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ON THE DIOPHANTINE EQUATION $(8n)^x + (15n)^y = (17n)^z$

ZHI-JUAN YANG and MIN TANG ✉

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Abstract

Let a, b, c be relatively prime positive integers such that $a^2 + b^2 = c^2$. Half a century ago, Jeśmanowicz [‘Several remarks on Pythagorean numbers’, *Wiadom. Mat.* **1** (1955/56), 196–202] conjectured that for any given positive integer n the only solution of $(an)^x + (bn)^y = (cn)^z$ in positive integers is $(x, y, z) = (2, 2, 2)$. In this paper, we show that $(8n)^x + (15n)^y = (17n)^z$ has no solution in positive integers other than $(x, y, z) = (2, 2, 2)$.

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1. Introduction

Let n be a positive integer and let (a, b, c) be a primitive Pythagorean triple such that $a^2 + b^2 = c^2$, $(a, b, c) = 1$, and $2 \mid b$. It is well known that $a = u^2 - v^2$, $b = 2uv$, $c = u^2 + v^2$ with $u > v > 0$, $2 \mid uv$ and $(u, v) = 1$. Clearly, the Diophantine equation

$$(na)^x + (nb)^y = (nc)^z \tag{1.1}$$

has the solution $(x, y, z) = (2, 2, 2)$. In 1956, Sierpiński [7] showed there were no other solutions when $n = 1$ and $(a, b, c) = (3, 4, 5)$, and Jeśmanowicz [2] proved that when $n = 1$ and $(a, b, c) = (5, 12, 13), (7, 24, 25), (9, 40, 41)$ or $(11, 60, 61)$, then (1.1) has only the solution $(x, y, x) = (2, 2, 2)$. Moreover, he conjectured that (1.1) has no positive integer solutions for any n other than $(x, y, z) = (2, 2, 2)$.

In 1998, Deng and Cohen [1] proved the following two theorems.

Theorem A. *Let* $a = 2k + 1$, $b = 2k(k + 1)$, $c = 2k(k + 1) + 1$, *for some positive integer* k. *Suppose that* a *is a prime power, and that the positive integer* n *is such that either* $C(b) \mid n$ *or* $C(n) \nmid b$, *where* $C(n)$ *is the product of distinct primes of* n. *Then the only solution of the Diophantine equation* $(na)^x + (nb)^y = (nc)^z$ *is* $x = y = z = 2$.

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THEOREM B. *For each of the Pythagorean triples* $(a, b, c) = (3, 4, 5)$, $(5, 12, 13)$, $(7, 24, 25)$, $(9, 40, 41)$ *and* $(11, 60, 61)$, *and for any positive integer n, the only solution of the Diophantine equation* $(na)^x + (nb)^y = (nc)^z$ *is* $x = y = z = 2$.

In 1999, Le Maohua [5] obtained certain conditions for (1.1) to have positive integer solutions (x, y, z) with $(x, y, z) \neq (2, 2, 2)$. For other related problems, see [3, 4, 6, 8].

In this paper, we consider (1.1) with $(a, b, c) = (8, 15, 17)$ and obtain the following result.

THEOREM. *For any positive integer n, the only solution of the Diophantine equation*

$$(8n)^x + (15n)^y = (17n)^z \tag{1.2}$$

is $(x, y, z) = (2, 2, 2)$.

2. Proofs

LEMMA 1 [1, Lemma 2]. *If* $z \geq \max\{x, y\}$, *then the Diophantine equation* $a^x + b^y = c^z$, *where a, b and c are any positive integers (not necessarily relatively prime) such that* $a^2 + b^2 = c^2$, *has no solution other than* $(x, y, z) = (2, 2, 2)$.

LEMMA 2 [9]. *The only solution of the Diophantine equation* $(4n^2 - 1)^x + (4n)^y = (4n^2 + 1)^z$ *is* $(x, y, z) = (2, 2, 2)$.

PROOF OF THEOREM. By Lemma 2, we know that the Diophantine equation $8^x + 15^y = 17^z$ has the single solution $(x, y, z) = (2, 2, 2)$. Suppose that (1.2) has solutions other than $x = y = z = 2$, and $n \geq 2$. By Lemma 1 we have $z < \max\{x, y\}$.

Case 1. $x > y$.

Subcase 1.1 $z \leq y < x$. Then

$$n^{y-z}(8^x n^{x-y} + 15^y) = 17^z. \tag{2.1}$$

If $(n, 17) = 1$, then by (2.1) and $n \geq 2$ we have $y = z$. Thus

$$8^x n^{x-y} + 15^y = 17^y. \tag{2.2}$$

We have $(-1)^y \equiv 1 \pmod 4$, so y is even. Write $y = 2y_1$. By (2.2),

$$8^x n^{x-y} = (17^{y_1} - 15^{y_1})(17^{y_1} + 15^{y_1}).$$

Noting that $(17^{y_1} - 15^{y_1}, 17^{y_1} + 15^{y_1}) = 2$, then

$$2^{3x-1} \mid 17^{y_1} - 15^{y_1}, \quad 2 \mid 17^{y_1} + 15^{y_1}, \tag{2.3}$$

or

$$2 \mid 17^{y_1} - 15^{y_1}, \quad 2^{3x-1} \mid 17^{y_1} + 15^{y_1}. \tag{2.4}$$

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However,

$$2^{3x-1} > 2^{3y-1} = 2^{6y_1-1} > 2^{5y_1} = (17+15)^{y_1} > 17^{y_1} + 15^{y_1} > 17^{y_1} - 15^{y_1},$$

which contradicts both (2.3) and (2.4).

If $(n, 17) = 17$, then write $n = 17^r n_1$, where $r \geq 1$ and $17 \nmid n_1$. By (2.1),

$$n_1^{y-z} 17^{r(y-z)} (8^x n_1^{x-y} 17^{r(x-y)} + 15^y) = 17^z.$$

Noting that $(17, n_1) = 1$ and $(8^x n_1^{x-y} 17^{r(x-y)} + 15^y, 17) = 1$, we know that $r(y-z) = z$. Thus $n_1^{y-z}(8^x n_1^{x-y} 17^{r(x-y)} + 15^y) = 1$. This is impossible.

Subcase 1.2. $y < z < x$. Then

$$15^y = n^{z-y}(17^z - 8^x n^{x-z}). \tag{2.5}$$

If $(n, 15) = 1$, then by (2.5) and $n \geq 2$ we have $y = z$, a contradiction.

If $(n, 15) > 1$, then write $n = 3^r 5^q n_1$, where $(15, n_1) = 1$ and $r + q \geq 1$. By (2.5),

$$15^y = 3^{r(z-y)} 5^{q(z-y)} n_1^{z-y} (17^z - 8^x 3^{r(x-z)} 5^{q(x-z)} n_1^{x-z}). \tag{2.6}$$

Thus $r(z-y) = q(z-y) = y$. Hence $r = q$. By (2.6),

$$1 = n_1^{z-y}(17^z - 8^x 15^{r(x-z)} n_1^{x-z}).$$

Thus $n_1 = 1$ and $17^z - 8^x 15^{r(x-z)} = 1$. Then $2^z \equiv 1 \pmod 3$ and $z \equiv 0 \pmod 2$. Write $z = 2z_1$. We have

$$2^{3x} 15^{r(x-z)} = (17^{z_1} - 1)(17^{z_1} + 1).$$

Noting that $(17^{z_1} - 1, 17^{z_1} + 1) = 2$, then

$$2^{3x-1} \mid 17^{z_1} - 1, \quad 2 \mid 17^{z_1} + 1, \tag{2.7}$$

or

$$2 \mid 17^{z_1} - 1, \quad 2^{3x-1} \mid 17^{z_1} + 1. \tag{2.8}$$

However,

$$2^{3x-1} > 2^{3z-1} = 2^{6z_1-1} > 2^{5z_1} > (17+1)^{z_1} > 17^{z_1} + 1^{z_1} > 17^{z_1} - 1^{z_1},$$

which contradicts both (2.7) and (2.8).

Case 2. $x = y$. Then

$$n^{x-z}(8^x + 15^x) = 17^z. \tag{2.9}$$

If $(n, 17) = 1$, then by (2.9) and $n \geq 2$ we have $x = z$, a contradiction.

If $(n, 17) = 17$, then write $n = 17^r n_1$, where $r \geq 1$ and $17 \nmid n_1$. By (2.9),

$$17^{r(x-z)} n_1^{x-z} (8^x + 15^x) = 17^z. \tag{2.10}$$

It follows that $n_1^{x-z} \mid 17^z$, so $n_1 = 1$. By (2.10),

$$8^x + 15^x = 17^{z-r(x-z)}.$$

By Lemma 2, $x = z - r(x-z) = 2$ which implies that $x = z = 2$, a contradiction.

Case 3. $x < y$.

Subcase 3.1. $z < x < y$. Then

$$n^{x-z}(8^x + 15^y n^{y-x}) = 17^z. \tag{2.11}$$

If $(n, 17) = 1$, then by (2.11) and $n \geq 2$ we have $x = z$, a contradiction.

If $(n, 17) = 17$, then write $n = 17^r n_1$, where $r \geq 1$ and $17 \nmid n_1$. By (2.11),

$$17^{r(x-z)} n_1^{x-z}(8^x + 15^y 17^{r(y-x)} n_1^{y-x}) = 17^z. \tag{2.12}$$

It follows that $n_1^{x-z} \mid 17^z$, so $n_1 = 1$. By (2.12),

$$17^{r(x-z)}(8^x + 15^y 17^{r(y-x)}) = 17^z.$$

Then $r(x - z) < z$ and $8^x + 15^y 17^{r(y-x)} = 17^{z-r(x-z)}$. Thus $17 \mid 8^x$, a contradiction.

Subcase 3.2. $x \leq z < y$. Then

$$2^{3x} + 15^y n^{y-x} = 17^z n^{z-x}. \tag{2.13}$$

If $(n, 2) = 1$, then by (2.13) and $n \geq 2$ we have $x = z < y$. Thus

$$8^x + 15^y n^{y-x} = 17^x. \tag{2.14}$$

Then $3^x \equiv 2^x \pmod 5$, so $x \equiv 0 \pmod 2$. Write $x = 2x_1$. By (2.14),

$$3^y 5^y n^{y-x} = (17^{x_1} - 8^{x_1})(17^{x_1} + 8^{x_1}).$$

Noting that $(17^{x_1} - 8^{x_1}, 17^{x_1} + 8^{x_1}) = 1$, we have $5^y \mid 17^{x_1} - 8^{x_1}$ or $5^y \mid 17^{x_1} + 8^{x_1}$.

However,

$$5^y > 5^x = 5^{2x_1} = 25^{x_1} = (17 + 8)^{x_1} > 17^{x_1} + 8^{x_1} > 17^{x_1} - 8^{x_1},$$

a contradiction.

If $(n, 2) = 2$, write $n = 2^r n_1$, where $r \geq 1$ and $2 \nmid n_1$. By (2.13),

$$2^{3x} = n^{z-x}(17^z - 15^y n^{y-z}) = 2^{r(z-x)} n_1^{z-x}(17^z - 15^y 2^{r(y-z)} n_1^{y-z}).$$

It follows that $n_1^{z-x} \mid 2^{3x}$, so that $n_1 = 1$ or $x = z$.

If $n_1 = 1$, then

$$2^{3x} = 2^{r(z-x)}(17^z - 15^y 2^{r(y-z)}).$$

It follows that $r(z - x) = 3x$ and $17^z - 15^y 2^{r(y-z)} = 1$. Then $2^z \equiv 1 \pmod 3$, so $z \equiv 0 \pmod 2$. Write $z = 2z_1$. Then

$$15^y 2^{r(y-z)} = (17^{z_1} - 1)(17^{z_1} + 1).$$

Noting that $(17^{z_1} - 1, 17^{z_1} + 1) = 2$, we have $5^y \mid 17^{z_1} - 1$ or $5^y \mid 17^{z_1} + 1$.

However,

$$5^y > 5^z = 5^{2z_1} = 25^{z_1} > (17+1)^{z_1} > 17^{z_1} + 1 > 17^{z_1} - 1,$$

a contradiction.

If $x = z$, then $8^x + 15^y n^{y-x} = 17^x$. Thus $3^x \equiv 2^x \pmod 5$, so $x \equiv 0 \pmod 2$. Write $x = 2x_1$. Then

$$3^y 5^y n^{y-x} = (17^{x_1} - 8^{x_1})(17^{x_1} + 8^{x_1}).$$

Noting that $(17^{x_1} - 8^{x_1}, 17^{x_1} + 8^{x_1}) = 1$, we have $5^y \mid 17^{x_1} - 8^{x_1}$ or $5^y \mid 17^{x_1} + 8^{x_1}$.

However,

$$5^y > 5^x = 5^{2x_1} = 25^{x_1} = (17+8)^{x_1} > 17^{x_1} + 8^{x_1} > 17^{x_1} - 8^{x_1},$$

a contradiction.

This completes the proof of the theorem. □

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ZHI-JUAN YANG, Department of Mathematics, Anhui Normal University,
Wuhu 241000, China

MIN TANG, Department of Mathematics, Anhui Normal University,
Wuhu 241000, China
e-mail: tmzzz2000@163.com