(\pi)}(\hat{\mathbf{Q}}\pi) \arrow[ur] \arrow[dr] & & L_3^P(\mathbf{Z}\pi) \arrow[ur, "\sigma_*"] \arrow[dr, "\chi"] & & L_0^{\overline{W}(\pi)}(\hat{\mathbf{Q}}\pi) \\
H^1(\tilde{K}_0(\mathbf{Z}\pi)) \arrow[ur] \arrow[rr, bend right] & & L_3^K(\mathbf{Z}\pi) \arrow[ur] \arrow[rr, bend right] & & L_3^K(\hat{\mathbf{Z}}\pi) \oplus L_3^K(\mathbf{Q}\pi) \arrow[ur] &
\end{tikzcd}$$

Remarks. 1) For n even,

$$L_n^P(\mathbf{Z}\pi) = L_n^p(\mathbf{Z}\pi)$$

while for n odd,

$$L_n^P(\mathbf{Z}\pi)/\left\langle \begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix} \right\rangle = L_n^p(\mathbf{Z}\pi).$$

In general it appears that the sequence

$$\ldots \to H^n(K_0(R)) \to L_n^K(R) \to L_n^p(R) \to H^{n-1}(K_0(R)) \to \ldots$$

is the most suitable for calculations since each term is invariant under

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Morita equivalence of anti-structures. On the other hand, the use of $\widetilde{K}_0(\mathbf{Z}\pi)$ is necessary for geometrical reasons so this forces the use of L^P in the above diagram.

2) The map labelled σ_* in (3.8) is geometrically the cohomology class of the finiteness obstruction in [**33**, Section 2]: let $(f, b):M \to X$ represent an element of $L_3^p(\mathbf{Z}\pi)$ as in Section 2, then

$$\sigma_*(f, b) = \{\sigma(X)\}.$$

3) The map labelled χ in (3.8) is closely related to "semi-characteristics". This connection will be discussed in Section 4.

For the proof of Theorem A we need to relate L^p and $L^{\langle -k\rangle}$. Since $K_{-i}(\mathbf{Z}\pi) = 0$ for $i > 1$ by [**6**], the exact sequences of [**35**] show that

$$(3.9) \quad L_n^{\langle -k\rangle}(\mathbf{Z}\pi) = L_n^{\langle -1\rangle}(\mathbf{Z}\pi)$$

for all $k \geqq 1$. For calculating $K_{-1}(\mathbf{Z}\pi)$ there is the exact sequence [**5**]:

$$(3.10) \quad 0 \to \widetilde{K}_0(\hat{\mathbf{Z}}\pi) \oplus \widetilde{K}_0(\mathbf{Q}\pi) \to \widetilde{K}_0(\hat{\mathbf{Q}}\pi) \to K_{-1}(\mathbf{Z}\pi) \to 0.$$

The cohomology sequence arising from this short exact sequence of $\mathbf{Z}/2$-modules fits in with the arithmetic sequences for L^p and $L^{\langle -1\rangle}$ as above.

PROPOSITION 3.11. *There is a commutative braid of exact sequences*:

$$\begin{tikzcd}
L_0^{\widetilde{K}}(\hat{\mathbf{Z}}\pi) \oplus L_0^{\widetilde{K}}(\mathbf{Q}\pi) \arrow[r, bend left] \arrow[dr] & L_0^{\widetilde{K}}(\hat{\mathbf{Q}}\pi) \arrow[r, bend left] \arrow[dr] & L_0^p(\hat{\mathbf{Q}}\pi) \arrow[r, bend left] \arrow[dr] & L_3^{\langle -1\rangle}(\mathbf{Z}\pi) \\
H^0(\widetilde{K}_0(\hat{\mathbf{Q}}\pi)) \arrow[ur] \arrow[dr] & L_0^p(\hat{\mathbf{Z}}\pi) \oplus L_0^p(\mathbf{Q}\pi) \arrow[ur] \arrow[dr] & L_3^p(\mathbf{Z}\pi) \arrow[ur] \arrow[dr] & H^1(\widetilde{K}_0(\hat{\mathbf{Q}}\pi)) \\
L_0^{\langle -1\rangle}(\mathbf{Z}\pi) \arrow[ur] \arrow[r, bend right] & H^0(K_{-1}(\mathbf{Z}\pi)) \arrow[ur] \arrow[r, bend right, "j"'] & H^1(\widetilde{K}_0(\hat{\mathbf{Z}}\pi) \oplus \widetilde{K}_0(\hat{\mathbf{Q}}\pi)) \arrow[ur] \arrow[r, bend right] & L_3^{\widetilde{K}}(\hat{\mathbf{Z}}\pi) \oplus L_3^{\widetilde{K}}(\mathbf{Q}\pi)
\end{tikzcd}$$

Remarks. 1) For any ring with involution

$$L_n^{\widetilde{K}}(R) = L_n^{0 \subseteq \widetilde{K}_0(R)}(R)$$

as in [**34**] but we noted above that $L_n^{\widetilde{K}}(R) \neq L_n^K(R)$ in general. This is the point where the transition is made from L-theory based on subgroups of K_1 to that based on subgroups of K_0.

2) In the orientable anti-structure the involution induced on the free abelian group $\widetilde{K}_0(\mathbf{Q}\pi)$ is -1. On $\widetilde{K}_0(\hat{\mathbf{Q}}\pi)$ the involution is -1 on some factors and interchanges others in pairs. Hence

$$H^0(\widetilde{K}_0(\hat{\mathbf{Q}}\pi)) = 0$$

and the map labelled j in the braid is an injection.

THEOREM 3.12. *The natural map* (*cf.* (2.6))

$$i_1 : L_3^p(\mathbf{Z}\pi) \to L_3^{\langle -1 \rangle}(\mathbf{Z}\pi)$$

is injective for any finite group π.

Proof. First we rewrite (3.10) using the fact that

$$\widetilde{K}_0(\hat{\mathbf{Z}}_l\pi) \to \widetilde{K}_0(\hat{\mathbf{Q}}_l\pi)$$

is an isomorphism when $l \nmid |\pi|$:

$$(3.13) \quad 0 \to \sum_{l \| \pi|}^{\oplus} \widetilde{K}_0(\hat{\mathbf{Z}}_l\pi) \oplus \widetilde{K}_0(\mathbf{Q}\pi) \to \sum_{l \| \pi|}^{\oplus} \widetilde{K}_0(\hat{\mathbf{Q}}_l\pi) \to K_{-1}(\mathbf{Z}\pi) \to 0.$$

From this it is easy to see that torsion elements in $K_{-1}(\mathbf{Z}\pi)$ arise from the simple (involution-invariant) factors S in $\mathbf{Q}\pi$ such that S is non-split but $\hat{S}_l = S \otimes \hat{\mathbf{Q}}_l$ is split at all finite primes. More precisely, let $[S] \in \widetilde{K}_0(\mathbf{Q}\pi)$ be the class of such an algebra and note that $[\hat{S}_l] = m_l[P_l]$ where P_l denotes the simple module over the matrix ring $\hat{D}_l$ and m_l is the local Schur index. In fact, to satisfy the conditions S must be type Sp and Morita equivalent to a quaternion algebra D over a totally real field F, say

$$D = \left(\frac{\alpha, \beta}{F}\right)$$

with the standard involution $-$. Hence $m_l = 2$ for all primes l and $[P_l]$ generates an element of order 2 in $K_{-1}(\mathbf{Z}\pi)$. It is also true that all torsion elements arise in this way [**6**]. The image of the torsion element just considered under the coboundary

$$j : H^0(K_{-1}(\mathbf{Z}\pi)) \to H^1(\widetilde{K}_0(\hat{\mathbf{Z}}\pi) \oplus \widetilde{K}_0(\mathbf{Q}\pi))$$

is clearly represented by $(0, [D])$. We note that the left-hand side is just Torsion $(K_{-1}(\mathbf{Z}\pi))$.

LEMMA 3.14. *The hermitian form* $\mathrm{tr}:D \times D \to D$ *defined by* $\mathrm{tr}(u, v) = u\bar{v}$ *is hyperbolic over* $(\hat{D}_l, -, 1)$ *if and only if* $\hat{D}_l$ *is split.*

Assuming this for the moment we can look at the braid (3.11). The class of (D, tr) in $L_0^p(\mathbf{Q}\pi)$ maps to $[D]$ in $H^1(\widetilde{K}_0(\mathbf{Q}\pi))$ and to zero in $L_0^p(\hat{\mathbf{Q}}\pi)$. Since j is an injection it follows that the image of

$$L_0^{\langle -1 \rangle}(\mathbf{Z}\pi) \to H^0(K_{-1}(\mathbf{Z}\pi))$$

contains the torsion element given by D.

Proof of (3.14). The anti-structure induced by the given one on $\mathbf{Q}\pi$ is $(D, -, 1)$ where "$-$" denotes the usual involution on

$$D = \left(\frac{\alpha, \beta}{F}\right) = \{F.1 + F.e_1 + F.e_2 + F.e_3 : e_1^2 = \alpha, e_2^2 = \beta\}$$

where $e_3 = e_1 e_2$ and the involution is given by

$$\bar{e}_i = -e_i \quad (1 \leqq i \leqq 3).$$

If $\not{p}|l$ is a prime in F then $D_{\not{p}}$ is split if and only if the equation $\alpha x^2 + \beta y^2 = 1$ has a solution in $F_{\not{p}}$. When a solution exists then an explicit isomorphism

$$D_{\not{p}} \cong M_2(F_{\not{p}})$$

can be given by sending

$$e_1 \to \begin{pmatrix} \alpha x & \alpha y \\ \beta y & -\alpha x \end{pmatrix} \quad \text{and} \quad e_2 \to \begin{pmatrix} -\beta y & \alpha x \\ \beta x & \beta y \end{pmatrix}.$$

The induced anti-structure is

$$(M_2(F_{\not{p}}), A \to XA^tX^{-1}, 1)$$

where

$$X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and is scale equivalent to

$$(M_2(F_{\not{p}}), A \to A^t, -1).$$

The form $\mathrm{tr}: D \times D \to D$ now becomes the form over $M_2(F_{\not{p}})$ with matrix X but this is Morita equivalent to the standard hyperbolic form

$$F_{\not{p}}^2 \times F_{\not{p}}^2 \xrightarrow{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} F_{\not{p}}.$$

It is now possible to summarize the Proof of Theorem A. If π has a semi-free action on $(\mathbf{R}^{n+k}, \mathbf{R}^k)$ for $k > 0$ then from [**1**] there is a normal invariant for

$$X \simeq (\mathbf{R}^{n+k} - \mathbf{R}^k)/\pi$$

with vanishing obstruction in $L_{n-1}^{\langle -k \rangle}(\mathbf{Z}\pi)$. We remark that if the action was proper this would follow from (2.10)(ii). From the factorization

(3.15) $\quad [X, G/\mathrm{TOP}] \xrightarrow{\lambda^p} L_{n-1}^p(\mathbf{Z}\pi) \xrightarrow{i_1} L_{n-1}^{\langle -1 \rangle}(\mathbf{Z}\pi) \xrightarrow{\approx} L_{n-1}^{\langle -k \rangle}(\mathbf{Z}\pi)$

and the injectivity of i_1 (Theorem (3.12)) it follows that the given normal invariant has vanishing λ^p-obstruction also. Hence (2.10)(i) gives the action on $(\mathbf{R}^n, 0)$. Since the converse is obvious from (3.15) and (2.10) the proof is complete.

We close this section with an example to show that Theorem A cannot be sharpened to say that each (proper) semi-free action on $(\mathbf{R}^{n+k}, \mathbf{R}^k)$ can be desuspended to $(\mathbf{R}^n, 0)$.

Consider the type IIK group $\pi = Q(8p)$. We have

(3.16) $\mathbf{Q}\pi = 2 \cdot \{\mathbf{Q}(\zeta_p)^t[x|x^2 = 1] \oplus \mathbf{Q}(\zeta_p)^t[x|x^2 = -1]\}$.

The first factor is split, the second is not, as it becomes quaternion at infinite primes. We examine when it is split at all finite primes. At 2,

$$\hat{\mathbf{Z}}_2\pi = M_4(\hat{\mathbf{Z}}_2 \otimes \mathbf{Z}[\eta_p][\mathbf{Z}/4])$$

so $\hat{\mathbf{Q}}_2\pi$ splits into matrix rings (cf. [**19**, Section 3]), whence $\mathbf{Q}(\zeta_p)^t[x|x^2 = -1]$ can only be non-split at p. Recall that a cyclic algebra $E^t[\sigma|\sigma^n = a]$ is split at a p-adic prime $\mathfrak{p}$ precisely when

$$a \in \mathrm{Image}(N:E_{\mathfrak{p}} \to F_{\mathfrak{p}})$$

where F is the center, $F = E^\sigma$.

Lemma 3.17. *The algebra* $\mathbf{Q}(\zeta_p)^t[x|x^2 = -1]$ *is split at all finite primes if and only if* $p \equiv 1 \pmod 4$.

Proof. It follows from [**38**, Section 3, Corollary 5] that

$$-1 \in \mathrm{Image}(\hat{\mathbf{Q}}_p(\zeta_p) \to \hat{\mathbf{Q}}_p(\eta_p))$$

if and only if -1 is a square in $\mathbf{F}_p$.

We now use the exact sequence (3.13)

$$0 \to \sum_{l|2p}^{\oplus} \tilde{K}_0(\hat{\mathbf{Z}}_l\pi) \oplus \tilde{K}_0(\mathbf{Q}\pi) \to \sum_{l|2p}^{\oplus} \tilde{K}_0(\hat{\mathbf{Q}}_l\pi) \to K_{-1}(\mathbf{Z}\pi) \to 0.$$

The term $\tilde{K}_0(\hat{\mathbf{Z}}_l\pi)$ maps into a direct summand of $\tilde{K}_0(\hat{\mathbf{Q}}_l\pi)$, so we get from 3.17

$$\begin{aligned} \text{Torsion } K_{-1}(\mathbf{Z}\pi) &= \mathbf{Z}/2 \oplus \mathbf{Z}/2 \quad \text{if } p \equiv 1 \pmod 4 \\ &= 0 \quad \text{if } p \equiv 3 \pmod 4, \end{aligned}$$

and

$$H^0(K_{-1}(\mathbf{Z}\pi)) = \text{Torsion } K_{-1}(\mathbf{Z}\pi).$$

Let V be an $\mathscr{F}$-representation of $\pi = Q(8p)$ of real dimension $8k + 4$ and let $X = S(V)/\pi$ be the associated Swan complex. We have the diagram (cf. (2.15)) and from Theorem 3.12, $d^{\langle -1 \rangle}$ is surjective. It follows that $\bar{i}$ is not surjective when $p \equiv 1 \pmod 4$, thus there exist non-desuspendable semi-free actions on $(\mathbf{R}^{8k+5}, \mathbf{R}^1)$. On the other hand, we have already remarked that proper semi-free actions on $(\mathbf{R}^{n+k}, \mathbf{R}^k)$ for $k > 1$ desuspend to $(\mathbf{R}^{n+1}, \mathbf{R}^1)$.

$$\begin{tikzcd}
 & H^1(K_{-1}(\mathbf{Z}\pi)) \arrow[d] \arrow[r, equal] & H^1(K_{-1}(\mathbf{Z}\pi)) \arrow[d] & \\
\to [\Sigma X_+, G/\mathrm{TOP}] \arrow[r] \arrow[d, equal] & L_0^p(\mathbf{Z}\pi) \arrow[r] \arrow[d, "i"] & S^p(X) \arrow[r] \arrow[d, "\bar{i}"] & {[X, G/\mathrm{TOP}]} \arrow[d, equal] \\
\to [\Sigma X_+, G/\mathrm{TOP}] \arrow[r] & L_0^{\langle -1 \rangle}(\mathbf{Z}\pi) \arrow[r] \arrow[d, "d^{\langle -1 \rangle}"] & S^{\langle -1 \rangle}(X) \arrow[r] \arrow[d] & {[X, G/\mathrm{TOP}]} \\
 & H^0(K_{-1}(\mathbf{Z}\pi)) \arrow[d] \arrow[r, equal] & H^0(K_{-1}(\mathbf{Z}\pi)) \arrow[d] & \\
 & 0 & 0 &
\end{tikzcd}$$

4. The δ-invariant. In the proof of Theorem C, it will be necessary to calculate a surgery obstruction in $L_{n-1}^p(\mathbf{Z}\pi)$ for a $\mathcal{P}$-group π of type IIM. Since the 2-hyperelementary groups in this family have

$$d(\pi) = 2 \quad \text{and} \quad n \equiv 0 \pmod{2d(\pi)},$$

the obstructions will be in $L_3^P(\mathbf{Z}\pi)$. We will now apply the techniques of Wall **[44]**, **[45]** to calculate the relevant part of the arithmetic sequence:

(4.1) $\quad L_0^K(\hat{\mathbf{Z}}\pi) \oplus L_0^K(\mathbf{Q}\pi) \to L_0^K(\hat{\mathbf{Q}}\pi) \to L_3^P(\mathbf{Z}\pi) \to L_3^K(\hat{\mathbf{Z}}\pi) \oplus L_3^K(\mathbf{Q}\pi).$

The main result (4.15) is that $L_3^P(\mathbf{Z}\pi)$ is detected by two "a priori" invariants, namely the natural map

(4.2) $\quad \chi: L_3^P(\mathbf{Z}\pi) \to L_3^K(\hat{\mathbf{Z}}_2\pi)$

and the δ-invariant

(4.3) $\quad \delta: \ker \chi \to H^0(Wh(\hat{\mathbf{Q}}\pi))/(L_0^K(\hat{\mathbf{Z}}\pi) \oplus L_0^K(\mathbf{Q}\pi))$

induced by the discriminant homomorphism

(4.4) $\quad d_0: L_0^K(\hat{\mathbf{Q}}\pi) \to H^0(Wh(\hat{\mathbf{Q}}\pi)).$

The first of these is related to R. Lee's semi-characteristic (see Section 2) while the second is related to the Reidemeister torsion invariants of **[21]** (see Section 8).

The first step is to show that the image of $L_3^P(\mathbf{Z}\pi)$ is zero in $L_3^K(\hat{\mathbf{Z}}_p\pi)$ for p odd and in $L_3^K(\mathbf{Q}\pi)$. But $L_3^K(\mathbf{Q}\pi) = 0$ for any finite group and:

PROPOSITION 4.5. *For any finite group π, and p odd,*

$$L^K_{2d-1}(\hat{\mathbf{Z}}_p\pi) \to L^K_{2d-1}(\hat{\mathbf{Q}}_p\pi)$$

is injective.

Proof. By Dress induction [**10**] and the fact that odd-dimensional L-groups are 2-torsion groups, it is enough to prove the result for π a 2-hyperelementary group. Since

$$L^K_{2d-1}(\hat{\mathbf{Z}}_p\pi) \to L^K_{2d-1}(\hat{\mathbf{Z}}_p[\pi/\rho])$$

is an isomorphism whenever $\rho \lhd \pi$ is a p-Sylow subgroup [**48**], we may also assume that $p \nmid |\pi|$. After Morita equivalence, it is enough to show that

(4.7) $\quad L^K_1(A, \alpha, u) \to L^K_1(E, \alpha, u)$

is injective when A is the ring of integers in an unramified extension field E of $\hat{\mathbf{Q}}_p$. In this situation, both sides are zero unless α = identity and $u = 1$ [**44**]. In the remaining case, both

$$d_1: L^K_1(A, 1, 1) \to H^1(A^\times)$$

and

$$H^1(A^\times) \to H^1(E^\times)$$

are isomorphisms so the result follows.

We now return to the δ-invariant (4.3) and complete our general discussion of the arithmetic sequence (4.1). Let S denote a central simple E-algebra in $\mathbf{Q}\pi$ where E is a number field and let (S, α, u) denote the anti-structure induced on S by our involution $g \to g^{-1}$. We note that when (S, α, u) is type O, then S is split at all real primes and $(S, \alpha, u) \otimes \mathbf{R}$ does not contain any factor Morita equivalent to $(\mathbf{C}, 1, 1)$. From [**45**] there is an exact sequence

(4.11) $\quad 0 \to L^S_n(S, \alpha, u) \to L^S_n(S_A, \alpha, u) \to CL_n(S) \to 0$

where $S_A = \hat{S} \oplus T$ with $T = S \otimes \mathbf{R}$ and $\hat{S}$ is the restricted product

$$\hat{S} = \lim_{\Omega} \left(\prod_{l \in \Omega} \hat{S}_l \times \prod_{l \notin \Omega} \hat{R}_l \right)$$

where $\hat{R}_l \subset \hat{S}_l$ is a maximal (involution-invariant) order. The limit is taken over finite sets Ω of finite primes.

Now let (S, α, u) have type O. Then $L^S_0(T)$ is a non-trivial direct sum of groups

$$L^S_0(\mathbf{R}, 1, 1) = 4\mathbf{Z}$$

each of which maps surjectively onto $CL_0(S) = \mathbf{Z}/2$ in (4.11). It follows that

(4.12) $L_0^S(S, \alpha, u) \to L_0^S(\hat{S}, \hat{\alpha}, \hat{u})$

is onto.

PROPOSITION 4.13.

$$\ker(d_0:L_0^K(\hat{\mathbf{Q}}\pi) \to H^0(Wh(\hat{\mathbf{Q}}\pi))) \subseteq \mathrm{Im}(L_0^K(\hat{\mathbf{Q}}\pi) \to L_0^K(\hat{\mathbf{Q}}\pi)).$$

Proof. Consider first the statement that

(4.14) $\ker(d_0^X:L_0^K(\hat{\mathbf{Q}}\pi) \to H^0(K_1(\hat{\mathbf{Q}}\pi))) \subseteq \mathrm{Im}(L_0^K(\mathbf{Q}\pi) \to L_0^K(\hat{\mathbf{Q}}\pi))$.

Let (S, α, u) be a simple component of $\mathbf{Q}\pi$ as above. If it is type O then

$$L_0^S(\hat{R}_l) = 0 \quad \text{when } l \nmid |\pi|$$

so

$$\ker(d_0^X:L_0^K(\hat{S}) \to H^0(K_1(\hat{S})))$$

is just

$$\sum^{\oplus}\{L_0^S(S_l):\hat{S}_l \text{ splits}\}.$$

It follows from (4.12) that this is in the image of $L_0^S(S)$. If (S, α, u) is type U or Sp then

$$\ker(d_0^X:L_0^K(\hat{S}) \to H^0(K_1(\hat{S}))) = 0$$

and (4.14) is proved.

Next we observe that $L_0^Y(\mathbf{Q}\pi)$ and $L_0^Y(\hat{\mathbf{Q}}\pi)$ have the same image in $H^0(\pm\pi/\pi')$ by comparing the two Rothenberg sequences:

$$\begin{array}{ccccc} L_0^Y(\mathbf{Q}\pi) & \to & H^0(\pm\pi/\pi') & \to & L_3^S(\mathbf{Q}\pi) \\ \downarrow & & \| & & \downarrow \\ L_0^Y(\hat{\mathbf{Q}}\pi) & \to & H^0(\pm\pi/\pi') & \to & L_3^S(\hat{\mathbf{Q}}\pi) \end{array}$$

using (4.11) for $n = 3$ and the fact that $L_3^s(T) = 0$. Now suppose that $x \in \ker d_0$, so $d_0^X(x)$ is in the image of

$$H^0(\pm\pi/\pi') \to H^0(K_1(\hat{\mathbf{Q}}\pi)).$$

Since

$$\ker d_0 = \mathrm{Im}(L_0^Y(\hat{\mathbf{Q}}\pi) \to L_0^K(\hat{\mathbf{Q}}\pi)),$$

we may adjust x by a suitable element x' of $L_0^Y(\mathbf{Q}\pi)$ so that

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$$x - x' \in \operatorname{Ker} d_0^X.$$

The result follows from (4.14).

COROLLARY 4.15. *The δ-invariant is injective. The invariants δ and χ detect $L_3^P(\mathbf{Z}\pi)$.*

Since the discriminant d_0 used in defining the δ-invariant (4.4) factors through $H^0(K_1(\hat{\mathbf{Q}}\pi))$ we can analyse the range of δ by using the splitting:

(4.16) $$K_1(\mathbf{Q}\pi) = K_1(\mathbf{Q}\pi)_+ \oplus K_1(\mathbf{Q}\pi)_- \oplus K_1(\mathbf{Q}\pi)_0$$

associated to the decomposition of the (orientable) anti-structure on $\mathbf{Q}\pi$ into type O, Sp and U anti-structures on simple algebras. The types can be easily recognized by computing

$$\sum_{g \in \pi} \xi(g^2)$$

for an irreducible complex character ξ associated to the simple algebra: type O, Sp, U corresponds to this quantity being > 0, < 0 or $= 0$ respectively. The splitting (4.16) induces a similar one for $K_1(\hat{\mathbf{Q}}_l\pi)$ with type GL included in $K_1(\hat{\mathbf{Q}}_p\pi)_0$ and for $K_1(\hat{\mathbf{Z}}_l\pi)$ if $l \nmid |\pi|$. If l is odd and $l \mid |\pi|$ then **[44]**

(4.17) $$H^0(K_1(\hat{\mathbf{Z}}_l\pi) \cong H^0(K_1(\hat{\mathbf{Z}}_l\pi/\mathrm{Rad}))$$

so the decomposition of $\hat{\mathbf{Z}}_l\pi/\mathrm{Rad}$ into anti-structures over finite fields induces a splitting for $H^0(K_1(\hat{\mathbf{Z}}_l\pi))$. Since the norm homomorphism is onto for finite fields it is clear that

$$H^0(K_1(\hat{\mathbf{Z}}_l\pi)_0) = 0.$$

PROPOSITION 4.18. *The splittings above induce an isomorphism*

$$H^0(K_1(\hat{\mathbf{Q}}\pi))/L_0^K(\hat{\mathbf{Z}}\pi) \oplus L_0^K(\mathbf{Q}\pi)$$
$$\cong H^0(K_1(\hat{\mathbf{Q}}\pi)_-) \oplus H^0(K_1(\hat{\mathbf{Q}}\pi)_+)/\tilde{I}$$

where

$$\tilde{I} = \operatorname{Im}(H^0(K_1(\mathbf{Q}\pi)_+) \oplus H^0(K_1(\hat{\mathbf{Z}}_{\mathrm{odd}}\pi)_+) \oplus L_0^K(\hat{\mathbf{Z}}_2\pi)$$
$$\to H^0(K_1(\hat{\mathbf{Q}}\pi)_+).$$

Proof. In the definition of $\tilde{I}$ the map on $L_0^K(\hat{\mathbf{Z}}_2\pi)$ is the composition

$$L_0^K(\hat{\mathbf{Z}}_2\pi) \to H^0(K_1(\hat{\mathbf{Z}}_2\pi)) \to H^0(K_1(\hat{\mathbf{Q}}\pi)) \to H^0(K_1(\hat{\mathbf{Q}}\pi)_+)$$

where the last map is the projection given by (4.16).

In order to see that the image of $L_0^K(\mathbf{Q}\pi) \oplus L_0^K(\hat{\mathbf{Z}}\pi)$ is contained in the summands

(4.19) $H^0(K_1(\hat{\mathbf{Q}}\pi)_+) \oplus H^0(K_1(\hat{\mathbf{Q}}\pi)_0)$

we recall from **[44][45]** that $L_0^K(S, \alpha, u) = 0$ whenever (S, α, u) is a type *Sp* anti-structure over a central simple E-algebra and E is a local or finite field (of char $\neq 2$). The remark now follows from the factorization

(4.20) $L_0^K(\hat{\mathbf{Z}}\pi) \oplus L_0^K(\mathbf{Q}\pi) \to L_0^K(\hat{\mathbf{Q}}\pi) \xrightarrow{d_0^X} H^0(K_1(\hat{\mathbf{Q}}\pi))$.

Next we check that the summand $H^0(K_1(\hat{\mathbf{Q}}\pi)_0)$ is factored out. In fact, if (S, α, u) is a type U anti-structure over a global field E then

$$H^0(K_1(S)) \to H^0(K_1(\hat{S}))$$

is an isomorphism by the Hasse Norm theorem and class field theory (cf. proof of (7.4)). Since the discriminant

(4.21) $d_0^X: L_0^K(S, \alpha, u) \to H^0(K_1(S))$

is also surjective, we see that $H^0(K_1(\hat{\mathbf{Q}}\pi)_0)$ is the image of $L_0^K(\mathbf{Q}\pi)_0$ under the map in (4.20).

The map d_0^X in (4.21) is surjective also for E finite, local or global and (S, α, u) type O. Therefore the image of

$$L_0^K(\hat{\mathbf{Z}}_{\text{odd}}\pi)_+ \oplus L_0^K(\mathbf{Q}\pi)_+$$

is the same as that of

$$H^0(K_1(\hat{\mathbf{Z}}_{\text{odd}}\pi)_+) \oplus H^0(K_1(\mathbf{Q}\pi)_+).$$

PROPOSITION 4.22. *The range of the δ-invariant is contained in $H^0(Wh(\hat{\mathbf{Q}}\pi)_+)/I$ where*

$$I = \operatorname{Im}\{H^0(Wh(\mathbf{Q}\pi)_+) \oplus H^0(Wh(\hat{\mathbf{Z}}_{\text{odd}}\pi)_+) \oplus L_0^K(\hat{\mathbf{Z}}_2\pi) \to H^0(Wh(\hat{\mathbf{Q}}\pi)_+\}.$$

Proof. Although $Wh(\mathbf{Q}\pi)$ does not split as in (4.16) we can define $Wh(\Lambda\pi)_+$ to be the image of $K_1(\Lambda\pi)_+$ under the natural projection (for $\Lambda = \hat{\mathbf{Z}}_{\text{odd}}$, $\mathbf{Q}$ or $\hat{\mathbf{Q}}$). Since δ factors through the induced map

(4.23) $$j: H^0(K_1(\hat{\mathbf{Q}}\pi))/L_0^K(\hat{\mathbf{Z}}\pi) \oplus L_0^K(\mathbf{Q}\pi) \to H^0(Wh(\hat{\mathbf{Q}}\pi))/L_0^K(\hat{\mathbf{Z}}\pi) \oplus L_0^K(\mathbf{Q}\pi),$$

we see from (4.18) and the fact that $L_0^K(\hat{S}, \hat{\alpha}, \hat{u}) = 0$ for type *Sp* antistructures that the range of δ is contained in a quotient of $H^0(Wh(\hat{\mathbf{Q}}\pi)_+)$. Thus the proof can be completed by checking that $j(\tilde{I}) = I$. But $j(\tilde{I}) \subseteq I$ clearly and

$$H^1(\pm\pi/\pi') \to H^1(K_1(\Lambda\pi)_+)$$

is injective for $\Lambda = \hat{\mathbf{Z}}_{\text{odd}}$, $\mathbf{Q}$ or $\hat{\mathbf{Q}}$ so that

$$H^0(K_1(\Lambda\pi)_+) \to H^0(Wh(\Lambda\pi)_+)$$

is onto.

5. The map from L_3^Y to L_3^P. This section contains the main calculational technique of the paper. First we ask when an element in $L_3^P(\mathbf{Z}\pi)$ is in the image of the natural map

(5.1) $\quad i^P:L_3^Y(\mathbf{Z}\pi) \to L_3^P(\mathbf{Z}\pi)$

and then we show how to relate $\delta(i^P(x))$ to its image $i_2^Y(x)$ where

(5.2) $\quad i_2^Y:L_3^Y(\mathbf{Z}\pi) \to L_3^Y(\hat{\mathbf{Z}}_2\pi)$.

The result is stated in (5.18).

The reason for the importance of this device will be clear in Section 7 where we prove that

(5.3) $\quad \mathrm{Res}:L_3^Y(\hat{\mathbf{Z}}_2\pi) \to \oplus\{L_3^Y(\hat{\mathbf{Z}}_2\rho):\rho \subsetneq \pi\}$

is injective on the image of i_2^Y for the groups $\pi = Q(8a, b)$ used in computing our L^P-surgery obstructions (see (7.7) and Section 9) to the existence of actions. It will turn out that we know enough about the surgery problems to give a formula (8.11) for the obstructions in $L_3^Y(\hat{\mathbf{Z}}_2\rho)$ when $\rho \subsetneq \pi$ so the injectivity of Res and the results of this section allow us to determine the obstruction in $L_3^P(\mathbf{Z}\pi)$.

We begin by considering the following diagram of exact sequences:

(5.4)
$$\begin{tikzcd}
 & H^0(Wh'(\mathbf{Z}\pi)) \arrow[d] & & \\
L_0^Y(\mathbf{Z}\pi \to \hat{\mathbf{Z}}_2\pi) \arrow[r] & L_3^Y(\mathbf{Z}\pi) \arrow[r, "i_2^Y"] \arrow[d, "i^K"] \arrow[dr, dashed, "i^P"] & L_3^Y(\hat{\mathbf{Z}}_2\pi) & \\
H^1(\tilde{K}_0(\mathbf{Z}\pi)) \arrow[r] & L_3^K(\mathbf{Z}\pi) \arrow[r] & L_3^P(\mathbf{Z}\pi) \arrow[r, "\sigma_*"] & H^0(\tilde{K}_0(\mathbf{Z}\pi))
\end{tikzcd}$$

Since $H^1(Wh'(\mathbf{Z}\pi)) = 0$ [**48**] it follows that i^K is surjective and so the image of i^P is just ker σ_*. For the same reason,

(5.5) $\quad H^0(Wh'(\hat{\mathbf{Z}}_2\pi))/H^0(Wh'(\mathbf{Z}\pi)) \to H^0(Wh'(\hat{\mathbf{Z}}_2\pi)/Wh'(\mathbf{Z}\pi))$

is an isomorphism. From the cohomology sequence associated to

(5.6) $\quad 0 \to Wh'(\hat{\mathbf{Z}}_2\pi)/Wh'(\mathbf{Z}\pi) \to K_1(\mathbf{Z}\pi \to \hat{\mathbf{Z}}_2\pi) \to \tilde{K}_0(\mathbf{Z}\pi) \to 0$

we obtain a homomorphism

(5.7) $\quad d^*:H^1(\tilde{K}_0(\mathbf{Z}\pi)) \to L_3^Y(\hat{\mathbf{Z}}_2\pi)/H^0(Wh'(\mathbf{Z}\pi))$

defined by the composition:

$$(5.8)\qquad \begin{tikzcd}[column sep=small]
& H^0(Wh'(\hat{\mathbf{Z}}_2\pi)/Wh'(\mathbf{Z}\pi)) & \\
H^1(\tilde{K}_0(\mathbf{Z}\pi)) \arrow[ur] & & H^0(Wh'(\hat{\mathbf{Z}}_2\pi))/H^0(Wh'(\mathbf{Z}\pi)) \arrow[ul, "\approx"'] \arrow[d] \\
L_3^Y(\hat{\mathbf{Z}}_2\pi)/H^0(Wh'(\mathbf{Z}\pi)) \arrow[u, "d^*"'] & & H^0(Wh'(\hat{\mathbf{Z}}_2\pi))/H^0(Wh'(\mathbf{Z}\pi)) + L_0^K(\hat{\mathbf{Z}}_2\pi) \arrow[dl, "\approx"] \\
& \ker(L_3^Y(\hat{\mathbf{Z}}_2\pi) \to L_3^K(\hat{\mathbf{Z}}_2\pi))/H^0(Wh'(\mathbf{Z}\pi)) \arrow[ul] &
\end{tikzcd}$$

PROPOSITION 5.9. *The map i^P induces an isomorphism*:

$$\underline{i}^P: \frac{\operatorname{Im}(L_3^Y(\mathbf{Z}\pi) \to L_3^Y(\hat{\mathbf{Z}}_2\pi))}{H^0(Wh'(\mathbf{Z}\pi)) + d^*(H^1(\tilde{K}_0(\mathbf{Z}\pi)))}$$

$$\xrightarrow{\approx} \frac{\ker(L_3^P(\mathbf{Z}\pi) \to H^0(\tilde{K}_0(\mathbf{Z}\pi)))}{\operatorname{Im}(L_0^Y(\mathbf{Z}\pi \to \hat{\mathbf{Z}}_2\pi) \to L_3^P(\mathbf{Z}\pi))}.$$

Proof. From (5.4) it is clear that i^P induces a well defined homomorphism $\underline{i}^P$ from

$$\operatorname{Im}(L_3^Y(\mathbf{Z}\pi) \to L_3^Y(\hat{\mathbf{Z}}_2\pi))/H^0(Wh'(\mathbf{Z}\pi))$$

onto the right-hand side. It remains to be seen that

$$d^*H^1(\tilde{K}_0(\mathbf{Z}\pi))$$

is the kernel of this map. If U denotes the subgroup of $K_1(\hat{\mathbf{Z}}_2\pi)$ generated by Y and $K_1(\mathbf{Z}\pi)$, then the Rothenberg sequences give commutative ladders (cf. [**37**])

$$(5.10)\qquad \begin{tikzcd}[column sep=small]
\dots \arrow[r] & L_0^P(\mathbf{Z}\pi) \arrow[r] \arrow[d] & H^1(\tilde{K}_0(\mathbf{Z}\pi)) \arrow[r] \arrow[d, "d^*"] \arrow[dr, phantom, "(a)"] & L_3^K(\mathbf{Z}\pi) \arrow[r] \arrow[d] & L_3^P(\mathbf{Z}\pi) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & L_0^K(\hat{\mathbf{Z}}_2\pi) \arrow[r] & H^0(Wh'(\hat{\mathbf{Z}}_2\pi)/Wh'(\mathbf{Z}\pi)) \arrow[r] & L_3^U(\hat{\mathbf{Z}}_2\pi) \arrow[r] & L_3^K(\hat{\mathbf{Z}}_2\pi) \arrow[r] & \dots
\end{tikzcd}$$

and

$$(5.11)\qquad \begin{tikzcd}[column sep=small]
\dots \arrow[r] & L_0^Y(\mathbf{Z}\pi \to \hat{\mathbf{Z}}_2\pi) \arrow[r] \arrow[d] & L_3^Y(\mathbf{Z}\pi) \arrow[r] \arrow[d] \arrow[dr, phantom, "(b)"] & L_3^Y(\hat{\mathbf{Z}}_2\pi) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & L_0^{K,U}(\mathbf{Z}\pi \to \hat{\mathbf{Z}}_2\pi) \arrow[r] & L_3^K(\mathbf{Z}\pi) \arrow[r] & L_3^U(\hat{\mathbf{Z}}_2\pi) \arrow[r] & \dots
\end{tikzcd}$$

It now follows from (5.5) that the natural maps

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$$\begin{tikzcd} H^0(Wh'(\hat{\mathbf{Z}}_2\pi))/H^0(Wh'(\mathbf{Z}\pi)) + L_0^K(\hat{\mathbf{Z}}_2\pi) \arrow[r, "\approx"] \arrow[d, "\approx"] & \ker(L_3^U(\hat{\mathbf{Z}}_2\pi) \to L_3^K(\hat{\mathbf{Z}}_2\pi)) \\ \ker(L_3^Y(\hat{\mathbf{Z}}_2\pi) \to L_3^K(\hat{\mathbf{Z}}_2\pi))/H^0(Wh'(\mathbf{Z}\pi)) & \end{tikzcd}$$

are isomorphisms, so the result $\mathrm{Im}\, d^* = \ker \underline{i}^P$ is immediate from (5.8), (5.10 (a)) and (5.11 (b)).

For use in the remainder of this section, it is convenient to introduce the notation

$$\widetilde{L}_3^P(\mathbf{Z}\pi) = \ker(\chi:L_3^P(\mathbf{Z}\pi) \to L_3^K(\hat{\mathbf{Z}}_2\pi))$$

for the domain of the δ-invariant and similarly to let

$$\widetilde{L}_3^Y(\mathbf{Z}\pi) = \ker(i_2^K:L_3^Y(\mathbf{Z}\pi) \to L_3^K(\hat{\mathbf{Z}}_2\pi)).$$

Since $\chi \cdot i^P = i_2^K$, it is clear that $\underline{\underline{i}}^P$ induces an isomorphism also from the subgroup

$$\mathrm{Im}(\widetilde{L}_3^Y(\mathbf{Z}\pi) \to \widetilde{L}_3^Y(\hat{\mathbf{Z}}_2\pi))/H^0(Wh'(\mathbf{Z}\pi)) + d^*H^1(\widetilde{K}_0(\mathbf{Z}\pi))$$

to the corresponding subgroup

$$\ker(\widetilde{L}_3^P(\mathbf{Z}\pi) \to H^0(\widetilde{K}_0(\mathbf{Z}\pi)))/\mathrm{Im}(L_0^Y(\mathbf{Z}\pi \to \hat{\mathbf{Z}}_2\pi) \to \widetilde{L}_3^P(\mathbf{Z}\pi))$$

on the right-hand side of (5.9). We will now show that

$$\mathrm{Im}(L_0^Y(\mathbf{Z}\pi \to \hat{\mathbf{Z}}_2\pi) \to L_3^P(\mathbf{Z}\pi)) = 0$$

so that $\underline{\underline{i}}^P$ actually expresses the L^P-obstructions (in $\ker \sigma_*$) in terms of L^Y.

Consider the homomorphism:

(5.12) $$\delta^Y:\widetilde{L}_3^Y(\mathbf{Z}\pi) \to H^0(Wh(\hat{\mathbf{Q}}\pi))/L_0^K(\hat{\mathbf{Z}}\pi) \oplus L_0^K(\mathbf{Q}\pi)$$

induced by the additive relation in diagram (3.7):

(5.13) $$\begin{tikzcd} L_3^Y(\mathbf{Z}\pi) \arrow[r] & L_3^Y(\hat{\mathbf{Z}}\pi) \oplus L_3^Y(\mathbf{Q}\pi) & \\ & H^0(Wh(\hat{\mathbf{Z}}\pi)) \oplus H^0(Wh(\mathbf{Q}\pi)) \arrow[u] \arrow[r] & H^0(Wh(\hat{\mathbf{Q}}\pi)). \end{tikzcd}$$

PROPOSITION 5.14. *For $x \in \ker(L_3^Y(\mathbf{Z}\pi) \to L_3^K(\hat{\mathbf{Z}}_2\pi))$, there is the relation*

$$\delta^Y(x) = \delta(i^P(x)).$$

Proof. The argument is a variant of arguments used in [**21**, Section 2] and is based on the algebraic surgery theory of Ranicki [**36**]. Let (C, ψ) be a free, based, quadratic Poincaré chain complex representing

$$x \in \ker(L_3^Y(\mathbf{Z}\pi) \to L_3^K(\hat{\mathbf{Z}}_2\pi)).$$

Then $(C \otimes \hat{\mathbf{Z}}, \psi \otimes 1)$ and $(C \otimes \mathbf{Q}, \psi \otimes 1)$ represent the images of x in $L_3^Y(\hat{\mathbf{Z}}\pi)$ and $L_3^Y(\mathbf{Q}\pi)$. Since

$$\ker(L_3^Y(\mathbf{Z}\pi) \to L_3^K(\hat{\mathbf{Z}}\pi) \oplus L_3^K(\mathbf{Q}\pi)) = \ker(L_3^Y(\mathbf{Z}\pi) \to L_3^K(\hat{\mathbf{Z}}_2\pi))$$

from (4.5) and (3.3), it follows that there exist quadratic algebraic cobordisms $\hat{D}$ and D^0 (we will suppress mention of the $\mathbf{Z}/2$-hyperhomology classes [**36**, Section 3] from now on) such that,

$$\partial\hat{D} = C \otimes \hat{\mathbf{Z}} - \hat{C}, \quad \partial D^0 = C \otimes \mathbf{Q} - C^0$$

with $\hat{C}$ and C^0 acyclic complexes over $\hat{\mathbf{Z}}\pi$ and $\mathbf{Q}\pi$ respectively. Let $\Delta(\hat{C})$ and $\Delta(C^0)$ be their Whitehead torsion invariants in $H^0(Wh(\hat{\mathbf{Z}}\pi))$ and $H^0(Wh(\mathbf{Q}\pi))$. According to [**21**, 2.4] the difference of their images in $H^0(Wh(\hat{\mathbf{Q}}\pi))$ represents $\delta^Y(x)$:

$$\delta^Y([C, \psi]) = \Delta(\hat{C} \otimes_{\hat{\mathbf{Z}}} \hat{\mathbf{Q}}) - \Delta(C^0 \otimes_{\mathbf{Q}} \hat{\mathbf{Q}}).$$

On the other hand we can form the union [**36**, p. 135] of $\hat{D} \otimes_{\hat{\mathbf{Z}}} \hat{\mathbf{Q}}$ and $\hat{D}^0 \otimes_{\mathbf{Q}} \hat{\mathbf{Q}}$ along $C \otimes_{\mathbf{Z}} \hat{\mathbf{Q}}$ to get

$$D = \hat{D} \otimes_{\hat{\mathbf{Z}}} \hat{\mathbf{Q}} \cup D^0 \otimes_{\mathbf{Q}} \hat{\mathbf{Q}}$$

with

$$\partial D = \hat{C} \otimes_{\hat{\mathbf{Z}}} \hat{\mathbf{Q}} - C^0 \otimes_{\mathbf{Q}} \hat{\mathbf{Q}}.$$

This boundary is acyclic so $(D, \partial D)$ represents an element in $L_0^K(\hat{\mathbf{Q}}\pi)$ which maps to $i^P(x)$ in $L_3^P(\mathbf{Z}\pi)$. This follows from the cobordism interpretation [**37**] of the Mayer-Vietoris sequence (3.8) for calculating $L_3^P(\mathbf{Z}\pi)$. Finally, the cobordism interpretation of the Rothenberg sequence implies that

$$d_0: L_0^K(\hat{\mathbf{Q}}\pi) \to H^0(Wh(\hat{\mathbf{Q}}\pi))$$

can be calculated from

$$d_0([D, \partial D]) = \Delta(\partial D) = \Delta(\hat{C} \otimes_{\hat{\mathbf{Z}}} \hat{\mathbf{Q}}) - \Delta(C^0 \otimes_{\mathbf{Q}} \hat{\mathbf{Q}}).$$

COROLLARY 5.15. *Under the map induced by* i^P,

$$\operatorname{Im}(L_0^Y(\mathbf{Z}\pi \to \hat{\mathbf{Z}}_2\pi) \to L_3^P(\mathbf{Z}\pi)) = 0.$$

Proof. From (5.4),

$$\operatorname{Im}(L_0^Y(\mathbf{Z}\pi \to \hat{\mathbf{Z}}_2\pi) \to L_3^Y(\mathbf{Z}\pi)) = \ker i_2^Y \subseteq \tilde{L}_3^Y(\mathbf{Z}\pi)$$

so that for any

$$x \in \operatorname{Im}(L_0^Y(\mathbf{Z}\pi \to \hat{\mathbf{Z}}_2\pi) \to L_3^Y(\mathbf{Z}\pi)),$$

we have $\delta^Y(x) = \delta(i^P(x))$. However, if l is an odd prime,

$$H^0(Wh(\hat{\mathbf{Z}}_l\pi)_-) \to L_3^Y(\hat{\mathbf{Z}}_l\pi)$$

and

$$H^0(Wh(\mathbf{Q}\pi)_-) \to L_3^Y(\mathbf{Q}\pi)$$

are both onto (note that $L^S \to L^K$ is zero for finite fields of odd characteristic), so that $\delta^Y(x)$ is represented by an element in $H^0(Wh(\hat{\mathbf{Q}}\pi)_-)$. But from (4.22), the range of the δ-invariant is a quotient of $H^0(Wh(\hat{\mathbf{Q}}\pi)_+)$. Therefore

$$\delta(i^P(x)) = 0$$

and since δ is injective (4.15), $i^P(x) = 0$ also.

Remark 5.16. The argument just given also proves that

$$\delta(i^P(x)) = \hat{i} \circ \delta_2^Y(x)$$

for all $x \in \tilde{L}_3^Y(\mathbf{Z}\pi) = \ker i_2^K$, where δ_2^Y is the composition:

(5.17) $$\tilde{L}_3^Y(\mathbf{Z}\pi) \to \tilde{L}_3^Y(\hat{\mathbf{Z}}_2\pi) \overset{\approx}{\leftarrow} H^0(Wh'(\hat{\mathbf{Z}}_2\pi))/L_0^K(\hat{\mathbf{Z}}_2\pi) \to H^0(Wh(\hat{\mathbf{Q}}_2\pi)_+)/L_0^K(\hat{\mathbf{Z}}_2\pi)$$

and

$$\hat{i}:H^0(Wh(\hat{\mathbf{Q}}_2\pi)_+)/L_0^K(\hat{\mathbf{Z}}_2\pi) \to H^0(Wh(\hat{\mathbf{Q}}\pi)_+)/I$$

maps to the range of the δ-invariant (4.22). Since $\hat{i}$ is not injective, we suggested in an earlier version of this paper that the invariant δ_2^Y should be investigated further. Recently it has been shown that i_2^K and δ_2^Y detect $L_3^Y(\hat{\mathbf{Z}}_2\pi)$ and that δ_2^Y is a "semi-torsion" invariant.

We can now state the main result of this section.

PROPOSITION 5.18. *The map* i^P *induces an isomorphism*:

$$\underline{\underline{i}}^P:\ker(L_3^P(\mathbf{Z}\pi) \to H^0(\tilde{K}_0(\mathbf{Z}\pi))) \cong \frac{\mathrm{Im}(L_3^Y(\mathbf{Z}\pi) \to L_3^Y(\hat{\mathbf{Z}}_2\pi))}{H^0(Wh'(\mathbf{Z}\pi)) + d^*(H^1(\hat{K}_0(\mathbf{Z}\pi)))}.$$

Proof. See (5.9) and (5.15).

6. Splitting of Mackey functors. From Dress' induction theorem [**11**] we know that the groups $L_n(\mathbf{Z}\pi)$ are detected by restrictions to the lattice of 2-hyperelementary subgroups of π. Since restriction for normal maps is just passing to a covering, this way of describing surgery obstructions is very natural geometrically.

The groups $L_n(\mathbf{Z}\pi)$ have only 2-primary torsion (and their free part is described by signature invariants) so there is no harm in replacing $L_n(\mathbf{Z}\pi)$ by its localization at 2,

$$L_n(\mathbf{Z}\pi)_{(2)} = L_n(\mathbf{Z}\pi) \otimes \mathbf{Z}_{(2)}.$$

In this section we consider the problem of splitting 2-local Mackey functors in a functorial fashion. The splittings exist in general for finite groups but are particularly valuable to us (and transparent) when π is 2-hyperelementary.

Later in the section we restrict to 2-hyperelementary groups but we begin with general finite groups. First recall Dress' definition of a Mackey functor:

Definition 6.1. A *2-local Mackey functor* is a bifunctor $\mathcal{M} = (\mathcal{M}^*, \mathcal{M}_*)$ from the category of finite groups with monomorphisms to the category of $\mathbf{Z}_{(2)}$-modules such that $\mathcal{M}^*(\pi) = \mathcal{M}_*(\pi)$ and

(i) $\mathcal{M}_*$, $\mathcal{M}^*$ send inner automorphisms to the identity

(ii) for any isomorphism $f:\pi \to \pi'$,

$$\mathcal{M}_*(f) \circ \mathcal{M}^*(f) = \mathrm{id}$$

(iii) if $\rho, \rho' \subseteq \pi$ then

$$\mathcal{M}(\rho') \xrightarrow{\mathcal{M}_*} \mathcal{M}(\pi) \xrightarrow{\mathcal{M}^*} \mathcal{M}(\rho)$$

is the sum over double cosets $\{\rho g \rho'\} \subseteq \pi$ of composites

$$\mathcal{M}(\rho') \xrightarrow{\mathcal{M}^*} \mathcal{M}(g^{-1}\rho g \cap \rho') \xrightarrow{C_g} \mathcal{M}(\rho \cap g\rho' g^{-1}) \xrightarrow{\mathcal{M}_*} \mathcal{M}(\rho)$$

where C_g is induced by "conjugation with g".

If $i:\rho \to \pi$ is a monomorphism, $\mathcal{M}_*(i)$ is denoted i_* or Ind_ρ^π and $\mathcal{M}^*(i)$ is denoted i^* or Res_π^ρ. Usually the covariant structure is defined also for maps which are not monomorphisms.

One important example of a Mackey functor is the 2-local *Burnside ring* which we denote by $\Omega(\pi)$. It is the Grothendieck ring (tensored with $\mathbf{Z}_{(2)}$) of finite π-sets. This functor is even a so called *Green functor*: each $\Omega(\pi)$ is a commutative ring with 1 and for any inclusion $i:\rho \to \pi$,

(6.2) $\quad i^*(xy) = i^*(x)i^*(y), \quad xi_*(y) = i_*(i^*(x)y).$

Any Mackey functor $\mathcal{M}$ is a Green module over Ω: let $x \in \Omega$ be represented by the π-set $[\pi/\rho]$ then

$$x \cdot a = I_*(I^*(a)) \quad \text{for any } a \in \mathcal{M}(\pi).$$

Each $\mathcal{M}(\pi)$ is therefore an $\Omega(\pi)$ module and the identities:

$$\mathcal{M}_*(i)(i^*(x)b) = x\mathcal{M}_*(i)(b)$$

$$\mathcal{M}_*(i)(yi^*(a)) = i_*(y) \cdot a$$

are satisfied for all $x \in \Omega(\pi)$, $y \in \Omega(\rho)$ and $a \in \mathcal{M}(\pi)$, $b \in \mathcal{M}(\rho)$. Any natural transformation $\theta:\mathcal{M} \to \mathcal{N}$ of Mackey functors is a homomorphism of Green modules over Ω.

We now recall some facts (mainly due to Dress) about $\Omega(\pi)$. The reader is referred to [**10**] for more details. Given a virtual π-set X and a conjugacy class $\rho \subset \pi$, set

$$\phi_\gamma(X) = \#(X^\gamma).$$

This is an additive and multiplicative operation and defines an injection homomorphism

$$\phi:\Omega(\pi) \to \prod \mathbf{Z}_{(2)}$$

where the product ranges over the set $C(\pi)$ of conjugacy classes of subgroups. The cokernel of ϕ has order

$$\prod\{\,|W_2(\gamma)|:\gamma \in C(\pi)\,\}$$

where $W_2(\gamma)$ is the Sylow 2-subgroup of the Weyl group $N(\gamma)/\gamma$. More precisely, the image of ϕ can be described by the following set of congruences, one for each $\gamma \in C(\pi)$:

$$(6.3) \qquad \sum_\tau n(\tau, \gamma)\phi_\tau(X) \equiv 0 \ (\mathrm{mod}\ |W_2(\gamma)|\,).$$

The sum ranges over $N_2(\gamma)$ conjugacy classes of groups $\gamma \lhd \tau$ with τ/γ a cyclic 2-group. ($N_2(\gamma) \subset N(\gamma)$ is the pre-image of $W_2(\gamma)$.) The integers $n(\tau, \gamma) \in \mathbf{Z}_{(2)}$ which appear in (6.3) are given by the formula:

$$(6.4) \qquad n(\tau, \gamma) = |W_2(\gamma):N(\tau/\gamma)|\,|(\tau/\gamma)^\times|.$$

The congruences (6.3) are the 2-local version of the usual congruences; they are satisfied by elements in $\Omega(\pi)$ by the counting of orbits in $X^\gamma/W_2(\gamma)$ in the orthogonality relations and go back to Burnside.

The prime ideal spectrum for $\Omega(\pi)$ was considered by Dress in [**10**]. Each maximal prime ideal has the form

$$q(\gamma, 2) = \{x \in \Omega(\pi):\phi_\gamma(x) \equiv 0 \ (\mathrm{mod}\ 2)\,\}.$$

Let $O(\gamma)$ denote the smallest normal subgroup of γ with $\gamma/O(\gamma)$ a 2-group. Then $q(\gamma, 2) = q(\gamma', 2)$ if and only if $O(\gamma)$ is conjugate to $O(\gamma')$.

Each connected component in Spec $\Omega(\pi)$ contains precisely one maximal ideal, so by a standard fact from commutative algebra $\Omega(\pi)$ is isomorphic to the product of its localizations at the maximal prime ideals

$$(6.5) \qquad \Omega(\pi) \cong \prod \Omega(\pi)_q.$$

We proceed to make this decomposition explicit for 2-hyperelementary groups; in fact for the remainder of this section all groups considered will be 2-hyperelementary unless otherwise indicated. Such a group can be written $\pi = \mathbf{Z}/n \rtimes \sigma$ with n odd and σ a 2-Sylow subgroup of π. Each conjugacy class of subgroups contain one of the form $\mathbf{Z}/d \rtimes \rho$ with $\rho \subset \sigma$ and

$$\mathbf{Z}/d \rtimes \rho \sim \mathbf{Z}/l \rtimes \rho'$$

if and only if

$$d = l \quad \text{and} \quad \rho \sim \rho'.$$

We denote by $\mathcal{O}_{\text{odd}}(\pi)$ the set of conjugacy classes of proper subgroups of odd index in π. We notice that $O(\pi) = \mathbf{Z}/n$ is the subgroup consisting of all elements of odd order. Then the set $\mathcal{O}_{\text{odd}}(\pi)$ is in one to one correspondence with subgroups of $O(\pi)$, or with the divisors of $n = |O(\pi)|$, and

$$q(\mathbf{Z}/d \rtimes \tau, 2) = q(\mathbf{Z}/d, 2)$$

for all $\tau \subset \sigma$.

The congruences (6.3) become particularly simple for 2-hyperelementary groups: for each divisor $d|n$,

$$(6.6) \qquad \sum n(\mathbf{Z}/d \rtimes \tau, \mathbf{Z}/d \rtimes \gamma)\phi_{\mathbf{Z}/d\rtimes\tau}(X) \equiv 0 \ (\text{mod}|N_\sigma(\gamma)/\gamma|\,)$$

where $\gamma \subset \sigma$ and the sum ranges over $N_\sigma(\gamma)$ conjugacy classes with τ/γ cyclic. Moreover,

$$n(\mathbf{Z}/d \rtimes \tau, \mathbf{Z}/d \rtimes \gamma) = n(\tau, \gamma)$$

as one may easily check from (6.4).

For $d|n$ let

$$\phi_d:\Omega(\pi) \rightarrow \prod_{C(\sigma)} \mathbf{Z}_{(2)}$$

be the map which to $X \in \Omega(\pi)$ associates

$$\phi_d(X) = (\phi_{\mathbf{Z}/d\rtimes\tau}(X)\,)_{\tau\in C(\sigma)}.$$

Since the congruences (6.6) for the various divisors d are independent

$$(6.7) \qquad \Omega(\pi) \cong \prod \phi_d(\Omega(\pi)\,),$$

and since $n(\mathbf{Z}/d \rtimes \tau, \mathbf{Z}/d \rtimes \gamma) = n(\tau, \gamma)$

$$\phi_d\Omega(\pi) \cong \Omega(\sigma).$$

The decomposition (6.7) is (of course) a special case of (6.5). Indeed

$$\phi_d:\Omega(\pi) \rightarrow \prod_{C(\sigma)} \mathbf{Z}_{(2)}$$

factors over the localization $\Omega(\pi)_q$ where $q = q(\mathbf{Z}/d, 2)$ because

$$\phi_d(\Omega(\pi) - q(\mathbf{Z}/d, 2)\,) \subseteq \prod \mathbf{Z}_{(2)}^{\times}$$

and it defines an isomorphism

$$(6.8) \qquad \phi_d:\Omega(\pi)_q \xrightarrow{\cong} \Omega(\sigma) \subset \prod \mathbf{Z}_{(2)}.$$

To see that ϕ_d localized at q is injective, notice that there exists an element $\beta \in \Omega(\pi) - q$ with $\phi_l(\beta) = 0$ for $l \neq d$; then $\phi(x\beta) = 0$ when $\phi_d(x) = 0$ so $x\beta = 0$ in $\Omega(\pi)$ and $x = 0$ in $\Omega(\pi)_q$.

The equivalent decompositions (6.5) and (6.7) imply that every 2-local Mackey functor $\mathcal{M}(\pi)$ will also decompose into summands: the orthogonal idempotents given by (6.7) split the $\Omega(\pi)$-module $\mathcal{M}(\pi)$. For our applications below it is necessary to describe these idempotents explicitly in terms of the transfer structure $(\mathcal{M}_*(i), \mathcal{M}^*(i))$.

Consider $\rho = \mathbf{Z}/d \rtimes \sigma \in \mathcal{O}_{\text{odd}}(\pi)$. The composite

$$\Omega(\rho) \xrightarrow{i_*} \Omega(\pi) \xrightarrow{i^*} \Omega(\rho)$$

is multiplication with $i^*[\pi/\rho]$. Indeed i^* is surjective (e.g. by (6.6)) and $\pi \times_\rho X \cong \pi/\rho \times X$ when X is π-set, so

$$i^* \circ i_*(i^*(x)) = i^*([\pi/\rho] \cdot x) = i^*[\pi/\rho] \cdot i^*(x).$$

Moreover, we claim that $i^*([\pi/\rho])$ is invertible in $\Omega(\rho)$. To see this it suffices to check that

$$i^*[\pi/\rho] \in \Omega(\rho) - q(\gamma, 2) \quad \text{for each } \gamma \subset \rho.$$

But for each subgroup γ of ρ,

$$(\pi/\rho)^\gamma \neq 0$$

so that $\phi_\gamma(\pi/\rho) = \#(\pi/\rho)^\gamma$ is odd, and the claim follows.

Let

$$y_\rho = i^*([\pi/\rho])^{-1}.$$

Then $i_*(y_\rho) \in \Omega(\pi)$ is an idempotent

$$E_d = i_*(y_\rho).$$

It is convenient to use the same letter for the induced projection operator,

$$E_d:\Omega(\pi) \to \Omega(\pi), \quad E_d(x) = i_*(y_\rho) \cdot x.$$

We now determine its character. Let

$$\gamma = \mathbf{Z}/l \rtimes \gamma_2.$$

If γ is not a subgroup of ρ,

$$\phi_\gamma(i_*(y_\rho)) = 0$$

and this happens when $d \nmid l$. If $d | l$,

$$\phi_\gamma(i_*(y_\rho)) = \phi_\gamma(i^*i_*(y_\rho)) = \phi_\gamma(1) = 1,$$

so altogether

$$\phi_\gamma(i_*(y_\rho)) = \begin{cases} 0 \text{ if } d \nmid l \\ 1 \text{ if } d | l \end{cases}$$

when $\rho = \mathbf{Z}/d \rtimes \sigma$. We see that E_d corresponds (under ϕ) to the idempotent

$$E_d: \prod_{l|n} \prod_{C(\sigma)} \mathbf{Z}_{(2)} \to \prod_{l|n} \prod_{C(\sigma)} \mathbf{Z}_{(2)}$$

which projects onto

$$\prod_{l|d} \prod_{C(\sigma)} \mathbf{Z}_{(2)}.$$

There is a corresponding family of orthogonal idempotents, which we index by $\mathcal{O}_{\text{odd}}(\pi)$. For $\rho = \mathbf{Z}/d \rtimes \sigma$ define

(6.9) $\quad E_\rho = E_d \circ \prod \{ (1 - E_l): l < d, l|d \} \in \Omega(\pi).$

Then $E_\rho E_{\rho'} = 0$ if $\rho \neq \rho'$ in $\mathcal{O}_{\text{odd}}(\pi)$ (i.e., $\rho \not\sim \rho'$) and $\sum E_\rho = 1$. The projection operator E_ρ corresponds under ϕ to projection onto the factor

$$\prod_{C(\sigma)} \mathbf{Z}_{(2)}$$

indexed by $d = |O(\rho)|$, so we get

(6.10) $\quad E_\rho\Omega(\pi) = \phi_d(\Omega(\pi)) = \Omega(\pi)_{q(\mathbf{Z}/d,2)}.$

Definition 6.11. Let $\mathcal{M}$ be a 2-local Mackey functor and $d|n$, $n = |O(\pi)|$. The *d-component* of $\mathcal{M}(\pi)$ is the subgroup

$$\mathcal{M}(\pi)(d) = E_\rho\mathcal{M}(\pi) = \text{Ind}_\rho^\pi(E_\rho \cdot \mathcal{M}(\rho))$$

where

$$\rho = \mathbf{Z}/d \rtimes \sigma \in \mathcal{O}_{\text{odd}}(\pi).$$

Since

$$\text{Res}_\pi^\rho(E_\rho) = E_\rho \in \Omega(\rho)$$

it follows that

$$\text{Res}_\pi^\rho(\mathcal{M}(\pi)(d)) = \mathcal{M}(\rho)(d) \quad \text{where } \rho = \mathbf{Z}/d \rtimes \sigma.$$

Thus it really suffices to consider the top component. It has the following alternative description which is often convenient to use:

(6.12) $\quad \mathcal{M}(\pi)(n) = \text{Ker}\{\text{Res}: \mathcal{M}(\pi) \to \prod \{\mathcal{M}(\rho): \rho \neq \pi, \rho \in \mathcal{O}_{\text{odd}}(\pi)\}\}.$

We summarize the splitting results for 2-hyperelementary groups.

PROPOSITION 6.13. *Let $\mathcal{M}$ be a 2-local Mackey functor. Then*

(i) $\mathcal{M}(\pi) = \sum^{\oplus} \{\mathcal{M}(\pi)(d) : d \mid |O(\pi)|\}$
(ii) *for any* $\rho \in \mathcal{O}_{\text{odd}}(\pi)$,

$$\mathrm{Res}^{\rho}_{\pi}(\mathcal{M}(\pi)(d)) = O \quad \text{if } d \nmid |O(\rho)|, \text{ and}$$

(iii) *if* $d \mid |O(\rho)|$ *for some* $\rho \in \mathcal{O}_{\text{odd}}(\pi)$, $\mathrm{Res}^{\rho}_{\pi}$ *is injective on* $\mathcal{M}(\pi)(d)$) *and has image* $\mathcal{M}(\rho)(d)$.

PROPOSITION 6.14. *Let* $\mathcal{M}$ *be a* 2-*local Mackey functor and* $\tau \subset \pi$ *be a normal subgroup of* 2-*power index. Then for any* $d \mid |O(\pi)| = |O(\tau)|$:

$$\text{(i) } \mathrm{Ind}^{\pi}_{\tau}(\mathcal{M}(\tau)(d)) \subseteq \mathcal{M}(\pi)(d)$$

and

$$\text{(ii) } \mathrm{Res}^{\tau}_{\pi}(\mathcal{M}(\pi)(d)) \subseteq \mathcal{M}(\tau)(d).$$

The results above will be applied to K or L-theory in the next sections. Since these are actually functors on orders (not just groups and monomorphisms), we can obtain a more easily computable description (6.22) of the components by using a modification of Fröhlich's description [13] of the class group.

Let $S \subseteq \mathbf{Q}$ be a subring and write

$$S\zeta_n = S \otimes \mathbf{Z}[\zeta_n].$$

We want to consider functors defined on a suitable category $\mathcal{C}_S$ containing Λ-orders in semi-simple K-algebras where Λ is one of the rings $S\zeta_n$ as n varies (or $\hat{S}_l\zeta_n$ for some $l \nmid n$) and K is its quotient field. Note that such an order is invariant (but not fixed) under the action of $\mathrm{Gal}(K/\mathbf{Q})$. It will be clear from our results exactly what objects are used so we will not need to formalize the definition of $\mathcal{C}_S$.

The most important examples of these Λ-orders for our purposes are $S[\mathbf{Z}/n \rtimes \sigma]$ and $S\zeta_d[\sigma]^t$ where as usual $d|n$ are odd and σ is a 2-group (acting on $S\zeta_n$ via a homomorphism $\sigma \to \mathrm{Gal}(\mathbf{Q}\zeta_n/\mathbf{Q})$). Let

$$i_d : S\zeta_d[\sigma]^t \to S\zeta_n[\sigma]^t$$

denote the inclusion map and

$$\mathrm{pr}(\chi) : S[\mathbf{Z}/n \times S] \to S\zeta_n[\sigma]^t$$

the projection map induced by a complex character χ of $\mathbf{Z}/n$. If we fix a generator T of $\mathbf{Z}/n$ and denote by χ_d the character such that $\chi_d(T) = \zeta_d$ then we obtain projection maps $\mathrm{pr}_d = \mathrm{pr}(\chi_d)$ indexed by $d|n$. If $\mathcal{M}$ is a functor defined on $\mathcal{C}_S$ then there is a pairing

$$(6.15) \quad \langle , \rangle : R(\mathbf{Z}/n) \times \mathcal{M}(S[\mathbf{Z}/n \rtimes \sigma]) \to \mathcal{M}(S\zeta_n[\sigma]^t)$$

where $R(\mathbf{Z}/n)$ denotes the complex character ring. For

$$\chi \in R(\mathbf{Z}/n) \quad \text{and} \quad a \in \mathcal{M}(S(\mathbf{Z}/n \rtimes \sigma))$$

set

$$\langle \chi, a \rangle = \mathrm{pr}(\chi)_*(a)$$

and extend by linearity to sums of characters. Note that $R(\mathbf{Z}/n)$ is also a Green module over the Burnside ring $\Omega(\mathbf{Z}/n)$.

We now show how to identify $\mathscr{M}(S\pi)(n)$ assuming that $\mathscr{M}$ is a *2-local Mackey functor on* $\mathscr{C}_S$. This means:

(6.16)

(i) $\mathscr{M}$ is an additive covariant functor from $\mathscr{C}_S$ to $\mathbf{Z}_{(2)}$-modules

(ii) There is a contravariant map $I^*:\mathscr{M}(\mathscr{B}) \to \mathscr{M}(\mathscr{A})$ defined whenever $\mathscr{A} \subseteq \mathscr{B}$ are in $\mathscr{C}_S$ and $\mathscr{B}$ is $\mathscr{A}$-projective.

(iii) $\mathscr{M}(S\pi)$ is a Mackey functor on subgroups of π using (I_*, I^*) and I^* is natural with respect to morphisms of $S\pi$ induced by group homomorphisms $\pi \to \pi'$.

(iv) For $\pi = \mathbf{Z}/n \rtimes \sigma$, $\mathscr{N}(\pi) = \mathscr{M}(S\zeta_n[\sigma]^t)$ is a Mackey functor on subgroups $\rho \subseteq \pi$ with $\rho \in \mathscr{O}_{\text{odd}}(\pi)$ and $\bar{n}|m|n$ (i.e., when the odd part $m = |O(\rho)|$ has the same prime divisors as n). The projections

$$\mathrm{pr}_n:S(\mathbf{Z}/n \rtimes \sigma) \to S\zeta_n[\sigma]^t$$

of orders induce a natural transformation of Mackey functors on this subcategory.

PROPOSITION 6.17. *Let $\mathscr{M}$ be a 2-local Mackey functor on $\mathscr{C}_S$ with $1/n \in S$. Then for $\pi = \mathbf{Z}/n \rtimes \sigma$ the composite*

$$\mu:\mathscr{M}(S\pi)(n) \subseteq \mathscr{M}(S\pi) \xrightarrow{\Sigma(\mathrm{pr}_d)_*} \sum_{\bar{n}|d|n}^{\oplus} \mathscr{M}(S\zeta_d[\sigma]^t) \xrightarrow{\Sigma(i_d)_*} \mathscr{M}(S\zeta_n[\sigma]^t)$$

is an isomorphism.

Proof. First we use the projection maps $J:\mathbf{Z}/n \to \mathbf{Z}/d$ for $d|n$ and $\left(d, \frac{n}{d}\right) = 1$ to split $\mathscr{M}(S\pi)$ into components indexed by subsets of set P of prime divisors of n (as in [**46**, Section 4]). Let $\mathscr{M}(S\pi)(P)$ denote the top component in the crude splitting and observe that the projection to this top component

$$\sum_{\bar{n}|m|n}^{\oplus} \mathscr{M}(S\pi)(m) \subseteq \mathscr{M}(S\pi) \to \mathscr{M}(S\pi)(P)$$

is an isomorphism. Since $1/n \in S$ we can identify

$$\mathscr{M}(S\pi) \cong \sum_{d|n}^{\oplus} \mathscr{M}(S\zeta_d[\sigma]^t)$$

using $\sum_{d|n} (\mathrm{pr}_d)_*$ and then $\mathscr{M}(S\pi)(P)$ is just

$$\sum_{\bar{n}|m|m}^{\oplus} \mathcal{M}(S\zeta_m[\sigma]^t).$$

On the other hand, by (6.16) (iii) applied to the diagram

$$\begin{tikzcd} \mathbf{Z}/n \rtimes \sigma \arrow[r, "J"] & \mathbf{Z}/d \rtimes \sigma \arrow[d, equal] \\ \mathbf{Z}/d \rtimes \sigma \arrow[u, phantom, "\cup"] \arrow[r, "J"] & \mathbf{Z}/d \rtimes \sigma \end{tikzcd}$$

for any $d|n$ with $(d, n/d) = 1$ we see that

$$\sum_{\bar{n}|m|n}^{\oplus} \mathcal{M}(S\pi)(m) \subseteq \bigcap_{(d,n/d)=1} \ker(J_*:\mathcal{M}(S\pi) \to \mathcal{M}(S(\mathbf{Z}/d \rtimes \sigma)))$$

$$= \mathcal{M}(S\pi)(P).$$

It follows that this inclusion is an equality and in particular, that for m with $\bar{n}|m|n$

$$(\mathrm{pr}_d)_*\mathcal{M}(S\pi)(m) = 0$$

unless $\bar{d} = \bar{n}$.

From (6.16) (iv) and (6.13) there is a splitting

$$\mathcal{M}(S\zeta_n[\sigma]^t) = \sum_{\bar{n}|d|n}^{\oplus} \mathcal{M}(S\zeta_n[\sigma]^t)(d)$$

using the Mackey functor structure I^*, I_*.

We now assume $\bar{n}|m|n$, $\bar{n}|d|n$ and apply (6.15) (iii) and (iv) to the commutative diagrams:

$$\begin{tikzcd} S\mathbf{Z}/n \arrow[r, "J"] & S\mathbf{Z}/d \arrow[r, "\mathrm{pr}_d"] & S\zeta_d \\ S\mathbf{Z}/m \arrow[u, hook] \arrow[r, "J"] & S\mathbf{Z}/l \arrow[u, hook] \arrow[r, "\mathrm{pr}_l"] & S\zeta_l \arrow[u, hook] \end{tikzcd}$$

for $l = d/\left(\frac{n}{m}, d\right)$. From these it follows that

(6.18) $(\mathrm{pr}_d)_*\mathcal{M}(S\pi)(m) \subseteq \mathcal{M}(S\zeta_d)(l)$

if $\bar{m} = \bar{l}$ and otherwise

$$(\mathrm{pr}_d)_*\mathscr{M}(S\pi)(m) = 0.$$

(Note that $l|m$ and that the correspondence $(d, m) \to (d, l)$ with $\bar{l} = \bar{m}$ is bijective.) However, our assumption on S means that $\Sigma(\mathrm{pr}_d)_*$ is an isomorphism and so the inequality (6.18) must be an equality. For the top component we obtain the isomorphism:

$$\sum_{\bar{n}|d|n} (\mathrm{pr}_d)_*:\mathscr{M}(S\pi)(n) \to \sum_{\bar{n}|d|n}^{\oplus} \mathscr{M}(S\zeta_d[\sigma]^t)(d)$$

which can be followed by the isomorphism

$$\Sigma(i_d)_*: \sum_{\bar{n}|d|n}^{\oplus} \mathscr{M}(S\zeta_d[\sigma]^t)(d) \to \sum_{\bar{n}|d|n}^{\oplus} \mathscr{M}(S\zeta_n[\sigma]^t)(d) = \mathscr{M}(S\zeta_n[\sigma]^t)$$

to give the result.

For our applications of these results it is important to observe that the K and L functors of $\mathbf{Q}\pi$, $\hat{\mathbf{Q}}_l\pi$ and $\hat{\mathbf{Z}}_l\pi$ (for $l \nmid n$) give 2-local Mackey functors in the sense of (6.15).

For the functors $\widetilde{K}_0(\_) \otimes \hat{\mathbf{Z}}_2$ and $K_1'(\_) \otimes \hat{\mathbf{Z}}_2$ this is well-known ([**13**], [**2**]) and follows from the arithmetic sequence in K-theory together with the description of K_1' in terms of reduced norms (recall that $K_1'(\mathscr{A})$ is the image of $K_1(\mathscr{A})$ in $K_1(\mathscr{A} \otimes K)$), or for $K_1'(\hat{\mathbf{Z}}_2\pi)$ from the results of Oliver [**30**]. Similarly, the functors $H^*(\widetilde{K}_0(\_))$ and $H^*(K_1'(\_))$ are 2-local Mackey functors where as usual H^* denotes Tate cohomology with respect to the involution.

PROPOSITION 6.19. *The L-functors L^X, L^K, L^P are 2-local Mackey functors on $S\pi$ for $S = \mathbf{Q}$, $\hat{\mathbf{Q}}_l$ and $\hat{\mathbf{Z}}_l(l \nmid n)$. Properties* (6.15) (i)-(iii) *are satisfied for $S = \mathbf{Z}$.*

Proof. The properties listed in (6.16) become clear when we describe the restriction map I^* for these functors. Let (R, α, u) and (R', α', u') be two rings with anti-structure [**49**]. A *hermitian* R-R' bimodule (M, b) is bimodule ${}_RM_{R'}$ together with a non-singular hermitian form

$$b: M \xrightarrow{\approx} \mathrm{Hom}_{R'}(M, R')$$

such that b is an isomorphism of R-R' bimodules where the natural R'-R structure on $\mathrm{Hom}_R(M, R')$ is twisted by α, α' to produce a R-R' structure. (M, b) is called *based* if M is R-based and $M^{\oplus k}$ is R'-based for some integer k depending on (R, R').

If $Q(R, \alpha, u)$ (resp. $BQ(R, \alpha, u)$) denotes the category of quadratic forms on projective (resp. based free) modules over (R, α, u), then a based hermitian R-R' bimodule gives functors:

$$\_ \otimes_R (M, b): Q(R, \alpha, u) \rightarrow Q(R', \alpha', u)$$
$$\_ \otimes_R (M, b): BQ_k(R, \alpha, u) \rightarrow BQ(R', \alpha', u')$$

where the subscript k indicates the cofinal subcategory of quadratic modules whose rank is a multiple of k. This bimodule formalism includes the usual restriction and induction structure as well as Morita equivalence. To check the properties (6.15) we need to consider the special cases:

(6.20) (i) $R = \mathbf{Z}\rho$, $R' = \mathbf{Z}\pi$ where $\rho \subseteq \pi$. With

$$M = {}_R\mathbf{Z}\pi_{R'} \quad \text{and} \quad b(x)(y) = \bar{x}y$$

we get I_* for the L-groups. With

$$M = {}_{R'}\mathbf{Z}\pi_R \quad \text{and} \quad b(x)(y) = \sum_{h \in \rho} \epsilon_1(\bar{x}yh)h^{-1}$$

we get I^* on L-groups.

(6.20) (ii) Let E/F be a finite Galois extension and $c:E \rightarrow E$ an involution fixing F. If (R, α) is an (F, c) algebra, set

$$(R', \beta) = (R \otimes_F E, \alpha \otimes_F c)$$

and consider

$$M = {}_F E_E, \quad b(x)(y) = c(x)y$$
$$M = {}_E E_F, \quad b(x)(y) = \mathrm{Tr}_E^F(c(x)y)$$

to get I_* and I^* for the L-groups.

Properties (6.15) (i)-(iii) are now immediate from (6.20) (i). For (6.15) (iv) and $S = \mathbf{Q}, \hat{\mathbf{Q}}_l$ or $\hat{\mathbf{Z}}_l(l \nmid n)$, consider the extension of fixed rings $(S\zeta_n)^\sigma/(S\zeta_d)^\sigma$ when $\bar{n}|d|n$. The degree is n/d and

(6.21) $\quad S\zeta_n = (S\zeta_n)^\sigma \otimes_{(S\zeta_d)^\sigma} S\zeta_d.$

It follows that the induction, restriction maps of (6.20) (ii) induce a Mackey functor structure. For $S = \hat{\mathbf{Z}}_l(l \nmid n)$ note that the extension $(S\zeta_n)^\sigma/(S\zeta_d)^\sigma$ is unramified and so the trace form is unimodular. One can check directly that Res $\circ$ Ind is a 2-local isomorphism: in fact, it is induced by tensoring with the $(S\zeta_d)^\sigma - (S\zeta_d)^\sigma$ bimodule $(S\zeta_n)^\sigma$ equipped with the trace form: For $S = \mathbf{Q}, \hat{\mathbf{Q}}_l$ or $\hat{\mathbf{Z}}_l(l \nmid n)$ the trace form is equivalent to $n/d \langle 1 \rangle$.

Our final goal in this section is a description of the top component of $K_1(S\pi)$ suitable for computation. Following **[13]**, we let $\Omega = \mathrm{Gal}(\mathbf{Q}\zeta_n/\mathbf{Q})$ and define

$$\theta_\pi : K_1(S\pi) \rightarrow \mathrm{Hom}_\Omega(R(\mathbf{Z}/n), K_1(S\zeta_n[\sigma]))$$

for $\pi = \mathbf{Z}/n \rtimes \sigma$ by the formula (cf. discussion before (6.15)):

$$\theta_\pi(a)(\chi) = \mathrm{pr}(\chi)_*(a).$$

On the right-hand side there is a Mackey functor structure given by induction and restriction of characters:

$$\mathrm{Ind}^* f(\psi) = f(\mathrm{Ind}\,\psi), \quad \mathrm{Res}_* g(\chi) = g(\mathrm{Res}\,\chi)$$

where $\psi \in R(\mathbf{Z}/m)$, $\chi \in R(\mathbf{Z}/n)$ and f, g are Galois-invariant homomorphisms.

PROPOSITION 6.22. *For $S \subseteq \mathbf{Q}$ or $\hat{\mathbf{Q}}_l$ and $m|n$ the following diagrams commute*:

$$\begin{tikzcd}
K_1(S(\mathbf{Z}/n \rtimes \sigma)) \arrow[r, "\theta_\pi"] \arrow[d, shift right, "I^*"'] & \mathrm{Hom}_\Omega(R(\mathbf{Z}/n), K_1(S\zeta_n[\sigma]^t)) \arrow[d, shift right, "\mathrm{Ind}^*"'] \\
K_1(S(\mathbf{Z}/m \rtimes \sigma)) \arrow[r, "\theta_\rho"] \arrow[u, shift right, "I_*"'] & \mathrm{Hom}_\Omega(R(\mathbf{Z}/m), K_1(S\zeta_m[\sigma]^t)) \arrow[u, shift right, "\mathrm{Res}_*"']
\end{tikzcd}$$

Proof. Note that for $G = \mathrm{Gal}(\mathbf{Q}\zeta_n/\mathbf{Q}\zeta_m) \subseteq \Omega$,

(6.23) $\quad K_1(S\zeta_n[\sigma]^t)^G = K_1(S\zeta_m[\sigma]^t)$

so that

$$\mathrm{Ind}^* f(\psi) \subseteq K_1(S\zeta_m[\sigma]^t).$$

Now the relation

$$\mathrm{Res}_* \circ \theta_\rho = \theta_\pi \circ I^*$$

is clear and for the other, we use the formula

(6.24) $\quad \mathrm{Ind}\,\psi = \sum_{i=0}^{k-1} \chi^{1+i(m/d)}$

where $\psi \in R(\mathbf{Z}/m)$, $\chi \in R(\mathbf{Z}/n)$ with $\mathrm{Res}\,\chi = \psi$, $k = n/m$ and $d = |\ker \psi|$. The commutativity now follows from:

(6.25) $\quad \mathrm{pr}(\mathrm{Ind}\,\psi)_*(a) = \mathrm{pr}(\psi_*)(I^*(a))$

for any $a \in K_1(S\pi)$, $\psi \in R(\mathbf{Z}/m)$.

Remarks. (i) If $1/n \in S$, θ_π is an isomorphism and (6.22) gives a computation of the top component (cf. Section 9). If this is combined with the isomorphism:

$$e{:}\mathrm{Hom}_\Omega(R(\mathbf{Z}/n), K_1(\mathbf{Q}\zeta_n[\sigma]^t)) \to \sum_{d|n}^{\oplus} K_1(\mathbf{Q}\zeta_d[\sigma]^t)$$

given by evaluation of characters $e(f) = (f(\chi_d))$, the formula (6.17) where

$$e \circ \theta_\pi = \sum_{d|n} (\mathrm{pr}_d)_*,$$

and the isomorphism

$$Nrd:K_1(\mathbf{Q}\zeta_d[\sigma]^t) \xrightarrow{\approx} ((\mathbf{Q}\zeta_d)^\sigma)^*,$$

the result is an explicit identification:

$$K_1(\mathbf{Q}(\mathbf{Z}/n \rtimes \sigma)(n) \xrightarrow{\approx} ((\mathbf{Q}\zeta_n)^\sigma)^*.$$

(ii) The properties needed for (6.22) can be abstracted as Galois invariance (6.23), and a compatibility condition (6.25). This is actually a Burnside ring invariance property for the pairing (6.15):

$$\langle \omega \cdot \chi, a \rangle = \langle \chi, i_*\omega \cdot a \rangle$$

where

$$\chi \in R(\mathbf{Z}/n), \omega \in \Omega(\mathbf{Z}/n) \quad \text{and} \quad a \in \mathcal{M}(S(\mathbf{Z}/n \times \sigma)).$$

7. Applications of the splitting. The methods and results of the last section can be applied to the functors appearing in the K and L-theory exact sequences. We first state the general result for L-theory and then list some consequences which will be used later. The proofs follow easily from (6.17) and (6.19). Another splitting for the special case of L^X with similar properties is due to Wall [**46**, Section 4].

THEOREM 7.1. *Let $\pi = \mathbf{Z}/n \rtimes \sigma$ be 2-hyperelementary. There is a splitting*

$$L_i^Y(\mathbf{Z}\pi) = \sum_{d|n}^{\oplus} L_i^Y(\mathbf{Z}\pi)(d)$$

such that

i) $L_i^Y(\mathbf{Z}\pi)(d) = L_i^X(\mathbf{Z}\pi)(d)$ *for* $d > 1$.

ii) *there is an exact sequence*

$$\ldots \to L_{i+1}^S(\hat{S}(d)) \to L_i^X(\mathbf{Z}\pi)(d)$$
$$\to \prod_{l|d} L_i^X(\hat{R}_l(d)) \oplus L_i^S(\hat{S}(d)) \to \ldots$$

there $R(d) = \mathbf{Z}\xi_d[\sigma]^t$ *and* $S(d) \otimes \mathbf{Q}$.

THEOREM 7.2. *Let $\pi = \mathbf{Z}/n \rtimes \sigma$ b 2-hyperelementary, then L^P splits as above so that*

i) $L_1^P(\mathbf{Z}\pi)(d) = L_i^P\mathbf{Z}\pi)(d)$ *if* $d > 1$

ii) *there is an exact sequence*

$$\begin{aligned} \ldots &\to L^K_{i+1}(\hat{S}(d)) \to L^P_i(\mathbf{Z}\pi)(d) \\ &\to \prod_{l\nmid d} L^K_i(\hat{R}_l(d)) \oplus L^K_i(S(d)) \to L^S_i(\hat{S}(d)) \to \ldots \end{aligned}$$

where $R(d)$ *is as in* (7.1).

Proof (i). The result follows from the exact sequence

$$0 \to \langle \tau \rangle \to L^P_{2k+1}(\mathbf{Z}\pi) \to L^p_{2k+1}(\mathbf{Z}\pi) \to 0$$

where

$$\tau = \begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix}$$

is in the image of $L^p_{2k+1}(\mathbf{Z}\sigma)$.

ii) Combine (6.25), (6.27) with the basic exact sequence (3.8). Note that

$$\prod_{l\nmid d} L^K_i(\hat{\mathbf{Z}}_l\pi)(d) = 0.$$

COROLLARY 7.3. *Let* π *be a* 2*-hyperelementary* $\mathcal{P}$*-group of type* IIM. *Then*

$$\mathrm{Res}: L^P_3(\mathbf{Z}\pi) \to \Sigma\{L^P_3(\mathbf{Z}\rho): \rho \subseteq \pi \text{ is a special type IIM group}\}$$

is injective.

Proof. The 2-hyperelementary IIM groups are described in the Introduction by Milnor's notation

$$\pi = Q(8a, b, c) \times \mathbf{Z}/m$$

where $a \geqq b \geqq c$, while the special IIM groups are of the form

$$(a = \alpha, b = \beta, c = m = 1): \rho = Q(8\alpha, \beta).$$

The result will follow from (6.20) and the following lemma.

LEMMA 7.4. *Let* $\pi = Q(8a, b, c) \times \mathbf{Z}/m$ *and either* $d|m$, $d > 1$ *or* $(a, d)(b, d)(c, d) > 1$. *Then*

$$L^P_3(\mathbf{Z}\pi)(d) = 0.$$

Proof. The assumptions on d have the effect that each involution-invariant summand in $S(d)$ or in $\hat{R}_2(d)/\mathrm{rad}$ has type U. The sequence 7.2 (ii) becomes somewhat simpler after (4.5):

$$\ldots \to L^K_0(\hat{S}(d)) \to L^P_3(\mathbf{Z}\pi)(d) \to L^K_3(\hat{R}_2(d)) \to \ldots$$

Since

$$L^K_3(\hat{R}_2(d)) \cong L^K_3(\hat{R}_2(d)/\mathrm{rad})$$

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it follows from [**49**, Section 6] that $L_3^K(\hat{R}_2(d)) = 0$. Moreover,

$$L_0^K(S(d)) \cong H^0(K_1(S(d)) \quad \text{and } L_0^K(\hat{S}(d)) \cong H^0(K_1(\hat{S}(d)))$$

so to prove that

$$L_0^K(S(d)) \to L_0^K(\hat{S}(d))$$

is surjective it suffices to show that $H^0(\hat{E}^\times/E^\times) = 0$ where E is a center field in $S(d)$ (with non-trivial involution). By class field theory $H^0(E_A^\times/E^\times) = \mathbf{Z}/2$ and $H^0(E_\infty^\times) = g_\infty \cdot \mathbf{Z}/2$ maps onto $\mathbf{Z}/2$, (see [7, VII]). Since $E_A^\times = \hat{E}^\times \oplus E_\infty^\times$ it follows that $H^0(\hat{E}^\times/E^\times) = 0$, whence $L_3^P(\mathbf{Z}\pi)(d) = 0$.

For the next result and later we need to identify the top components of certain K_1-groups (see Section 6) for $\pi = \mathbf{Z}/n \rtimes \sigma$: By (6.27),

(7.5) $\quad K_1(S\pi)(n) \cong K_1(S\zeta_n[\sigma]^t)$

for $S = \mathbf{Q}$, $\hat{\mathbf{Q}}_l$ and for $\hat{\mathbf{Z}}_l$ if $l \nmid n$. Similarly,

(7.6)
$$\begin{aligned} L_3^X(\hat{\mathbf{Z}}_l\pi)(n) &\cong L_3^X(\hat{\mathbf{Z}}_l\zeta_n[\sigma]^t) \\ L_3^K(\hat{\mathbf{Z}}_l\pi)(n) &\cong L_3^K(\hat{\mathbf{Z}}_l\zeta_n[\sigma]^t). \end{aligned}$$

In Section 5 we described how the group $L_3^P(\mathbf{Z}\pi)$ is related to $L_3^Y(\hat{\mathbf{Z}}_2\pi)$. Our final calculations will rest on the fact that for $\pi = Q(8a, b)$ the relevant part of $L_3^Y(\hat{\mathbf{Z}}_2\pi)$ is detected by restriction to $\tau \subseteq \pi$ where

$$\tau = \mathbf{Z}/ab \rtimes \mathbf{Z}/4$$

is the "diagonal" subgroup with $\mathbf{Z}/4$ acting on $\mathbf{Z}/ab$ by inversion. This group is usually denoted $Q(4ab)$.

PROPOSITION 7.7. *Let $\pi = Q(8a, b)$ and $\tau = Q(4ab)$. Then the image in $L_3^Y(\hat{\mathbf{Z}}_2\pi)$ of the kernel of*

$$i_{\text{odd}}^Y: L_3^Y(\mathbf{Z}\pi)(ab) \to L_3^Y(\hat{\mathbf{Z}}_{\text{odd}}\pi)(ab)$$

is mapped injectively by

$$\text{Res}: L_3^Y(\hat{\mathbf{Z}}_2\pi)(ab) \to L_3^Y(\hat{\mathbf{Z}}_2\tau)(ab).$$

Proof. This is proved in [**21**, 4.19] but stated slightly incorrectly there, so we give the following outline. Recall that we introduced the ring $A = \mathbf{Z}[\eta_a, \eta_b]$ and its quotient field F in our discussion of condition $C(a, b)$. Since

(7.8) $\quad \hat{\mathbf{Z}}_2 \otimes \mathbf{Z}\zeta_{ab}[Q8]^t \cong M_4(\hat{A}_2[\mathbf{Z}/2])$

it follows from (7.5) and an examination of the Rothenberg sequence [**21**, 4.11] that

(7.9) $\quad L_3^Y(\hat{\mathbf{Z}}_2\pi)(ab) = H^0(\hat{A}_2[\mathbf{Z}/2]^\times)/L_0^K(\hat{\mathbf{Z}}_2\pi)(ab) \cong A/2A \oplus H^0(\hat{A}_2^\times).$

In fact the first factor is the first summand of

$$H^0(K_1(\hat{\mathbf{Q}}_2\pi)_+(ab))/L_0^K(\hat{\mathbf{Z}}_2\pi)(ab) \cong A/2A \oplus H^0(\hat{F}_2^\times/\hat{A}_2^\times)$$

and the second factor injects into

$$H^0(K_1(\hat{\mathbf{Q}}_2\pi)_-(ab)) \cong H^0(\hat{F}_2^\times)$$

under the map induced by the injection

$$H^0(K_1'(\mathbf{Z}_2\pi)(ab)) \rightarrowtail H^0(K_1(\hat{\mathbf{Q}}_2\pi)(ab)).$$

We also use the calculation at odd primes:

$$(7.10)\quad L_3^Y(\hat{\mathbf{Z}}_{\text{odd}}\pi)(ab) = \prod_{l\nmid 2ab} H^0(\hat{A}_l^\times)$$

where each factor injects into the corresponding part of

$$\prod_{l\nmid 2ab} H^0(K_1(\hat{\mathbf{Q}}_l\pi)_-(ab)) \cong \prod_{l\nmid 2ab} H^0(\hat{F}_l^\times).$$

The image needed will be obtained from [**21**, 4.14]:

$$(7.11)\quad \operatorname{Im}(L_3^Y(\mathbf{Z}\pi)(ab) \to L_3^Y(\hat{\mathbf{Z}}\pi)(ab))$$
$$= A/2A \oplus \ker(F^{*(2)}/F^{\times 2} \to H^0((A/ab)^\times))$$

where $F^{*(2)} \subset F^{(2)}$ denotes the elements with positive valuation at all primes (infinite or finite) and the map is induced by

$$\Phi_A : F^{(2)} \to (A/ab)^\times/\text{squares} = H^0((A/ab)^\times)$$

defined in the Introduction: note that $A^\times$ has trivial involution. The image of

$$i_2^Y : L_3^Y(\mathbf{Z}\pi)(ab) \to L_3^Y(\hat{\mathbf{Z}}_2\pi)(ab)$$

is just the first factor $A/2A$ together with the image of the second factor in (7.11) under the reduction map

$$F^{*(2)}/F^{\times 2} \to H^0(\hat{A}_2^\times).$$

Similarly, the image of i_{odd}^Y is the image of the second factor under the reduction

$$F^{*(2)}/F^{\times 2} \to \prod_{l\nmid 2ab} H^0(\hat{A}_l^\times).$$

The global square theorem now implies that

$$(7.12)\quad \operatorname{Im}(i_2^Y|\ker i_{\text{odd}}^Y) = A/2A.$$

In order to see that this subgroup is mapped injectively by Res, we list the calculation of $L_3^Y(\hat{\mathbf{Z}}_2\tau)(ab)$ obtained in [**21**, 4.18], [**19**]: Let $B = \mathbf{Z}[\eta_{ab}]$ then

(7.13) $L_3^Y(\hat{\mathbf{Z}}_2\tau)(ab) = B/2B \oplus H^0(\hat{B}_2^\times)$

and the map Res on the first summand of (7.10) is just the inclusion $A/2A \subset B/2B$.

Our final application of the splitting technique in this section will be a "calculation" of the map

$$\chi: L_3^P(\mathbf{Z}\pi) \to L_3^K(\hat{\mathbf{Z}}_2\pi)$$

used in (4.15). The result is based on the relationship found by Pardon [**32**] between R. Lee's semi-characteristic invariant (cf. 1.3) and 2-local surgery obstructions.

Let $\pi = \mathbf{Z}/n \rtimes \sigma$ be a 2-hyperelementary group and let $R_{GL,ev}(\mathbf{F}_2\pi)$ be R. Lee's (unreduced) Grothendieck group of $\mathbf{F}_2[\pi]$-modules modulo those which admit an "even π-quadratic form" [**20**, pp. 191-2]. Let $R_{GL,ev}(\mathbf{F}_2\zeta_d[\sigma]^t)$ denote the subgroup of $R_{GL,ev}(\mathbf{F}_2[\pi])$ generated by the modules belonging to the block

$$\mathbf{F}_2\zeta_d[\sigma]^t \subseteq \mathbf{F}_2[\pi].$$

Then

(7.14) $R_{GL,ev}(\mathbf{F}_2[\pi]) = \sum_{d|n}^{\oplus} R_{GL,ev}(\mathbf{F}_2\zeta_d[\sigma]^t)$

where the splitting is induced by projection maps.

PROPOSITION 7.15. [**32**, 2.23]. *Let $\pi = \mathbf{Z}/n \rtimes \sigma$ be 2-hyperelementary where σ acts faithfully on $\mathbf{Z}/n$. Then the difference of semi-characteristics in domain and range of a 2-local normal map induces an injection*

$$\chi_{1/2}: L_3^K(\mathbf{Z}_{(2)}\pi)(n) \to R_{GL,ev}(\mathbf{F}_2\zeta_n[\sigma]^t).$$

This result can be used to identify ker χ: given any $\pi = \mathbf{Z}/n \rtimes \sigma$ and $d|n$, let

$$\bar{\pi}_d = \mathbf{Z}/d \rtimes \sigma/\sigma_d$$

where σ_d is the kernel of the action of σ restricted to $\mathbf{Z}/d \subseteq \mathbf{Z}/n$ and define χ as the composite:

$$\begin{tikzcd}
L_3^P(\mathbf{Z}\pi) \arrow[r, "\tilde{\chi}"] \arrow[d] & \sum_{d|n}^{\oplus} R_{GL,ev}(\mathbf{F}_2\zeta_d[\sigma/\sigma_d]^t). \\
\sum_{d|n}^{\oplus} L_3^K(\mathbf{Z}_{(2)}\pi)(d) \arrow[r, "\approx"] & \sum_{d|n}^{\oplus} L_3^K(\mathbf{Z}_{(2)}\bar{\pi}_d)(d) \arrow[u]
\end{tikzcd}$$

The first map is induced by $\mathbf{Z} \subseteq \mathbf{Z}_{(2)}$ and the splitting (6.15). The second is induced by restrictions and projections, while the third is the sum of the semi-characteristics.

COROLLARY 7.17. *We have*

$$\operatorname{Ker} \chi = \operatorname{Ker} \widetilde{\chi}.$$

Proof. Clearly

$$\ker \widetilde{\chi} = \ker(L_3^P(\mathbf{Z}\pi) \to L_3^K(\mathbf{Z}_{(2)}\pi)),$$

and the natural map

$$L_3^K(\mathbf{Z}_{(2)}\pi) \to L_3^K(\hat{\mathbf{Z}}_2\pi)$$

is an injection [**32**, 2.12].

COROLLARY 7.18. [**20**, **31**, **32**]. *Let π be a $\mathscr{P}$-group and*

$$(f, b):M \to X^{4l-1}$$

a normal map where X is a $(4l - 1)$-dimensional Swan complex for π. Then $\lambda^p(f, b) \in \ker \chi$ if and only if every subgroup $\rho \subseteq \pi$ of order $2p$ is cyclic (p any prime) and, when l is odd, π has no type IIL *subgroups.*

Proof. By Dress induction we may assume that π is 2-hyperelementary. The result now follows from Lee's examples [**20**, 4.14-4.17] and [**32**, 4.1, 5.1, 6.1].

Remark. Recently J. Davis [**9**] has given a different expression for the map χ valid for any finite group π. He defines a "surgery semi-characteristic" using homology with twisted coefficients with values in $H^0(\widetilde{K}_0(\hat{\mathbf{Z}}_2\pi))/L_0^p(\hat{\mathbf{Z}}_2\pi)$ and proves that the map χ is just the surgery semi-characteristic of a normal map followed by the natural identification of the group above with $L_3^h(\hat{\mathbf{Z}}_2\pi)$.

8. The idelic Reidemeister torsion. In this section we show how the L^p-surgery obstruction of certain space-form problems can be calculated using Reidemeister torsion invariants.

Let $X^{n-1} = X(\pi)$ be a Swan complex for a $\mathscr{P}$-group π. The homotopy type of X is determined by the chain homotopy type of an associated periodic projective resolution (P_i $0 \leqq i \leqq n - 1$ are $\mathbf{Z}\pi$-projectives):

$$0 \to \mathbf{Z} \to P_{n-1} \to \ldots \to P_0 \to \mathbf{Z} \to 0.$$

An orientation for X fixes a base for the homology of the complex P_*. Moreover by Swan's theorem $\mathbf{Z}\pi$-projectives are locally free. So after choosing bases $\mathbf{b}$ for $\hat{\mathbf{Z}}_l \otimes P_*$ (for all primes l) we obtain the idelic Reidemeister torsion invariant

(8.1) $\quad \hat{\Delta}(X, \mathbf{b}) \in K_1(\hat{\mathbf{Q}}\pi)$

used in [**47**, Section 9] and [**21**, Section 6]. If the bases $\mathbf{b}$ are changed, the torsion is multiplied by an element of $K_1'(\hat{\mathbf{Z}}\pi)$ and the class of $\hat{\Delta}(X, \mathbf{b})$ is then a well-defined invariant of the homotopy type of X in the quotient group:

(8.2) $\quad \hat{\Delta}(X) \in K_1(\hat{\mathbf{Q}}\pi)/K_1'(\hat{\mathbf{Z}}\pi).$

This invariant has the important property that

$$\partial\hat{\Delta}(X) = \sigma(X) \in \widetilde{K}_0(\mathbf{Z}\pi)$$

where $\sigma(X)$ is the Wall finiteness obstruction for X and the map ∂ comes from the exact sequence (cf. (3.4)):

(8.3) $\quad 0 \to Wh(\mathbf{Q}\pi)/Wh'(\mathbf{Z}\pi) \to Wh(\hat{\mathbf{Q}}\pi)/Wh'(\hat{\mathbf{Z}}\pi) \xrightarrow{\partial} \widetilde{K}_0(\mathbf{Z}\pi) \to 0.$

In the important special case when P_* is actually free over $\mathbf{Z}\pi$, the torsion is denoted $\Delta(X)$ and

$$\Delta(X) \in K_1(\mathbf{Q}\pi)/K_1'(\mathbf{Z}\pi).$$

Also, in this case the choice of a base $\mathbf{b}$ over $\mathbf{Z}\pi$ corresponds to a particular finite cell structure on X. If X is a compact smooth or PL manifold, then the (smooth) triangulation $\mathbf{c}_X$ defines a unique element:

(8.4) $\quad \Delta(X) = \Delta(X, \mathbf{c}_X) \in Wh(\mathbf{Q}\pi).$

To define the idelic torsion for more general spaces than Swan complexes it is necessary to make more choices. A chain complex C of finitely generated projective $\mathbf{Z}\pi$-modules is *based* if a base $\mathbf{b}$ is chosen for $C \otimes \hat{\mathbf{Z}}$. (We will call this a *base* for C.) The complex C has *based homology* if a base $\mathbf{h}$ is chosen for $H_*(C \otimes \mathbf{Q})$. The finiteness obstruction for C is

$$\sigma(C) = \Sigma(-1)^i[C_i] \in \widetilde{K}_0(\mathbf{Z}\pi).$$

These definitions are extended to finitely dominated spaces with $\pi_1 Y = \pi$ by considering the chain complex of its universal cover. Notice that $H_*(C \otimes \hat{\mathbf{Q}}_l)$ is still only projective over $\hat{\mathbf{Q}}_l\pi$ so the bases $\mathbf{h}$ referred to above are over each simple algebra in $\hat{\mathbf{Q}}_l\pi$ separately.

From [**21**, Section 2] following the original definition of Milnor there is an idelic torsion invariant

(8.5) $\quad \hat{\Delta}(C, \mathbf{b}, \mathbf{h}) \in K_1(\hat{\mathbf{Q}}\pi)$

associated to any based complex C with based homology. We now concentrate on the 2-adic component

$$\hat{\Delta}_2(C, \mathbf{b}, \mathbf{h}) \in K_1(\hat{\mathbf{Q}}_2\pi)$$

of the idelic torsion and list some of its properties (following [**21**, Section 2]). It is clear from the definition that $\hat{\Delta}_2$ is really an invariant of based free $\hat{\mathbf{Z}}_2\pi$-complexes. In this setting an acyclic based complex has

$$\hat{\Delta}_2(C, \mathbf{b}) \in K_1'(\hat{\mathbf{Z}}_2\pi)$$

and we call its projection into $Wh'(\hat{\mathbf{Z}}_2\pi)$ the *Whitehead torsion* by analogy with the usual invariant. Notice that the notation is simplified by omitting $\mathbf{h}$. A based quadratic Poincaré complex [**36**] is called *simple* if the duality map has zero Whitehead torsion. Since

$$Wh'(\mathbf{Z}\pi) \to Wh'(\hat{\mathbf{Z}}_2\pi)$$

is an injection, this definition agrees with the ordinary one on free $\mathbf{Z}\pi$-complexes.

Let (C, ψ) be a Poincaré complex of formal dimension n. In [**21**, 2.17] the concept of a *PD* base for $H_*(C \otimes \hat{\mathbf{Q}})$ was introduced: if n is odd, $\mathbf{h}$ is a *PD* base when the dual base $\mathbf{h}^*$ for cohomology and $\mathbf{h}$ correspond under Poincaré duality, and *PD* bases always exist. If $n = 2m$ is even, assume in addition that the intersection form on $H_m(C \otimes \hat{\mathbf{Q}})$ is hyperbolic and $\mathbf{h}$ is a symplectic base.

The usefulness of this concept rests on an observation in [**21**, Section 2]: when C is a simple Poincaré complex (dim $C = n$) with base $\mathbf{b}$ and $\mathbf{h}$ is a *PD* base, then $\hat{\Delta}(C, \mathbf{b}, \mathbf{h})$ defines an element in $H^{n+1}(K_1(\hat{\mathbf{Q}}\pi))$ independent of the choice of *PD* base. We denote the cohomology class by $\{\hat{\Delta}(C, \mathbf{b}, \mathbf{h})\}$.

The main property of $\hat{\Delta}_2$ needed for surgery obstructions is the following.

PROPOSITION 8.6. [**21**, 2.4]. *Let (C, ψ) be a simple quadratic Poincaré complex of free $\hat{\mathbf{Z}}_2\pi$-modules with formal dimension n. If C is acyclic then*

$$\lambda_2^Y(C, \psi) \in L_n^Y(\hat{\mathbf{Z}}_2\pi)$$

is the image of $\{\hat{\Delta}_2(C, \mathbf{b})\}$ *under the map in* (3.3):

$$H^{n+1}(Wh'(\hat{\mathbf{Z}}_2\pi)) \to L_n^Y(\hat{\mathbf{Z}}_2\pi).$$

The assumption that (C, ψ) admits a simple base is sometimes unnecessary: for example if C is a free $\mathbf{Z}\pi$-complex and n is odd. In general, if $\mathbf{b}$ is a base for C_* and $\mathbf{h}$ a *PD* base for its homology take $\mathbf{b}^*$, $\mathbf{h}^*$ for the cochain complex C^*, then the Whitehead torsion of the duality map $\psi:C^* \to C_*$ is given by:

$$(8.7) \qquad \tau(\psi) = u + (-1)^n\bar{u}, \quad u = \hat{\Delta}(C, \mathbf{b}, \mathbf{h}).$$

This follows immediately from the additivity formula [**21**, 2.16] for $\hat{\Delta}$. Given a based exact sequence $0 \to C' \to C \to C'' \to 0$ with based homology groups:

$$(8.8) \qquad \hat{\Delta}(C, \mathbf{b}, \mathbf{h}) = \hat{\Delta}(C', \mathbf{b}', \mathbf{h}') + \hat{\Delta}(C'', \mathbf{b}', \mathbf{h}'') + \hat{\Delta}(\mathscr{H}, \mathbf{h}),$$

where $(\mathscr{H}, \mathbf{h})$ is the based acyclic complex of homology groups. Now to derive (8.7) we apply (8.8) to the based sequence

$$0 \to C^* \to C_* \to C_*(\psi) \to 0$$

where $C_*(\psi)$ is the mapping cone complex. Since

$$\tau(\psi) = \hat{\Delta}(C_*(\psi), [\mathbf{b}^*/\mathbf{b}])$$

by definition the result follows from the relation

(8.9) $\quad \hat{\Delta}(C^*, \mathbf{b}^*, \mathbf{h}^*) = (-1)^{n+1}\bar{u}, \quad u = \hat{\Delta}(C_*, \mathbf{b}, \mathbf{h}).$

In our applications the group ring $\hat{\mathbf{Q}}_2\pi$ will have no type U factors. Then the conjugation is trivial on $Wh(\hat{\mathbf{Q}}_2\pi)$ so any odd-dimensional Poincaré complex of free $\hat{\mathbf{Z}}_2\pi$-modules is automatically simple.

PROPOSITION 8.10. *Let (D, C) be a Poincaré pair of free $\hat{\mathbf{Z}}\pi$-modules with* $\dim C = n$. *If $n = 2m + 1$ assume that $H_{m+1}(D) = 0$. Then there exists a simple base for (D, C) and a PD base $\mathbf{h}$ for $H_*(C \otimes \hat{\mathbf{Q}})$. For any such bases*

$$\hat{\Delta}(C, \mathbf{b}, \mathbf{h}) \in \{u + (-1)^{n+1}\bar{u}: u \in K_1(\hat{\mathbf{Q}}\pi)\}$$

where $\mathbf{b}$ is the base for C.

Proof. Consider the diagram of chain complexes:

$$\begin{array}{ccccccccc} 0 & \to & D_* & \to & C_*(i) & \to & C_{*-1} & \to & 0 \\ & & \uparrow{\scriptstyle\Phi} & & \uparrow{\scriptstyle\tilde{\Phi}} & & \uparrow{\scriptstyle\varphi} & & \\ 0 & \to & C^*(i) & \to & D^* & \to & C^* & \to & 0 \end{array}$$

where $C_*(i)$ is the mapping cone of the inclusion $i:C \to D$, the lower sequence is the dual of the upper and the vertical maps are the PD chain equivalences. Let $\mathbf{d}$ be a base for D_* and choose a base $\mathbf{e}$ for $C_*(i)$ so that

$$\Phi:(C^*(i), \mathbf{e}) \to (D_*, \mathbf{d})$$

is simple. Then

$$\tilde{\Phi}:(D^*, \mathbf{d}^*) \to (C_*(i), \mathbf{e})$$

is simple also and we choose a base $\mathbf{b}$ for C_* such that the upper sequence is based.

Now we claim that when $n = 2m$ the homology groups can be based so that the torsions of Φ_*, $\tilde{\Phi}_*$ and φ_* all vanish. Look at the diagram:

$$
\begin{tikzcd}
\cdots \arrow[r] & H_{m+1}(D) \arrow[r] & H_{m+1}(D, C) \arrow[rr] \arrow[dr] & & H_m(C) \arrow[rr] \arrow[dr] & & H_m(D) \arrow[r] & \cdots \\
& & & K_{m+1} \arrow[ur] & & K_m \arrow[ur] & & \\
\cdots \arrow[r] & H^m(D, C) \arrow[r] \arrow[uu, "\Phi_*"] & H^m(D) \arrow[rr] \arrow[dr] \arrow[uu, "\tilde{\Phi}_*"] & & H^m(C) \arrow[rr] \arrow[dr] \arrow[uu, "\varphi_*"] & & H^{m+1}(D, C) \arrow[r] \arrow[uu] & \cdots \\
& & & K^*_m \arrow[ur] \arrow[uu] & & K^*_{m+1} \arrow[ur] \arrow[uu] & &
\end{tikzcd}
$$

where

$$K_m = \ker(H_mD \to H_m(D, C)), \quad K_{m+1} = \ker(H_mC \to H_mD)$$

and the vertical maps are all isomorphisms. We start with bases for

$$0 \to K_m \to H_m(D) \to \dots \to H_0(D, C) \to 0$$

so that this acyclic complex has zero torsion, then apply duality to base all the other groups except for $H_m(C)$ and $H^m(C)$. Then take the induced base $\mathbf{h}_m$ on $H_m(C)$ from

$$0 \to K_{m+1} \to H_m(C) \to K_m \to 0$$

and note that with these choices the horizontal sequences and the vertical maps all have zero torsion. Furthermore $\mathbf{h}_m$ is a symplectic base for $H_m(C)$ with respect to the intersection form so $\mathbf{h}$ is a PD base for $H_*(C)$.

If $n = 2m + 1$ a similar argument works since $H_{m+1}(D) = 0$ to give simple bases for all the homology groups. Apply the additivity formula (8.8) to our diagram above: if $\tau(\varphi)$ denotes the Whitehead torsion of φ with respect to the given bases then

$$\tau(\tilde{\Phi}) = \tau(\Phi) - \tau(\varphi)$$

shows that $\tau(\varphi) = 0$ and (D, C) is a simple pair. Furthermore (suppressing the bases) we have:

$$\hat{\Delta}(C_*(i)) = \hat{\Delta}(D_*) - \hat{\Delta}(C_*),$$

and

$$\tau(\hat{\Phi}) = \hat{\Delta}(C_*(i)) - \hat{\Delta}(D^*).$$

Since $\tau(\tilde{\Phi}) = 0$,

$$\hat{\Delta}(C_*) = \hat{\Delta}(D_*) - \hat{\Delta}(D^*).$$

Now from (8.9) we get

$$\hat{\Delta}(C_*, \mathbf{b}, \mathbf{h}) = u + (-1)^{n+1}\bar{u} \quad \text{for } u = \hat{\Delta}(D^*).$$

Remark. In general the image of $L_0^K(\hat{\mathbf{Q}}\pi)$ in $H^0(K_1(\hat{\mathbf{Q}}\pi))$ must be divided out to get bordism invariance for the cohomology class of $\hat{\Delta}(C, \mathbf{b}, \mathbf{h})$ [**21**, 2.29].

PROPOSITION 8.11. *Let* (D', C') *and* (D'', C'') *be based free Poincaré pairs over* $\hat{\mathbf{Z}}\pi$ *of odd formal dimension. Suppose also that* (D'', C'') *is simple and that* $\alpha: C' \to C''$ *is a simple homotopy equivalence. Then the torsion of* $D = D' \cup_\alpha D''$ *is*:

$$\hat{\Delta}(D, \mathbf{b}, \mathbf{h}) = \hat{\Delta}(D', \mathbf{b}', \mathbf{h}') + \hat{\Delta}(D'', \mathbf{b}'', \mathbf{h}'')$$
$$\in K_1(\hat{\mathbf{Q}}\pi)/\{u - \bar{u}: u \in K_1(\hat{\mathbf{Q}}\pi)\}$$

where $\mathbf{h}$ *is a PD base,* $\mathbf{h}'$ *depends on the choice of* $\mathbf{h}$, $\mathbf{h}''$ *and*

$$0 \to C' \to D' \oplus D'' \to D \to 0$$

is based exact.

Proof. By (8.10) there exists a *PD* base $\mathbf{h}_0$ for H_*C'' and

$$\hat{\Delta}(C'', \mathbf{b}_0, \mathbf{h}_0) \in \{u - \bar{u}: u \in K_1(\hat{\mathbf{Q}}\pi)\}.$$

We note that this involves a choice of base $\mathbf{h}''$ for $H_*(D'')$. Since $D = D' \cup D''$ is odd dimensional $H_*(D)$ admits a *PD* base [**21**, 2.18], so we must choose a base $\mathbf{h}'$ for $H_*(D')$ such that the based acyclic complex

$$\dots \to H_{i+1}(D) \to H_i(C') \to H_i(D') \oplus H_i(D'') \to H_i(D) \to \dots$$

has zero torsion. The result then follows from (8.8).

To do this consider the diagram ($\dim D = 2m + 1$):

$$\begin{tikzcd}
\dots \arrow[r] & H_{m+1}(D) \arrow[r] & H_{m+1}(D, D') \arrow[r] & H_m(D') \arrow[r] & H_m(D) \arrow[r] & \dots \\
\dots \arrow[r] & H^m(D) \arrow[r] \arrow[u] & H^m(D'') \arrow[r] \arrow[u] & H^{m+1}(D, D'') \arrow[r] \arrow[u] & H^{m+1}(D) \arrow[r] \arrow[u] & \dots
\end{tikzcd}$$

obtained by Poincaré duality and excision. We have already chosen bases for $H_*(D)$ and $H_*(D'')$ so (using the dual bases for cohomology) choose a base for $H^*(D, D'')$ such that the lower sequence has zero torsion. This gives $\mathbf{h}'$ on $H_*(D')$ so that the upper sequence has zero torsion since the vertical maps are simple isomorphisms.

With the choices the Mayer-Vietoris sequence above is based exact and has zero torsion, and we can apply the additivity (8.8).

We now introduce another invariant for certain finitely-dominated Poincaré complexes. Consider the cohomology sequence of (8.3):

$$0 \to \frac{H^0(Wh(\mathbf{Q}\pi)/Wh'(\mathbf{Z}\pi))}{d_0^* H^1(\tilde{K}_0(\mathbf{Z}\pi))}$$
$$\xrightarrow{j_*} H^0(Wh(\hat{\mathbf{Q}}\pi)/Wh'(\hat{\mathbf{Z}}\pi)) \xrightarrow{\partial_*} H^0(\tilde{K}_0(\mathbf{Z}\pi))$$

and recall that

$$\partial_*\{\hat{\Delta}(X)\} = \{\sigma(X)\}.$$

Definition 8.12. Let X be a finitely-dominated odd-dimensional Poincaré complex with

$$\pi_1 X = \pi \quad \text{and} \quad \{\sigma(X)\} = 0 \in H^0(\widetilde{K}_0(\mathbf{Z}\pi)).$$

Then

$$\Delta_0(X) \in H^0(Wh(\mathbf{Q}\pi)/Wh'(\mathbf{Z}\pi))/\mathrm{Im}\, d_0^*$$

is defined by the condition

$$j_*(\Delta_0(X)) = \{\hat{\Delta}(X)\}.$$

This invariant is closely related to the $\hat{\Delta}_2$-invariant when X is actually finite. As before the answer will be expressed for chain complexes since Definition (8.19) can be rephrased in terms of $C_*(\widetilde{X})$.

PROPOSITION 8.13. *Let C be an odd-dimensional Poincaré complex of free $\mathbf{Z}\pi$-modules. If* $\mathbf{b}$ *is a $\mathbf{Z}\pi$-base for C and* $\mathbf{b}$ *is a PD base for $H_*(C \otimes \mathbf{Q})$ then $\{\hat{\Delta}_2(C, \mathbf{b}, \mathbf{h})\}$ and $\Delta_0(C)$ have the same image in*

$$H^0(Wh(\hat{\mathbf{Q}}_2\pi))/H^0(Wh'(\mathbf{Z}\pi) + d^*H^1(\widetilde{K}_0(\mathbf{Z}\pi))$$

where d^* is defined in (5.7).

Proof. The result (including the fact that both invariants end up in the given group) follows from the diagrams

$$\begin{tikzcd}
 & H^0(Wh(\mathbf{Q}\pi)/Wh'(\mathbf{Z}\pi)) \arrow[r, "j_*"] \arrow[d] & H^0(Wh(\mathbf{Z}\pi \to \mathbf{Q}\pi)) \arrow[d] \arrow[dr, "\partial_*"] & \\
H^1(\widetilde{K}_0(\mathbf{Z}\pi)) \arrow[ur, "d_0^*"] \arrow[r, "d_2^*"] \arrow[dr, "d^*"'] & H^0(Wh(\hat{\mathbf{Q}}_2\pi)/Wh'(\mathbf{Z}\pi)) \arrow[r] & H^0(Wh(\mathbf{Z}\pi \to \hat{\mathbf{Q}}_2\pi)) \arrow[r] & H^0(\widetilde{K}_0(\mathbf{Z}\pi)) \\
 & H^0(Wh(\hat{\mathbf{Z}}_2\pi)/Wh'(\mathbf{Z}\pi)) \arrow[r] \arrow[u] & H^0(Wh(\mathbf{Z}\pi \to \hat{\mathbf{Z}}_2\pi)) \arrow[u] \arrow[ur] &
\end{tikzcd}$$

(8.14)

and

$$\begin{tikzcd}
H^0(Wh(\mathbf{Q}\pi)) \arrow[r] \arrow[d] & H^0(Wh(\mathbf{Q}\pi))/H^0(Wh'(\mathbf{Z}\pi) \arrow[r, "\approx"] \arrow[d] & H^0(Wh(\mathbf{Q}\pi)/Wh'(\mathbf{Z}\pi)) \arrow[d] \\
H^0(Wh(\hat{\mathbf{Q}}_2\pi) \arrow[r] & H^0(Wh(\hat{\mathbf{Q}}_2\pi))/H^0(Wh'(\mathbf{Z}\pi) \arrow[r] & H^0(Wh(\mathbf{Q}\pi)/Wh'(\mathbf{Z}\pi))
\end{tikzcd}$$

(8.15)

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The isomorphism in (8.15) follows from

$$H^1(Wh'(\mathbf{Z}\pi)) = 0$$

and the upper sequence in (8.14) is just the sequence used in defining Δ_0 since

$$Wh(\mathbf{Z}\pi \to \mathbf{Q}\pi) \cong Wh(\hat{\mathbf{Q}}\pi)/Wh'(\hat{\mathbf{Z}}\pi).$$

Now C is a free $\mathbf{Z}\pi$-complex, so

$$\hat{\Delta}_2(C, \mathbf{b}, \mathbf{h}) = \Delta(C, \mathbf{b}, \mathbf{h}) \in K_1(\mathbf{Q}\pi).$$

Its image in $K_1(\hat{\mathbf{Q}}\pi)/K_1'(\hat{\mathbf{Z}}\pi)$ is $\hat{\Delta}(C)$ by definition hence the image of $\{\Delta(C, \mathbf{b}, \mathbf{h}\}$, in

$$H^0(Wh(\mathbf{Q}\pi)/Wh'(\mathbf{Z}\pi))/\mathrm{Im}\ d_0^*$$

must be $\Delta_0(C)$. From (8.14) Im d_0^* and Im d^* have the same image in

$$H^0(Wh(\hat{\mathbf{Q}}_2\pi)/Wh'(\mathbf{Z}\pi)).$$

The final result of this section relates Δ_0 to the L^P surgery obstruction of a normal map $(f, \hat{f}):M \to X$ where M is a closed manifold and

$$\sigma_*(\lambda^P(f, \hat{f})) = \{\sigma(X)\} = 0 \quad \text{in } H^0(\widetilde{K}_0(\mathbf{Z}\pi)).$$

Recall that a smooth or PL manifold N has a preferred triangulation and hence (if dim N is odd and $\pi_1 N = \tau$) we can simply write

$$\{\Delta(N)\} \in H^0(Wh(\mathbf{Q}\tau))$$

without ambiguity. This class has image

$$\{\hat{\Delta}(N)\} \in H^0(Wh(\hat{\mathbf{Q}}\tau)).$$

PROPOSITION 8.16. *Let X be a $(4k - 1)$-dimensional Swan complex $(k > 1)$ for a $\mathcal{P}$-group π and*

$$(f, \hat{f}):M \to X$$

be a normal map with

$$\sigma_*(\lambda^P(f, \hat{f})) = 0.$$

Suppose for some $\tau \subseteq \pi$ there exists a closed manifold N and a normal cobordism from $(f_\tau, \hat{f}_\tau)$ to a homotopy equivalence

$$(g, \hat{g}):N \to X_\tau.$$

Then for any $x \in L_3^Y(\mathbf{Z}\pi)$ such that $i^P(x) = \lambda^P(f, \hat{f})$,

$$\mathrm{Res}_\pi^\tau(\delta_2^Y(x)) = \mathrm{Res}_\pi^\tau(\Delta_0(X)) - \{\Delta(N)\} \in H^0(Wh(\hat{\mathbf{Q}}_2\tau)_+)/I$$

where δ_2^Y is defined in (5.17) *and*

$$I = L_0^K(\hat{\mathbf{Z}}_2\tau) + \operatorname{Res}_\pi^\tau[H^0(Wh'(\mathbf{Z}\pi)) + d^*H^1(\tilde{K}_0(\mathbf{Z}\pi))].$$

Proof. From the condition

$$\sigma_*(\lambda^P(f, \hat{f})) = 0$$

and [**33**, Section 3], there exists a finitely-dominated Poincaré pair (Q, P) of dimension $4k$ with $\pi_1 P = \pi_1 Q = \pi$ such that

$$\sigma(X) = -\sigma(P) = -(\sigma(Q) + \overline{\sigma(Q)}) \quad \text{in } \tilde{K}_0(\mathbf{Z}\pi).$$

In addition the Spivak fibration is reducible so there exists a normal map

$$(\Phi, \hat{\Phi}): V \to Q$$

whose boundary is

$$\partial(\Phi, \hat{\Phi}) = (\varphi, \hat{\varphi}): U \to P.$$

We note that P may be constructed so that $H_{2k}(Q) = 0$.

Next we use the result of Hodgson [**17**] or Jones [**18**] that any finite Poincaré complex of dimension $\geqq 7$ has a handle decomposition. We will need it only up to 2-handles in which case the result holds for finitely-dominated complexes with finite 2-skeleton.

More precisely, let Z be a finitely-dominated (finite) Poincaré complex with $\dim Z \geqq 7$ and $\pi_1 Z = \pi$. Then there exists a finitely-dominated (finite) Poincaré pair $(Z', \partial Z')$ and a compact manifold pair $(Z'', \partial Z'')$ such that

(8.17) (i) all fundamental groups are isomorphic to π,

(ii) Z'' is homotopy equivalent to a 2-complex,

(iii) there is a simple homotopy equivalence

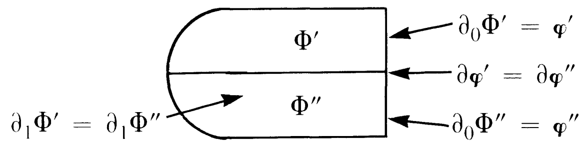
$$\alpha: \partial Z' \to \partial Z''$$

(iv) Z is simple homotopy equivalent (measured in $Wh(\hat{\mathbf{Q}}\pi)$) to $Z' \cup_\alpha Z''$.

We will apply this splitting result to X and P above and its obvious relative version to Q. The $\pi - \pi$ theorem now implies that the normal maps $(f, \hat{f})$, $(\varphi, \hat{\varphi})$ and $(g, \hat{g})$ also can be split: $f = f' \cup f''$ where

$$f'': (M'', \partial M'') \to (X'', \partial X'')$$

is a simple equivalence of pairs. In fact the explicit construction of (Q, P) in [**33**] allows us to assume that $M'' = P'' = U''$. Therefore the surgery problem $(f', \hat{f}')$ rel. boundary gives an element of $L_3^P(\mathbf{Z}\pi)$ and similarly for the others. Any normal cobordism from $(f_\tau, \hat{f}_\tau)$ to $(g, \hat{g})$ can also be split so that $(f'_\tau, \hat{f}'_\tau)$ is normally cobordant to $(g', \hat{g}')$ relative to the boundary. For $(\Phi, \hat{\Phi})$ we have triads:



Again using the π-π theorem, we assume that $(\Phi'', \partial_0\Phi'', \partial_1\Phi'')$ is a simple homotopy equivalence of manifold triads. By construction $(\varphi', \hat{\varphi}') = \partial_0(\Phi', \Phi')$ is a normal cobordism of both domain and range (rel. $\partial\varphi' = \partial\partial_1\Phi'$) to the simple homotopy equivalence $\partial_1\Phi' = \partial_1\Phi''$ of manifold pairs. Since the range spaces Q', $\partial_0 Q' = P'$ are just finitely-dominated, the conclusion is that

$$\lambda^P(\varphi', \hat{\varphi}') = 0.$$

Although X is perhaps not finite,

$$\sigma(X) = \sigma(X') = -\sigma(P')$$

and so the union $X' \cup P'$ along the boundary has

$$\sigma(X' \cup P') = 0.$$

To identify $\partial X'$, $\partial P'$ use the simple homotopy equivalence

$$\partial X' \simeq \partial X'' \simeq \partial P'' \simeq \partial P',$$

arising from (8.17). Let $\underline{\underline{X' \cup P'}}$ denote a finite cell structure on $X' \cup P'$ giving a $\mathbf{Z}\pi$-base for $C_*(X' \cup P')$ and choose a PD base $\mathbf{h}$ for $H_*(X' \cup P', \mathbf{Q})$. Now pick $\hat{\mathbf{Z}}_2\pi$-bases $\mathbf{b}'$, $\mathbf{c}'$ for $C_*(X')$, $C_*(P')$ such that

$$0 \to C_*(\partial X') \to C_*(X') \oplus C_*(P') \to C_*(X' \cup P') \to 0$$

is based exact. (Each manifold part X'', P'' has a given $\mathbf{Z}\pi$-base from the PL structure so $C_*(\partial X')$ is already based.) The bases $\mathbf{b}'$, $\mathbf{c}'$ then induce bases $\mathbf{b}$, $\mathbf{c}$ for $C_*(X)$ and $C_*(P)$ compatible with the PL structures. A similar discussion holds for the homology bases.

It follows that the normal map obtained by union on the boundary (composed with a homotopy equivalence):

(8.18) $(\psi, \hat{\psi}) = (\hat{f}' \cup \hat{\varphi}', \hat{f}' \cup \hat{\varphi}'): M' \cup U' \to X' \cup P' \simeq \underline{\underline{X' \cup P'}}$

defines an element $\lambda^Y(\psi, \hat{\psi})$ in $L_3^Y(\mathbf{Z}\pi)$. This is the element x in the statement:

$$i^P(\lambda^Y(\psi, \hat{\psi})) = \lambda^P(f, \hat{f}) \quad \text{since } \lambda^P(\varphi', \hat{\varphi}') = 0.$$

We now calculate

$$\delta_2^Y \circ \mathrm{Res}_\pi^\tau(\lambda^Y(\psi, \hat{\psi}))$$

following the definition (5.17).

In the τ-covering spaces we have:

(i) a finite structure $\underline{\underline{X}}'_\tau$ for X'_τ since

$$\sigma(X'_\tau) = \text{Res}\,\sigma(X) = 0, \quad \text{and}$$

(ii) a finite cell structure $\underline{\underline{P}}'_\tau$ for P'_τ (since $\sigma(P) = -\sigma(X)$) such that

$$\underline{\underline{(X' \cup P')}}_\tau = \underline{\underline{X}}'_\tau \cup \underline{\underline{P}}'_\tau$$

is a simple homotopy equivalence.

Therefore

$$(\varphi'_\tau, \varphi'_\tau): U'_\tau \to \underline{\underline{P}}'_\tau$$

defines an element in $L^Y_3(\mathbf{Z}\tau)$ and

$$i^Y_2(\lambda^Y(\varphi'_\tau, \hat{\varphi}'_\tau)) = \lambda^Y_2((\varphi'_\tau, \hat{\varphi}'_\tau) \otimes \hat{\mathbf{Z}}_2) = 0 \in L^Y_3(\hat{\mathbf{Z}}_2\tau)$$

because $(\varphi'_\tau, \varphi'_\tau) = \partial(\Phi'_\tau, \Phi'_\tau)$ and we can find a simple $\hat{\mathbf{Z}}_2\pi$-base for the pair (Q'_τ, P'_τ) extending the $\mathbf{Z}\pi$-base from the finite structure $\underline{\underline{P}}'_\tau$. By homological surgery we conclude that $(\varphi'_\tau, \hat{\varphi}'_\tau)$ is normally cobordant (rel ∂) to a $\mathbf{Z}_{(2)}$-homology equivalence with π_1-isomorphism

$$(h, \hat{h}): U'_+ \to P'_\tau$$

where $(U'_+, \partial U'_+)$ is a compact manifold pair. This normal cobordism together with the one relating $(f'_\tau, \hat{f}'_\tau)$ to

$$(g', \hat{g}'): N' \to \underline{\underline{X}}'_\tau$$

imply that $\text{Res}(\psi, \hat{\psi}) = (\psi_\tau, \hat{\psi}_\tau)$ is normally cobordant to the $\mathbf{Z}_{(2)}$-homology equivalence:

$$(8.19) \quad (g' \cup h, \hat{g}' \cup \hat{h}): N' \cup U'_+ \to \underline{\underline{(X' \cup P')}}_\tau.$$

Apply (8.6) to the quadratic mapping cone complex

$$C(g' \cup h) \otimes \hat{\mathbf{Z}}_2:$$

this is simple and acyclic (since $C(g' \cup h)$ is simple and $\mathbf{Z}_{(2)}$-acyclic). Therefore

$$(8.20) \quad \{\text{Res}(\hat{\Delta}_2(\underline{\underline{X' \cup P'}})) - \hat{\Delta}_2(N' \cup U'_+)\} \in H^0(Wh'(\hat{\mathbf{Z}}_2\tau))/L^K_0(\hat{\mathbf{Z}}_2\tau)$$

has image

$$i^Y_2(\text{Res}(\lambda^Y(\psi, \hat{\psi}))) = \text{Res}\; i^Y_2(\lambda^Y(\psi, \hat{\psi})) \in L^Y_3(\hat{\mathbf{Z}}_2\tau).$$

Now from (8.13) it follows that

$$(8.21) \quad \{\Delta_2(\underline{\underline{X' \cup P'}})\} = \Delta_0(X' \cup P) \in H^0(Wh(\hat{\mathbf{Q}}_2\pi))/H^0(Wh'(\mathbf{Z}\pi)) + d^*H'(\widetilde{K}_0(\mathbf{Z}\pi)).$$

But

$$0 \to C_*(\partial X') \to C_*(X', \mathbf{b}') \oplus C_*(P', \mathbf{c}') \to C_*(\underline{\underline{X' \cup P'}}) \to 0$$

is based exact so we can use (8.10), (8.11) to relate $\Delta_0(\underline{\underline{X' \cup P'}})$ to $\Delta_0(X)$ and $\Delta_0(P)$. The rest of the argument will now take place in $K_1(\hat{\mathbf{Q}}\pi)_+$ since $u = \bar{u}$ for these factors. Therefore by (8.7) $\mathbf{b}$, $\mathbf{c}$ are simple bases for $C_*(X)$, $C_*(P)$ and the formula of (8.11) is valid without indeterminancy. Using (8.11) we get:

$$\begin{aligned}\hat{\Delta}(\underline{\underline{X' \cup P'}}) &= \hat{\Delta}(X', \mathbf{b}') + \hat{\Delta}(P', \mathbf{c}') \\ &= \hat{\Delta}(X, \mathbf{b}) + \hat{\Delta}(P, \mathbf{c}) \in K_1(\hat{\mathbf{Q}}\pi)_+.\end{aligned}$$

From (8.10) applied to the pair (Q, P),

$$\{\hat{\Delta}(P, \mathbf{c})\} = 0 \quad \text{in } H^0(K_1(\hat{\mathbf{Q}}\pi)_+)$$

and so:

(8.22) $$\Delta_0(\underline{\underline{X' \cup P'}}) = \Delta_0(X) \in H^0(Wh(\mathbf{Q}\pi)_+)/H^0(Wh'(\mathbf{Z}\pi)) + d^*H^1(\widetilde{K}_0(\mathbf{Z}\pi)).$$

Finally, a similar argument for $\hat{\Delta}_2(N' \cup U'_+)$ gives:

(8.23) $$\{\Delta_2(N' \cup U_+)\} = \{\Delta(N)\} \in H^0(Wh'(\mathbf{Q}_2\tau)_+).$$

The formula

$$\operatorname{Res}^{\tau}_{\pi}(\delta^Y_2(x)) = \operatorname{Res}^{\tau}_{\pi}(\Delta_0(X)) - \{\Delta(N)\} \in H^0(Wh_2(\hat{\mathbf{Q}}_2\tau)_+)/I$$

now follows from (8.20)-(8.23).

Remark 8.24. This result will be applied in Section 9 for

$$\pi = Q(8a, b) \quad \text{and} \quad \tau = Q(4ab).$$

It will be shown that $L^Y_3(\hat{\mathbf{Z}}_2\pi)$ splits into ($\pm$) parts (cf. (7.9), (7.10)) and that the composite

$$\begin{aligned}L^Y_3(\hat{\mathbf{Z}}_2\pi)_+(ab) \xleftarrow{\approx} H^0(Wh'(\hat{\mathbf{Z}}_2\pi)_+(ab))/L^K_0(\hat{\mathbf{Z}}_2\pi)(ab) \\ \to H^0(Wh(\hat{\mathbf{Q}}_2\pi)_+(ab))/L^K_0(\hat{\mathbf{Z}}_2\pi)(ab)\end{aligned}$$

is injective. Furthermore we will show that

$$\operatorname{Res}^{\tau}_{\pi}: L^Y_3(\hat{\mathbf{Z}}_2\pi)_+(ab) \to L^Y_3(\hat{\mathbf{Z}}_2\tau)_+(ab)$$

is injective (cf. (7.7)) and conclude from (5.18) that Proposition 8.16 just established actually calculates

$$\lambda^P(f, \hat{f}) \in L^P_3(\mathbf{Z}\pi) \quad \text{if } \{\sigma(X)\} = 0 \in H^0(\widetilde{K}_0(\mathbf{Z}\pi)).$$

9. Calculation of the surgery obstruction. Throughout this section $\pi = Q(8a, b)$ is a special type IIM group and X is an $(8l + 3)$-dimensional

Swan complex for $\pi(l > 0)$ with almost-linear k-invariant and

$$\tau = Q(4ab) \subset Q(8a, b).$$

The main conclusion will be that condition $C(a, b)$ is equivalent to the vanishing of the top component of the L^p-surgery obstruction for a suitable normal map $(f, \hat{f}):M \to X$. This will establish Theorem C (by (6.20) and (7.3)).

In order to do this calculation (following (8.16)) we must first describe more precisely the top components in the exact sequence:

(9.1) $$0 \to K_1'(\mathbf{Z}\pi)(ab) \to K_1(\mathbf{Q}\pi)(ab) \to K_1(\hat{\mathbf{Q}}\pi)(ab)/K_1'(\hat{\mathbf{Z}}\pi)(ab) \to \widetilde{K}_0(\mathbf{Z}\pi)(ab) \to 0.$$

For convenience in this section we will assume (usually without further comment) that all K-groups are localized at 2.

PROPOSITION 9.1. ([**21**, Sections 2, 6]).

(i) $K_1(\mathbf{Q}\pi)_+(ab) = K_1(M_4(F)) \cong F^\times$ *and*

$$K_1(\mathbf{Q}\pi)_-(ab) = K_1(M_2(D)) \cong F^*$$

where $F = \mathbf{Q}[\eta_a, \eta_b]$ *and* D *is a (non-split) division algebra with centre* F. *(*$F^* \subset F^\times$ *denotes the units which are positive at the infinite primes.)*

(ii) $K_1(\hat{\mathbf{Q}}\pi)_\pm(ab)$ *is determined by the (split) exact sequences:*

$$0 \to \hat{D}(\pi)_\pm(ab) \to K_1(\hat{\mathbf{Q}}\pi)_\pm(ab)/K_1'(\hat{\mathbf{Z}}\pi)_\pm(ab) \to \hat{F}^\times/\hat{A}^\times \to 0$$

where $A = \mathbf{Z}[\eta_a, \eta_b]$ *and*

$$\hat{D}(\pi)_\pm(ab) \cong (A/ab)^\times.$$

(iii) *The exact sequence* (9.1) *splits into* $\pm$ *parts such at*

$$0 \to K_1'(\mathbf{Z}\pi)_+(ab) \to A^\times \xrightarrow{\hat{\Phi}_A} (A/ab)^\times \to \widetilde{K}_0(\mathbf{Z}\pi)_+(ab) \to \Gamma_A \to 0$$

and

$$0 \to K_1'(\mathbf{Z}\pi)_-(ab) \to A^* \xrightarrow{\hat{\Phi}_A} (A/ab)^\times \to \widetilde{K}_0(\mathbf{Z}\pi)_-(ab) \to \Gamma_A^* \to 0$$

where $\hat{\Phi}_A$ *is the reduction map (see Introduction) and* $\Gamma_A(\Gamma_A^*)$ *denotes the class group (strict class group) of* A.

(iv) *There is a similar splitting for* $\tau = Q(4ab)$ *with* F *replaced by* $E = \mathbf{Q}[\eta_{ab}]$ *and* A *replaced by* $B = \mathbf{Z}[\eta_{ab}]$. *The restriction map induces the inclusion* $F^\times \subset E^\times$.

Proof. Since this is given a detailed proof in [**21**] we make only a few remarks. The splitting into $\pm$ parts is clear for $K_1(\mathbf{Q}\pi)$, $K_1(\hat{\mathbf{Q}}\pi)$ and $K_1(\hat{\mathbf{Z}}_p\pi)$ if $p \neq 2$. When $p|ab$ we write $ab = p^n \cdot c$ where $p \nmid c$ and set

$$R = \mathbf{Z}(\zeta_c)[\mathbf{Z}/p^n \rtimes (\mathbf{Z}/2 \times \mathbf{Z}/2)].$$

Then

$$K_1'(\hat{\mathbf{Z}}_p\pi)_\pm(ab) \cong K_1'(\hat{\mathbf{Z}}_p \otimes R)$$

and there is an exact sequence

$$(1 + J)^\times \to K_1'(\hat{\mathbf{Z}}_p \otimes R) \to K_1'(\hat{\mathbf{Z}}_p(\zeta_c)[\mathbf{Z}/2 \times \mathbf{Z}/2]^t) \to 0$$

where

$$J = \langle 1 - T : T \in \pi \text{ has order } p^n \rangle.$$

After applying reduced rooms we get

$$Nrd_i(1 + J) \subseteq U^1(F_i) \quad \text{for each field } F_i = \mathbf{Q}(\eta_{p^i\alpha}\eta_b)$$

and (since J is topologically nilpotent),

$$Nrd(1 + J)^\times \supseteq U^{m_i}(F_i) \quad \text{for some } m_i \geqq 1.$$

Therefore after localizing at 2,

$$Nrd_i(1 + J)^\times = U^1(F_i)$$

and so

$$K_1(\hat{\mathbf{Q}}_p\pi)_\pm(ab)/K_1'(\hat{\mathbf{Z}}_p\pi)_\pm(ab) = \hat{F}_p^\times/\hat{A}_p^\times \times (A/pA)^\times.$$

For $p = 2$, the splitting into $\pm$ parts is a calculation:

$$K_1'(\hat{\mathbf{Z}}_2\pi)_+(ab) \cong (1 + 2\hat{A}_2)^\times, \quad K_1'(\hat{\mathbf{Z}}_2\pi)_-(ab) \cong \hat{A}_2^\times.$$

The sequences in (iii) now follow from (ii) and the definition:

(9.3) $$1 \to F^\times/A^\times \to \hat{F}^\times/\hat{A}^\times \to \Gamma_A \to 0.$$

COROLLARY 9.4. (cf. (7.7), (7.9)). (i) *There is a splitting of* $L_3^Y(\hat{\mathbf{Z}}_2\pi)(ab)$ *into* $\pm$ *parts compatible with the splitting of* $H^0(K_1'(\mathbf{Z}\pi)(ab))$ *and the restriction to* $\tau = Q(4ab)$ *such that*

$$L_3^Y(\hat{\mathbf{Z}}_2\pi)_+(ab) = A/2A, \quad L_3^Y(\hat{\mathbf{Z}}_2\pi)_-(ab) = H^0(\hat{A}_2^\times).$$

(ii) Res: $L_3^Y(\hat{\mathbf{Z}}_2\pi)_+(ab) \to L_3^Y(\hat{\mathbf{Z}}_2\tau)_+(ab)$ *is injective.*

We are now ready to specify the Swan complex X and the surgery problem $(f, \hat{f}): M \to X$. Consider the subgroups

$$\mathbf{Z}/2ab \subset Q(4ab) \subset Q(8a, b)$$

and let χ be a faithful character of $\mathbf{Z}/2ab$. The $\mathscr{F}$-representation with character $\chi + \chi^{-1}$ extends to $Q(4ab)$ and has k-invariant

$$g_0 = c_2(\chi + \chi^{-1}) \in H^4(Q(4ab); \mathbf{Z}).$$

One can check that there exists a generator

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$$g_1 \in H^4(Q(8a, b); \mathbf{Z})$$

whose restriction to $Q(4ab)$ is g_0 and (hence) whose restriction to the 2-Sylow subgroup $Q8$ is the Chern class of the standard $\mathscr{F}$-representation. Now let $X = X(g)$ the $(8l + 3)$-dimensional Swan complex for π whose k-invariant is

$$g = g_1^{2l+1} \in H^{8l+4}(\pi; \mathbf{Z}).$$

By construction g is an almost linear k-invariant. In fact Res g is a linear k-invariant on every type I or IIK subgroup of π.

Next, by induction of normal invariants as in [**22**], there exists a normal invariant for $X(g)$ which restricts to the normal invariant of the smooth space forms over $Q(4ab)$ and $Q8$. Let

$$(f, \hat{f}): M \to X$$

be a normal map with this normal invariant. Then in particular, if N denotes the space form arising from the $\mathscr{F}$-representation for $\tau = Q(4ab)$,

$$(f_\tau, \hat{f}_\tau): M_\tau \to X_\tau$$

is normally cobordant to a homotopy equivalence

$$(g, \hat{g}): N \to X_\tau.$$

Also by (6.20), for this normal map all the components

$$\lambda^P(f, \hat{f})(d) \in L_3^P(\mathbf{Z}\pi)(d)$$

are zero except possibly when $d = \alpha\beta$ where $1 \neq \alpha | a$ and $1 \neq \beta | b$. It is therefore enough (by (7.3)) to calculate the top component $\lambda^P(f, \hat{f})(ab)$.

Our first goal will be to examine the condition

$$\sigma_*(\lambda^P(f, \hat{f})) = 0$$

needed to apply (8.16). Since

$$\sigma_*(\lambda^P(f, \hat{f})) = \{\sigma(X)\} = \partial_*\{\hat{\Delta}(X)\} \in H^0(\widetilde{K}_0(\mathbf{Z}\pi))$$

we need to calculate the top component of $\hat{\Delta}(X)$. This will be done following the method of [**21**], [**2**].

PROPOSITION 9.5. *For* $\pi = Q(8a, b)$ *and* $\rho = \mathbf{Z}/2ab$ *the restriction map*

$$\mathrm{Res}: K_1(\hat{\mathbf{Q}}\pi)/K_1'(\hat{\mathbf{Z}}\pi) \to K_1(\hat{\mathbf{Q}}\rho)/K_1'(\hat{\mathbf{Z}}\rho)$$

is a 2-*local injection in the top component.*

Proof. From (9.2) the restriction map on the top component is calculated from the diagram (of split exact sequences):

$$\begin{tikzcd}
0 \to (A/ab)^{\times}_{(2)} \arrow[r] \arrow[d] & K_1(\hat{\mathbf{Q}}\pi)/K_1'(\hat{\mathbf{Z}}\pi)_{\pm}(ab) \arrow[r] \arrow[d, "\mathrm{Res}"] & \hat{F}^{\times}/\hat{A}^{\times} \to 0 \arrow[d] \\
0 \to (C/ab)^{\times}_{(2)} \arrow[r] & K_1(\hat{\mathbf{Q}}\rho)/K_1'(\hat{\mathbf{Z}}\rho)_{\pm}(ab) \arrow[r] & \hat{L}^{\times}/\hat{C}^{\times} \to 0
\end{tikzcd}$$

where $C = \mathbf{Z}[\zeta_{ab}]$ and $L = \mathbf{Q}[\zeta_{ab}]$. Since the restriction map on

$$K_1(\hat{\mathbf{Q}}\pi)_{\pm}(ab) = \hat{F}^{\times}$$

is just induced by the inclusion

$$\mathbf{Q}[\eta_a, \eta_b] \subset \mathbf{Q}[\zeta_{ab}]$$

of centres, both

$$(A/ab)^{\times}_{(2)} \to (C/ab)^{\times}_{(2)} \quad \text{and}$$
$$\hat{F}^{\times}/\hat{A}^{\times} \to \hat{L}^{\times}/\hat{C}^{\times}$$

are injective.

The basic diagram used to analyse $\hat{\Delta}(X)(ab)$ is

$$\begin{tikzcd}
0 \to K_1(\mathbf{Q}\rho)/K_1'(\mathbf{Z}\rho) \arrow[r, "j_\rho"] & K_1(\hat{\mathbf{Q}}\rho)/K_1'(\hat{\mathbf{Z}}\rho) \arrow[r, "\partial"] & \widetilde{K}_0(\mathbf{Z}\rho) \to 0 \\
0 \to K_1(\mathbf{Q}\pi)/K_1'(\mathbf{Z}\pi) \arrow[r, "j_\pi"] \arrow[u, "\mathrm{Res}"] & K_1(\hat{\mathbf{Q}}\pi)/K_1'(\hat{\mathbf{Z}}\pi) \arrow[r, "\partial"] \arrow[u, "\mathrm{Res}"] & \widetilde{K}_0(\mathbf{Z}\pi) \to 0 \arrow[u, "\mathrm{Res}"]
\end{tikzcd}$$

(9.6)

Consider the top components in this diagram and use the description (6.24) for the middle terms:

(9.7) $\quad K_1(\mathbf{Q}\rho)_{\pm}(ab) = \mathrm{Hom}_{\Omega}(R(\mathbf{Z}/ab), \mathbf{Q}[\zeta_{ab}]^{\times})_{\pm}$

where

$$\Omega = \mathrm{Gal}(\mathbf{Q}(\zeta_{ab})/\mathbf{Q})$$

and $R(\mathbf{Z}/ab)$ is the complex character ring. Let $\chi_{\alpha\beta,\pm 1}$ denote the characters of $\mathbf{Z}/2ab$ corresponding to divisors $\alpha|a$, $\beta|b$ with $\alpha \cdot \beta \neq 1$. If $T \in \mathbf{Z}/2ab$ is a generator then

(9.8) $\quad \chi_{\alpha\beta,\pm 1}(T) = \pm\zeta_{\alpha\beta} = \pm e^{2\pi i/\alpha\beta}.$

Since the k-invariant $g \in H^{8l+4}(\pi; \mathbf{Z})$ of our Swan complex X satisfies the relation

$$\mathrm{Res}\, g = [c_2(\chi + \chi^{-1})]^{2l+1}$$

where $\chi = \chi_{ab,-1}$ is a faithful character of $\mathbf{Z}/2ab$, the Reidemeister torsion is given by

$$\mathrm{Res}\,\hat{\Delta}(X) = [\Delta(L^3(\chi + \chi^{-1}))]^{2l+1}.$$

From [23] or [2], the torsion of this lens space $L^3(\chi + \chi^{-1})$ (expressed as a homomorphism via (9.7)) is given by

$$\text{(9.9)}\quad \begin{aligned} &f(\chi_{\alpha\beta,\pm 1}) = 2 \mp \eta_{\alpha\beta} \qquad \alpha \cdot \beta \neq 1 \\ &f(\chi_{1,1}) = 1/4a^2b^2, \qquad f(\chi_{1,-1}) = 4. \end{aligned}$$

Let $E^{\pm}$ denote the top component idempotents acting on

$$\mathrm{Hom}_{\Omega}(R(\mathbf{Z}/ab), \mathbf{Q}[\zeta_{ab}]^{\times})_{\pm},$$

then from (6.17)

$$\text{(9.10)}\quad \Delta(L^3(\chi + \chi^{-1}))_{\pm}(n) = \prod_{\bar{n}|\alpha\beta|n} Ef(\chi_{\alpha\beta,\pm 1}) \in \mathbf{Q}[\zeta_{ab}]^{\times}$$

where $n = ab$. Note that when translating from $K_1(\mathbf{Q}[\zeta_{ab}])$ to $\mathbf{Q}[\zeta_{ab}]^{\times}$ it is natural to switch to multiplication, hence the product sign. If

$$n = ab = p_1^{l_1} \dots p_t^{l_t}$$

is the primary decomposition, let $n_i = n/p_i$ and denote by $E_i^{\pm}$ the idempotent on

$$\mathrm{Hom}_{\Omega}(R(\mathbf{Z}(ab), \mathbf{Q}[\zeta_{ab}]^{\times})$$

corresponding to $\mathbf{Z}/n_i \subseteq \mathbf{Z}/ab$.

LEMMA 9.11.

$$\text{(i)}\quad E^{\pm} = \prod_{i=1}^{r} (1 - E_i^{\pm})$$

(ii) *For any* $f \in \mathrm{Hom}_{\Omega}(R(\mathbf{Z}/ab), \mathbf{Q}[\zeta_{ab}]^{\times})$ *and any* $\chi \in R(\mathbf{Z}/ab)$,

$$[E_i^{\pm} f(\chi)]^{p_i} = \begin{cases} N_i(f(\chi)), & \text{if } p_i | (n_i/d) \\ f(\chi^{p_i})^{\omega_i} N_i(f(\chi)), & \text{if } p_i \nmid (n_i/d). \end{cases}$$

Here $\omega_i \in \Omega = (\mathbf{Z}/ab)^{\times}$ *has image* $p_i^{-1} \in (\mathbf{Z}/n_i)^{\times}$, $d = |\ker \chi|$ *and*

$$N_i : \mathbf{Q}\zeta_{ab}^{\times} \to \mathbf{Q}\zeta_{n_i}^{\times}$$

is the norm map.

Proof. (i) This is clear from (6.9).

(ii) From (6.22) and (6.24) and set $m = n/d$, $m_i = n_i/d$:

$$[E_i^{\pm} f(\chi)]^{p_i} = \mathrm{Ind}^* f(\chi) = f\left(\sum_{l=0}^{p_i - 1} \chi^{lm_i + 1}\right).$$

When $p_i | m_i$, then

$$\mathrm{Gal}(\mathbf{Q}\zeta_m/\mathbf{Q}\zeta_{m_i}) \to (\mathbf{Z}/m)^{\times} \to (\mathbf{Z}/m_i)^{\times} \to 0$$

is just represented by $\{lm_i + 1 : 0 \leqq l \leqq p_i - 1\}$ so the formula follows. If $p_i \nmid m_i$ then for some l, $p_i | lm_i + 1$ and ω_i is represented by $1/p_i \in (\mathbf{Z}/m_i)^{\times}$.

These formulas allow us to calculate (9.10). The answer can be most neatly expressed when we note that elements in

$$\ker(\phi_B : E^{(2)} \to B/2B)$$

will not matter in our applications. Therefore we can neglect elements of $B^{\times 2}$ when working in $K_1(\mathbf{Q}\rho)(ab)$. We can also neglect squares from $(A/ab)^{\times}$ when calculating in $(C/ab)^{\times}$.

LEMMA 9.12. (i) *If $r|ab$ is not a prime power,*

$$\eta_r - 2 \in B^{\times} \cap C^{\times 2},$$

(ii) *For any $r|ab$,*

$$\eta_r + 2 \in B^{\times 2}.$$

Proof. Let $k = \dfrac{1 - r}{2}$ then

$$\eta_r \pm 2 = (\zeta_r^k \pm \zeta_r^{-k})^2.$$

Although $\eta_r + 2 \in B^{\times}$ always, $\eta_r - 2 \in B^{\times}$ only when r is composite.

An easy calculation gives:

LEMMA 9.13. *Let $N : \mathbf{Q}(\zeta_{pm}) \to \mathbf{Q}(\zeta_m)$ be the norm map and $m = p^a k$ with $(p, k) = 1$ and $a \geqq 0$. Then for $a = 0$,*

$$N(2 \pm \eta_{pk}) = \frac{2 \pm \eta_k}{(2 \pm \eta_k)^{\omega_p}}$$

where

$$\omega_p \in \mathrm{Gal}(\mathbf{Q}(\zeta_k)/\mathbf{Q}) = (\mathbf{Z}/k)^{\times}$$

is represented by r, $rp \equiv 1 \pmod{k}$. For $a > 0$,

$$N(2 \pm \eta_{pm}) = 2 \pm \eta_m.$$

Proof. If $a = 0$ then $\mathrm{Gal}(\mathbf{Q}(\zeta_{pk})/\mathbf{Q}(\zeta_k))$ is represented by

$$\{x \in (\mathbf{Z}/pk)^{\times} : x \equiv 1 \pmod{k}\}.$$

Let l_0 be defined by the equations $l_0 k + 1 = rp$, $0 \leqq l_0 \leqq p - 1$ then

$$N(1 - \zeta_{pk}) = \prod_{l \neq l_0} (1 - \zeta_{pk}^{lk+1})$$

$$= \zeta_k \cdot \frac{\prod_{l=0}^{p-1} (\zeta_{pk}^{-1} - \zeta_p^l)}{(1 - \zeta_{pk}^{l_0 k+1})}$$

$$= \zeta_k \frac{(\zeta_k^{-1} - 1)}{(1 - \zeta_k^r)}.$$

Therefore

$$N(2 - \eta_{pk}) = \frac{2 - \eta_k}{(2 - \eta_k)^{\omega_p}}$$

as required. The argument is similar for $2 + \eta_{pk}$. When $a > 0$, $\mathrm{Gal}(\mathbf{Q}(\zeta_{pm})/\mathbf{Q}(\zeta_m))$ is represented by $\{lk + 1:0 \leqq l \leqq p - 1\}$ so that

$$N(1 - \zeta_{pm}) = \prod_{l=0}^{p-1} (1 - \zeta_{pm}^{lm+1}) = \zeta_m(\zeta_m^{-1} - 1).$$

In order to state the next result we recall some notation: $F = \mathbf{Q}[\eta_a, \eta_b]$ and $\alpha \| n$ means that α is a full prime power divisor of n. Refer to (9.6) for the maps j_π and j_ρ.

In the statement below, the element

$$(v_\pm(a), v_\pm(b)) \in (A/ab)^\times \subseteq (C/ab)^\times$$

has values at the primes $p|a$ given by

$$\prod\{2 \mp \eta_\alpha:\alpha\|ab \text{ and } (\alpha, p) = 1\}$$

and similarly for $p|b$. For any integer n, $\bar{n}$ denotes the product of the distinct primes dividing n.

PROPOSITION 9.14. (i) *There exist elements* $w_+ \in F^\times$ *and* $w_- \in F^*$ *such that the image of*

$$\Delta(L^3(\chi + \chi^{-1}))_\pm(ab) \cdot j_\rho(\mathrm{Res}\ w_\pm)$$

in the top component of $K_1(\hat{\mathbf{Q}}\rho)/K_1'(\hat{\mathbf{Z}}\rho)$ *lies in the subgroup*

$$\hat{D}(\rho)_\pm(ab) = (C/ab)^\times.$$

Modulo squares from $(A/ab)^\times$ *its image is* $(v_\pm(\bar{a}), v_\pm(\bar{b}))$.

(ii) *Let r be the number of distinct primes dividing ab. Then both*

$$(-1)^{r+1} \cdot \Delta(L^3(\chi + \chi^{-1}))_+(ab) \cdot j_\rho(\mathrm{Res}\ w_+)$$

and

$$\Delta(L^3(\chi + \chi^{-1}))_-(ab) \cdot j_\rho(\mathrm{Res}\ w_-)$$

lie in $B^\times \cap C^{\times 2}$.

Proof. Let

$$n = ab = p_1^{l_1} \dots p_r^{l_r}$$

as above, denote the set of distinct primes dividing n by P and identify it with $(1, \dots, r)$. Write

$$P = P' \cup P'' \quad \text{where}$$

$$P' = \{p_i \in P: l_i = 1\} \quad \text{and} \quad P'' = \{p_i \in P: l_i > 1\}.$$

If $S = \{i_1, \dots, i_s\} \subseteq P$ then

$$N_S = N_{i_1} \circ N_{i_2} \circ \dots \circ N_{i_s} \quad \text{and}$$

$$E_S = E_{i_1} \circ E_{i_2} \circ \dots \circ E_{i_s}$$

(the $\pm$ will be suppressed here). For a homomorphism

$$f \in \mathrm{Hom}_\Omega(R(\mathbf{Z}/ab), \mathbf{Q}\zeta_{ab}^\times),$$

f_S denotes $f(\chi_{ab,\pm 1}^{||S||})$ where

$$||S|| = \prod\{p_i \in S\}.$$

Similarly

$$\omega(S) = \prod\{\omega_i: p_i \in S\} \quad \text{and} \quad \eta_S = \eta_{n/||S||}.$$

Notice that

$$\chi_{ab}^{||S||} = \chi_{ab/||S||}$$

from (9.8). Finally,

$$\eta_{\hat{S}} = \eta_{n/||\hat{S}||} \quad \text{where } ||\hat{S}|| = \prod\{p_i^{l_i}: p_i \in S\}.$$

By $|S|$ we mean the cardinality of S and when $S = \emptyset$, note E_S, N_S, $\omega(S)$ are the identity and $\eta_S = \eta_n$.

Using these notations and (9.11) (i) we see that the first aim must be to calculate

$$Ef(\chi_{ab,\pm 1}) = \prod_S E_S f(\chi_{ab,\pm 1})^{(-1)^{|S|}}.$$

To begin we assume that $n = ab$ is square-free and concentrate on the $(+)$ component. Then (9.11) (ii) gives

$$\text{(9.15)} \quad E_S f(\chi_{ab})^{||S||} = \prod_{(S_1,S_2)} \{N_{S_1}(f_{S_2})^{\omega(S_2)}: S_1 \cup S_2 = S\}$$

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by induction. Here the product is over ordered pairs (S_1, S_2) of disjoint subsets of S. Next by (9.13) for $S = S_1 \cup S_2$,

$$(9.16) \quad N_{S_1}(2 - \eta_{S_2}) = \prod_{T \subseteq S_1} (2 - \eta_S)^{\omega(T)(-1)^{|T|}}$$

provided that $S \neq P$. Putting these together we obtain (for $S \neq P$):

$$\prod_{(S_1, S_2)} N_{S_1}(f_{S_2})^{\omega(S_2)} = (2 - \eta_S)^{\omega}$$

where

$$\omega = \sum_{(S_1, S_2)} \sum_{T \subseteq S_1} \omega(T \cup S_2)(-1)^{|T|} = 1$$

so the result is

$$(9.17) \quad Ef(\chi_{ab,+1}) = \prod_S \{ (2 - \eta_S)^{(-1)^{|S|}} : |S| \leqq |P| - 2\} \cdot j_\rho(\mathrm{Res}\, w_+)^{-1}$$

where

$$(9.18) \quad w_+(ab) = \prod_{|S| \geqq |P|-1} \prod_{(S_1, S_2)} \{N_{S_1}(f_{S_2})^{\omega(S_2)} : S_1 \cup S_2 = S\}^{-1}.$$

Notice that

$$N_{S_1}(f_{S_2}) \in \mathbf{Q}(\eta_S) \subseteq \mathbf{Q}(\eta_a, \eta_b) = F$$

whenever $|S| \geqq |P| - 1$. By the same calculation

$$(9.19) \quad Ef(\chi_{ab,-1}) \cdot j_\rho(\mathrm{Res}\, w_-(ab))$$
$$= \prod_S \{ (2 + \eta_S)^{(-1)^{|S|}} : |S| \leqq |P| - 2\}$$

with $w_-(ab)$ given by (9.18) also.

Now we consider the general case where $P = P' \cup P''$. Let $S' = S \cap P'$, $S'' = S \cap P''$ and then

$$(9.20) \quad E_S f(\chi_{ab,\pm 1}) = \prod_{(S_1, S_2)} \{N_{S_1 \cup S''}(f_{S_2})^{\omega(S_2)} : S_1 \cup S_2 = S'\}.$$

Using the same method as above we get:

$$(9.21) \quad Ef(\chi_{ab,\pm 1}) = \prod_S \{N_{S''}(2 \mp \eta_{S'})^{(-1)^{|S|}} : S' \cup S'' = S\}.$$

From this it follows by (9.13) that

$$(9.22) \quad Ef(\chi_{ab,\pm 1}) = \prod_S (2 \mp \eta_S)^{(-1)^{|S|}}.$$

Note that since $P'' \neq \emptyset$, $n/\|S\|$ is never a prime power so

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$$\eta_S \mp 2 \in B^\times \cap C^{\times 2}$$

by (9.12) for any $S \subseteq P$. Therefore

$$Ef(\chi_{ab,\pm 1}) \in B^\times \cap C^{\times 2}$$

(the number of terms in the product is 2^r). Similarly the number of terms in the product of (9.17) is $2^r - r - 1$ so that (when ab is squarefree):

$$(-1)^{r+1} Ef(\chi_{ab,+1}) \cdot j_\rho(\text{Res } w_+(ab)) \in B^\times \cap C^{\times 2}.$$

Part (ii) now follows from the formula

$$(9.23) \quad \Delta(L^3(\chi + \chi^{-1}))_\pm(n) = \prod_{\bar{n}|\alpha\beta|n} Ef(\chi_{\alpha\beta,\pm 1})$$

when we define $w_\pm = w_\pm(\overline{ab})$ as in (9.18).

To obtain part (i) we must calculate the effect of the reduction maps

$$r_{\not{p}_i}: \mathbf{Z}[\zeta_{ab}]^\times \to (\mathbf{Z}[\zeta_{ab}]/p_i)^\times$$

for each $\not{p}_i | p_i$ on the right-hand side of (9.17) or (9.22). For this write $S = \{i\} \cup S_0$ when $i \in S \subseteq P$. For each $\not{p}_i | p_i$:

$$r_{\not{p}_i}(2 \mp \eta_S) = \begin{cases} 2 \mp \eta_{S \cup \{\hat{i}\}} & i \notin S \\ 2 \mp \eta_{S_0 \cup \{\hat{i}\}} & i \in S. \end{cases}$$

From this we obtain (for ab squarefree):

$$r_{\not{p}_i}(Ef(\chi_{ab,\pm 1}) \cdot j_\rho(\text{Res } w_\pm(ab))) = \prod_{j \neq i} (2 \mp \eta_{\not{p}_j}) \in (A/ab)^\times.$$

When $P'' \neq \emptyset$ (ab not squarefree):

$$r_{\not{p}_i}(Ef(\chi_{ab,\pm 1})) = \prod_{i \in S} (2 - \eta_{S_0 \cup \{\hat{i}\}})^{(-1)^{|S|}}$$

$$\times \prod_{i \notin S} (2 - \eta_{S \cup \{\hat{i}\}})^{(-1)^{|S|}} = 1.$$

Therefore from (9.23) we obtain

$$r_{\not{p}_i}(\Delta(L^3(\chi + \chi^{-1}))_\pm(n) \cdot j_\rho(\text{Res } w_\pm)) = (v_\pm(\bar{a}), v_\pm(\bar{b})).$$

To obtain $\hat{\Delta}(X)(ab)$ from this calculation we need one further result about $A = \mathbf{Z}[\eta_a, \eta_b]$.

LEMMA 9.24. [**2**, 3.2]. *There exists a unit $v_{a,b} \in A^\times$ such that for each $\not{p}|p$,*

$$r_{\not{p}}(v_{a,b}) = \begin{cases} v_p(ab) & \text{if } p|a \\ (-1) \cdot v_p(ab) & \text{if } p|b \end{cases}$$

where

$$v_p(ab) = \prod_{\alpha \| ab} \{ (2 - \eta_\alpha):(\alpha, p) = 1\}.$$

Remark. In the notation of the Introduction (see the discussion of condition $C(a, b)$) and recall $r = \#$ primes dividing ab):

$$\Phi_A((-1)^{r+1} \cdot v_{a,b}) = (1, -1) \cdot V(a, b) \in (A/ab)^\times.$$

PROPOSITION 9.25. (i) *The top component $\hat{\Delta}(X)_\pm(ab)$ lies in*

$$\hat{D}(\pi)_\pm(ab) = (A/ab)^\times_{(2)} \subseteq K_1(\hat{\mathbf{Q}}\pi)(ab)/K_1'(\hat{\mathbf{Z}}\pi)(ab).$$

$$\text{(ii)} \quad \hat{\Delta}(X)_+(ab) = (1, -1) \cdot \Phi_A(u_{a,b})/j_\pi(w_+)$$

modulo squares in $(A/ab)^\times$ where $u_{a,b} = v_{\bar{a},\bar{b}}$ from (9.24).

$$\text{(iii)} \quad \hat{\Delta}(X)_-(ab) = (4, 4) \cdot \Phi_A(u'_{a,b})/j_\pi(w_-)$$

modulo squares in $(A/ab)^\times$ where

$$u'_{a,b} = \prod\{ (2 + \eta_\alpha):\alpha\|\overline{ab}\}.$$

Proof. From (9.14) and the fact that

$$\text{Res } \hat{\Delta}(X) = [\Delta(L^3(\chi + \chi^{-1}))]^{2l+1}$$

we get the relation

$$\text{Res}(\hat{\Delta}(X)_+(ab) \cdot j_\pi(w_+)/\Phi_A(u_{a,b})) = (1, -1)$$

modulo squares from $(A/ab)^\times_{(2)}$ and the assertions now follow from (9.5). The argument is similar for $\hat{\Delta}(X)_-(ab)$.

Now that $\hat{\Delta}(X)(ab)$ has been calculated in $H^0((A/ab)^\times)$, we can proceed to the first step in determining $\lambda^P(f, \hat{f})$ for the normal map $(f, \hat{f}):M \to X$ described above. This is the cohomology finiteness obstruction:

$$\sigma_*(\lambda^P(f, \hat{f})) = \{\sigma(X) \} = \partial_*\{\hat{\Delta}(X) \} \in H^0(\widetilde{K}_0(\mathbf{Z}\pi))$$

where the map ∂ is from (9.6).

PROPOSITION 9.26. (i) *The cohomology finiteness obstruction $\{\sigma(X) \}$ is zero if and only if there exists an element $U_+ \in F^{(2)}$ such that*

$$\Phi_A(U_+) = (1, -1) \in H^0((A/ab)^\times).$$

(ii) *The condition in* (i) *is satisfied precisely when either a or b is divisible only by primes $p \equiv 1(4)$.*

Proof. It is convenient to divide the exact sequences of (9.2) (iii) into short exact sequences by setting

(9.27) $D(\pi)_+(ab) = \operatorname{coker}(\hat{\Phi}_A : A^\times \to (A/ab)^{\hat{\times}}_{(2)})$

and similarly

$$D(\pi)_-(ab) = \operatorname{coker}(\hat{\Phi}_A : A^* \to (A/ab)^{\times}_{(2)}).$$

From diagram (9.6) and (9.25), it follows that

$$\sigma_*(\lambda^P(f, \hat{f}))_\pm(ab)$$

is represented by the class of $(1, -1)$ or $(4, 4)$ in

$$\operatorname{Im}(H^0(D(\pi)_\pm(ab)) \to H^0(\widetilde{K}_0(\mathbf{Z}\pi)_\pm(ab)).$$

Since $(4, 4)$ is a square in $D(\pi)_-(ab)$,

$$\sigma_*(\lambda^P(f, \hat{f}))_-(ab) = 0$$

and it remains to evaluate the $(+)$-component.

Consider the following diagram of exact sequences:

$$\begin{tikzcd}
H^0(F^\times/\ker \Phi) \arrow[r, bend left] \arrow[dr] & H^0(F^\times/A^\times) \arrow[r, bend left] \arrow[dr] & H^0(\hat{F}^\times/\hat{A}^\times) \\
H^1(\Gamma_A) \arrow[ur] \arrow[dr] & H^0(\hat{F}^\times/\hat{A}^\times \times (A/ab)^\times) \arrow[ur] \arrow[dr] & H^1(A^\times/\ker \Phi) \\
H^0((A/ab)^\times) \arrow[ur] \arrow[r, bend right] & H^0(D(\pi)_+(ab)) \arrow[ur] \arrow[r, bend right] & H^0(\widetilde{K}_0(\mathbf{Z}\pi)_+(ab)) \\
F^{(2)} \arrow[u, "\Phi_A"] & &
\end{tikzcd}$$

(9.28)

arising from the cohomology sequences of (9.2) (iii) and (9.3). We fix a splitting of (9.2) (ii) (which appears in the middle of our diagram) and note that the coboundary

$$d^* : H^1(\Gamma_A) \to H^0(F^\times/A^\times)$$

is injective and has image $F^{(2)}F^{\times 2}A^\times$. By a diagram chase, it can now be checked that

$$(1, -1) \in \operatorname{Im}(H^1(\Gamma_A) \to H^0(D(\pi)_+(ab)))$$

if and only if

$$(1, -1) \in \operatorname{Im}(\Phi_A : F^{(2)} \to H^0((A/ab)^\times)).$$

For part (ii), we notice that the condition (i) is clearly satisfied when either a or b is only divisible by $p \equiv 1(4)$ since then $(1, -1)$ or $(-1, 1)$ is a square in $(A/ab)^{\times}$. Conversely, if there exist $p|a$, $q|b$ with $p \equiv q \equiv 3(4)$ then restrict to $Q(8p, q) \subseteq (8a, b)$ and consider the norms:

$$\begin{tikzcd} \mathbf{Q}(\eta_p, \eta_q)^{(2)} \arrow[r, "\Phi_A"] \arrow[d, "N"] & (\mathbf{Z}[\eta_q]/p)^{\times} \times (\mathbf{Z}[\eta_p]/q)^{\times} \arrow[d, "N"] \\ \mathbf{Q}^{(2)} \arrow[r, "\Phi_{\mathbf{Z}}"] & \mathbf{F}_p^{\times} \times \mathbf{F}_q^{\times} \end{tikzcd}$$

Since $\mathbf{F}_p^{\times} \times \mathbf{F}_q^{\times} \subset (A/pq)^{\times}$ is the diagonal inclusion, $(1, -1) \in \operatorname{Im} \Phi_A$ if and only if $(1, -1) \in \Phi_{\mathbf{Z}}$. But $\mathbf{Q}^{(2)} = \mathbf{Z}^{\times}$ so clearly $(1, -1) \notin \Phi_{\mathbf{Z}}$.

The preceeding results amount to a calculation of the top component of the surgery obstruction $\lambda^p(f, \hat{f})(ab)$. Recall from (8.16) that it is necessary to calculate

$$\operatorname{Res}^{\tau}_{\pi}(\Delta_0(X)) - \{\Delta(N)\} \quad \text{where } \tau = Q(4ab)$$

and N is the orbit space (under τ) of the $\mathscr{F}$ representation $\chi + \chi^{-1}$. Clearly

$$\operatorname{Res}^{\rho}_{\tau}(\Delta(N)) = (\Delta(L^3(\chi + \chi^{-1}))^{2l+1}$$

determines $\Delta(N)$.

PROPOSITION 9.29. *Suppose that*

$$\sigma_*(\lambda^P(f, \hat{f}))(ab) = 0$$

and that $U_+ \in F^{(2)}$ *is chosen with* $\Phi_A(U_+) = (1, -1)$. *Then*

$$\operatorname{Res}^{\tau}_{\pi}(\Delta_0(X)_+(ab)) - \{\Delta(N)_+(ab)\}$$
$$= \operatorname{Res}^{\tau}_{\pi}\{(-1)^{r+1} \cdot U_+ \cdot u_{a,b}\}$$

in $H^0(K_1(\hat{\mathbf{Q}}_2\tau)_+(ab))/I$ *where* $r = \#$ *primes dividing ab and*

$$I = \operatorname{Res}^{\tau}_{\pi}[H^0(Wh'(\mathbf{Z}\pi)) + d^*H^1(\widetilde{K}_0(\mathbf{Z}\pi))] + L^K_0(\hat{\mathbf{Z}}_2\tau).$$

Proof. We begin by calculating the top component of I. Since

$$H^0(Wh'(\mathbf{Z}\pi)_+(ab)) = \ker(\Phi_A : A^{\times}/A^{\times 2} \to H^0((A/ab)^{\times}))$$

by (9.2) (iii), the remaining part is $\operatorname{Im} d^*$. For this recall that from (8.14), $\operatorname{Im} d^* = \operatorname{Im} d_0^*$ in $H^0(Wh(\hat{\mathbf{Q}}_2\pi))$ where d_0^* is the coboundary in the cohomology sequence of (8.3) used in defining Δ_0. On the top-component this sequence (8.3) localized at 2 becomes:

(9.30) $$0 \to F^{\times}/K_1'(\mathbf{Z}\pi)_+(ab) \to \hat{F}^{\times}/\hat{A}^{\times} \times (A/ab)^{\times} \to \widetilde{K}_0(\mathbf{Z}\pi)_+(ab) \to 0$$

from which it is clear that

(9.31) $\quad \operatorname{Im} d_0^* = \ker(\Phi_A : F^{(2)}/F^{\times 2} \to H^0((A/ab)^{\times}))$.

The calculation of the proposition will therefore take place in

$$\operatorname{Im}(\varphi_B : E^{(2)}/E^{\times 2} \cdot \ker \Phi_A \to (B/2B)/\ker \Phi_A).$$

Compare (7.9) to see that φ_B is the map induced by the inclusion

$$K_1(\mathbf{Q}\tau) \to K_1(\hat{\mathbf{Q}}_2\tau).$$

Since the indeterminacy in $\Delta(N)$ lies in $B^{\times} \cap C^{\times 2} \subseteq \ker \varphi_B$ it can be neglected. The result now follows directly from (9.14) (ii) and (9.25) (ii).

The proof of Theorem C is now complete: from (8.16) and the formula just established we have (set $\bar{U}_+ = (-1)^{r+1} \cdot U_+$ for short)

(9.32) $\quad \operatorname{Res}_\pi^\tau(\delta_2^Y(x)) = \operatorname{Res}_\pi^\tau(\bar{U}_+ \cdot u_{a,b}) \in H^0(Wh(\hat{\mathbf{Q}}_2\tau)_+)/I$

where $x \in L_3^Y(\mathbf{Z}\pi)$ is an element so that

$$i^P(x) = \lambda^P(f, \hat{f}) \quad \text{in } L_3^P(\mathbf{Z}\pi).$$

But both sides of (9.32) lie in the image of

$$H^0(Wh'(\hat{\mathbf{Z}}_2\pi)_+)/L_0^K(\hat{\mathbf{Z}}_2\pi)$$

on which $\operatorname{Res}_\pi^\tau$ is injective by (9.4), therefore

(9.33) $\quad \delta_2^Y(x) = \{\bar{U}_+ \cdot u_{a,b}\} \in H^0(Wh'(\hat{\mathbf{Z}}_2\pi)_+)/L_0^K(\hat{\mathbf{Z}}_2\pi)$

$$+ H^0(Wh'(\mathbf{Z}\pi)) + d^*H^1(\tilde{K}_0(\mathbf{Z}\pi)).$$

Now the identification $\mathbf{i}^P$ of

$$\ker(L_3^P(\mathbf{Z}\pi) \to H^0(\tilde{K}_0(\mathbf{Z}\pi)))$$

with a subquotient of $L_3^Y(\hat{\mathbf{Z}}_2\pi)$ given in (5.18) shows that

$$\lambda^P(f, \hat{f})_+(ab) = 0 \quad \text{if and only if} \quad \{\bar{U}_+ \cdot u_{a,b}\} = 0$$

in the indicated range of (9.33). By our calculations above (cf. (9.4) and (9.31)) this range is $(A/2A)/\ker \Phi_A$ and the element $\bar{U}_+ \cdot u_{a,b} \in F^{(2)}$ is mapped in by φ_A. Since $\ker \Phi_A$ is just the indeterminacy in the choice of U_+, it follows that

$$\lambda^P(f, \hat{f})_+(ab) = 0$$

if and only if

(9.34) $\quad (-1)^{r+1} \cdot (1, -1) \cdot \Phi_A(u_{a,b}) \in \operatorname{Im}(\Phi_A | \ker \varphi_A)$.

When ab is squarefree,

$$(-1)^{r+1} \cdot (1, -1) \cdot \Phi_A(u_{a,b}) = V(a, b)$$

and (9.34) is just the condition $C(a, b)$ in Definition (0.3). In general (9.34) says that

$$\lambda^p(f, \hat{f})_+(ab) = 0$$

if and only if condition $C(\bar{a}, \bar{b})$ is satisfied. But the norm map

$$\mathbf{Z}[\eta_a, \eta_b]^\times \to \mathbf{Z}[\eta_{\bar{a}}, \eta_{\bar{b}}]^\times$$

is a 2-local isomorphism so that condition $\mathbf{C}(\bar{a}, \bar{b})$ is satisfied if and only if condition $C(a, b)$ is satisfied.

Remark. In (9.31) we have given the (+)-part of the indeterminacy $\mathrm{Im}\, d^* = \mathrm{Im}\, d_0^*$. A similar calculation shows

(9.35) $\quad (\mathrm{Im}\, d^*)_- = \ker (F^{*(2)}/F^{\times 2} \to H^0((A/ab)^\times)).$

By comparing with (7.11) we see that this is just the (−)-part of

$$\mathrm{Im}(L_3^Y(\mathbf{Z}\pi)(ab) \to L_3^Y(\hat{\mathbf{Z}}\pi)(ab)).$$

This checks the result of (5.16)-(5.18) that the type *Sp* factors of $\mathbf{Q}\pi$ do not contribute to the L^P-obstruction.

10. Appendix: The condition $C(a, b)$. Although the arithmetic condition $C(a, b)$ seems difficult to check in general, it is possible to derive some easier conditions when $a = p$ and $b = q$ are both primes. The first of these was obtained by R. J. Milgram in his work on the space-form problem [**25**, **26**].

PROPOSITION 10.1. (Milgram). *A necessary condition for the condition* $C(p, q)$ *to be satisfied is that the Legendre symbols* $\left(\frac{p}{q}\right) = 1$ *and* $\left(\frac{q}{p}\right) = 1.$

Proof. See [**26**, Theorem B] (or [**21**, 7.8]). Note that the result must be symmetric in p and q since

$$Q(8p, q) \cong Q(8q, p).$$

On the other hand, let $\mathrm{Ord}_q(p)$ denote the order of p in $\mathbf{F}_q^\times$.

PROPOSITION 10.2. *The condition* $C(p, q)$ *is satisfied if* $\mathrm{Ord}_p(q)$ *and* $\mathrm{Ord}_q(p)$ *are both odd.*

Proof. See [**21**, 7.9]. The point is that

$$V(p, q) = (\eta_q - 2, \eta_p - 2)$$

is a square in $(A/pq)^\times$ under the given conditions.

Unfortunately these are not equivalent in general (e.g. $p = 5$, $q = 19$). A more detailed study of condition $C(p, q)$ has been made in [**3**]: when

$\left(\frac{p}{q}\right) = +1$, let $\left(\frac{p}{q}\right)_4 = +1$ if $p \in \mathbf{F}_q^{\times 4}$ and $\left(\frac{p}{q}\right)_4 = -1$ otherwise.

PROPOSITION 10.3. (Bentzen). (i) *If condition $C(p, q)$ is satisfied then*

$$\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right) = +1 \quad \text{and} \quad \left(\frac{p}{q}\right)_4 \left(\frac{q}{p}\right)_4 = +1$$

provided $p \not\equiv 1 \pmod{16}$ *and* $q \not\equiv 1 \pmod{16}$.

(ii) *If* $p \equiv 3(4)$, *then condition $C(p, q)$ is satisfied if and only if* $q \equiv 1(4)$ *and* $\mathrm{Ord}_q(p)$ *is odd.*

(iii) *If* $p \equiv q \equiv 5(8)$ *then condition $C(p, q)$ is equivalent to the conditions in* (i).

(iv) *If* $p \equiv 5(8)$ *and* $q \equiv 1(8)$ *and condition $C(p, q)$ is satisfied then* $\mathrm{Ord}_p(q)$ *is odd and* $4 \nmid \mathrm{Ord}_q(p)$.

(v) *If* $p \equiv 1(8)$ *and* $q \equiv 1(8)$ *and condition $C(p, q)$ is satisfied then*

$$\left(\frac{p}{q}\right)_4 = -1 \rightarrow 4 \mid \mathrm{Ord}_p(q) \text{ and } \left(\frac{q}{p}\right)_4 = -1 \rightarrow 4 \mid \mathrm{Ord}_q(p).$$

From Dirichlet's theorem on primes in an arithmetic progression we then get

COROLLARY 10.4. *There are an infinite number of primes (p, q) with* $p \equiv 3(4)$ *and* $q \equiv 1(4)$ *such that condition $C(p, q)$ is satisfied.*

Using the criteria of (10.3) Bentzen has examined the ratio of prime pairs (p, q) which satisfy condition $C(p, q)$. For the approximately 18,200 pairs in the range $pq < 100{,}000$ between 11.2% and 12.4% satisfy the condition and the corresponding groups $Q(8p, q)$ act semi-freely on $(\mathbf{R}^{8l+4}, 0)$.

In the range $pq < 2{,}000$ Bentzen has found all groups $Q(8p, q)$ which act semi-freely on $(\mathbf{R}^{8l+4}, 0)$. There are 42 groups corresponding to the pairs:

$$p = 3, q = 13, 109, 181, 229, 277, 313, 421, 433, 541, 601$$

$$p = 5, q = 11, 29, 31, 71, 101, 131, 151, 181, 191, 211, 229, 251, 271, 311, 331, 349$$

$$p = 7, q = 29, 37, 109$$

$$p = 13, q = 53, 61, 79, 101, 107, 131, 139$$

$$(p, q) = (11, 157), (17, 103), (19, 101), (23, 29), (29, 59), (37, 47).$$

(10.5)

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In the same range $pq < 2{,}000$ Bentzen also studies which groups $Q(8p, q)$ can act freely on S^{8l+3} (cf. Theorem $\hat{C}$ of Section 0). He proves that there are only 4 such groups, namely the ones corresponding to the prime pairs:

(10.6) $(p, q) = (3, 313), (3, 433), (3, 601)$ and $(17, 103)$.

In these cases $p \equiv 3(4)$ and $q \equiv 1(8)$, but there are also groups $Q(8p, q)$ with $p \equiv q \equiv 5(8)$ which act freely on S^{8l+3}, for example in the case $(p, q) = (5, 461)$. The smallest group $Q(8p, q)$ where the results from **[3]** do not decide if it can act freely is $(p, q) = (5, 401)$.

Note that the actions given in the 38 cases from (10.5) which are not in (10.6) are topologically interesting since they do not arise from "coning" a free action on S^{8l+3} and hence have no invariant $S^{8l+3} \subset \mathbf{R}^{8l+4} - \{0\}$.

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