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A WEAKLY UNIFORMLY ROTUND DUAL OF A BANACH SPACE

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Abstract

A nonreflexive Banach space may have a weakly uniformly rotund dual. The aim of this paper is to determine alternative characterisations and study further implications of this property in higher duals.

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1. Introduction

A Banach space X is said to be *weakly uniformly rotund (WUR)* if for each $f \in S(X^*)$, given $\varepsilon > 0$, there exists $\delta(\varepsilon, f) > 0$ such that for $x, y \in S(X)$,

$$|f(x-y)| < \varepsilon \quad \text{when } \|x+y\| > 2-\delta.$$

Hájek [9] solved a longstanding problem showing that a WUR Banach space is an Asplund space. A simpler proof due to Godefroy appears in [6, page 397]. By equivalently renorming James space J to have WUR dual, he showed that a nonreflexive Banach space may have WUR dual. These results suggest that the WUR property might warrant further study as a dual property.

In Section 2 we characterise the WUR dual property by equivalent geometrical conditions on the space. In Section 3 we follow the effect the WUR dual has with fundamental relations expressed in terms of the natural embeddings of the space.

The norm of a Banach space X is *Gâteaux differentiable* at $x \in S(X)$ if

$$\lim_{\lambda \to 0} \frac{\|x+\lambda y\| - \|x\|}{\lambda} \quad \text{exists for all } y \in S(X)$$

or equivalently

$$\lim_{\lambda \to 0} \frac{\|x+\lambda y\| + \|x-\lambda y\| - 2}{\lambda} = 0 \quad \text{for all } y \in S(X)$$

and is *uniformly Gâteaux differentiable (UG)* if given $y \in S(X)$ the limit is approached uniformly for all $x \in S(X)$ (see [4, pages 2 and 63]).

A Banach space X has *weak* uniformly rotund* (W^*UR) dual X^* if for each $x \in S(X)$, given $\varepsilon > 0$, there exists $\delta(\varepsilon, x) > 0$ such that for $f, g \in S(X^*)$,

$$|(f-g)(x)| < \varepsilon \quad \text{when } \|f+g\| > 2-\delta.$$

It is well known that a Banach space X is WUR if and only if the dual norm of X^* is UG and that a Banach space X has UG norm if and only if the dual X^* is W^*UR [4, page 63].

Differentiability properties of the norm can be characterised by continuity of associated mappings. For $x \in S(X)$, consider the set $D(x) \equiv \{f \in S(X^*) : f(x) = 1\}$. We call the mapping $x \mapsto f_x$ of X into X^* a *support mapping* if, for each $x \in S(X)$ and real $\lambda > 0$, we have $f_x \in D(x)$ and $f_{\lambda x} = \lambda f_x$.

Proposition 1.1. *For a Banach space X with dual X^* and second dual X^{**}:*

(i) *the norm of X is Gâteaux differentiable at $x \in S(X)$ if and only if there exists a support mapping $x \mapsto f_x$ of X into X^* such that for each $y \in S(X)$ the real-valued mapping $x \mapsto f_x(y)$ is continuous at x [5, page 22];*

(ii) *the norm of X is UG if and only if for each $y \in S(X)$ the real-valued mapping $x \mapsto f_x(y)$ is uniformly continuous on $S(X)$ [7, page 394];*

(iii) *the norm of X^{**} is Gâteaux differentiable at $\widehat{x} \in S(\widehat{X})$ if and only if there exists a support mapping $x \mapsto f_x$ of X into X^* such that for each $F \in S(X^{**})$ the real-valued mapping $x \mapsto \widehat{f_x}(F)$ is continuous at x [8, page 105];*

(iv) *the norm of X^{**} is UG if and only if for each $F \in S(X^{**})$ the real-valued mapping $x \mapsto \widehat{f_x}(F)$ is uniformly continuous on $S(X)$.*

The proof of (iv) follows from Theorem 2.1 below.

2. Alternative characterisation of *WUR* dual

As in the nondual case, the characterisations are in terms of uniform Gâteaux differentiability of the norm and continuity of support mappings on the space. But we need a modified definition of UG on X^{**}.

We say that the norm of X^{**} is *UG on $S(\widehat{X})$* if for each $F \in S(X^{**})$ the limit

$$\lim_{\lambda} \frac{\|\widehat{x} + \lambda F\| - \|\widehat{x}\|}{\lambda}$$

is approached uniformly for all $x \in S(X)$.

Theorem 2.1. *Given a Banach space X, the following are equivalent:*

(i) *X^* is WUR;*

(ii) *there exists a support mapping $x \mapsto f_x$ of X into X^* such that for each $F \in S(X^{**})$ the real-valued mapping $x \mapsto \widehat{f_x}(F)$ is uniformly continuous on $S(X)$;*

(iii) *X^{**} is UG on $S(\widehat{X})$.*

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PROOF. (i) ⇒ (ii). For any support mapping $x \mapsto f_x$ of X into X^*,

$$4 \le \|f_x + f_y\|\,\|x + y\| + \|f_x - f_y\|\,\|x - y\| \quad \text{for all } x, y \in S(X).$$

Consider any support mapping $x \mapsto f_x$ of X into X^*. For sequences $\{x_n\}$ and $\{y_n\}$ in $S(X)$ such that $\|x_n - y_n\| \to 0$, $\|f_{x_n} + f_{y_n}\| \to 2$. So, if X^* is WUR, given $F \in S(X^{**})$, we have $F(f_{x_n} - f_{y_n}) \to 0$; that is, the uniform continuity property holds.

(ii) ⇒ (iii). For any $F \in S(X^{**})$, given $\varepsilon > 0$, there exists $\delta(\varepsilon, F) > 0$ such that for $x, y \in S(X)$,

$$|F(f_x - f_y)| < \varepsilon \quad \text{when } \|x - y\| < \delta.$$

We extend this uniform continuity property from X to a partially uniformly continuous support mapping on X^{**}. We begin by choosing $0 < \delta < \varepsilon < 1/2$. Consider $x \in S(X)$ and $G \in S(X^{**})$ such that $\|\widehat{x} - G\| < \delta^2/8$ and $\mathfrak{F}_G \in D(G)$. Then

$$|\mathfrak{F}_G(\widehat{x}) - 1| = |\mathfrak{F}_G(\widehat{x}) - \mathfrak{F}_G(G)| \le \|\widehat{x} - G\| < \delta^2/8.$$

Consider a $\sigma(X^{***}, X^{**})$ neighbourhood of $\mathfrak{F}_G$ determined by F, $\widehat{x}$ and $\delta^2/8$. Since $B(\widehat{X^*})$ is $\sigma(X^{***}, X^{**})$ dense in $B(X^{***})$, there exists $f \in B(X^*)$ such that

$$|\mathfrak{F}_G(\widehat{x}) - f(x)| < \delta^2/8 \quad \text{and} \quad |\mathfrak{F}_G(F) - F(f)| < \delta^2/8,$$

so

$$|f(x) - 1| \le |f(x) - \mathfrak{F}_G(\widehat{x})| + |\mathfrak{F}_G(\widehat{x}) - 1| < \delta^2/4.$$

By the Bishop–Phelps–Bollobás theorem [1], there exist $y \in S(X)$ and $f_y \in D(y)$ such that $\|x - y\| < \delta$ and $\|f_y - f\| < \delta$. So, by the uniform continuity property, $|F(f_x - f_y)| < \varepsilon$. Then

$$|F(f - f_x)| \le \|f - f_y\| + |F(f_x - f_y)| < \delta + \varepsilon < 2\varepsilon$$

and

$$|(\mathfrak{F}_G - \widehat{f_x})(F)| \le |(\mathfrak{F}_G - \widehat{f})(F)| + |F(f - f_x)| < \delta^2/8 + 2\varepsilon < 3\varepsilon.$$

So, we have established for the support mapping $F \mapsto \mathfrak{F}_F$ of X^{**} into X^{***} the property that for any $F \in S(X^{**})$, given $\varepsilon > 0$, there exists $\delta(\varepsilon, F) > 0$ such that for any $x \in S(X)$ and $G \in S(X^{**})$,

$$|(\mathfrak{F}_G - \widehat{f_x}(F)| < 3\varepsilon \quad \text{when } \|\widehat{x} - G\| < \delta^2/8.$$

Now for this support mapping we have the general inequality

$$\left|\frac{\|\widehat{x} + \lambda F\| - \|\widehat{x}\|}{\lambda} - \widehat{f_x}(F)\right| \le \left|\left(\frac{\mathfrak{F}_{\widehat{x}+\lambda F}}{\|\widehat{x} + \lambda F\|} - \widehat{f_x}\right)(F)\right| \quad \text{for real } \lambda \ne 0.$$

By the uniform continuity property,

$$\left|\left(\frac{\mathfrak{F}_{\widehat{x}+\lambda F}}{\|\widehat{x} + \lambda F\|} - \widehat{f_x}\right)(F)\right| < 3\varepsilon \quad \text{when } \left\|\frac{\widehat{x} + \lambda F}{\|\widehat{x} + \lambda F\|} - \widehat{x}\right\| < \delta^2/8$$

and this is so when $|\lambda| < \delta^2/17$. So, the norm of X^{**} is UG on $S(\widehat{X})$.

(iii) ⇒ (i). If X^* is not WUR, then for some $F \in S(X^{**})$ there are some $r > 0$ and sequences $\{f_n\}$ and $\{g_n\}$ in $S(X^*)$ such that $\|f_n + g_n\| \to 2$ but $F(f_n - g_n) > r$ for all $n \in \mathbb{N}$. Consider a sequence of positive real numbers $\{\lambda_n\}$ with $\lambda_n \to 0$ such that $2 - \|f_n + g_n\| \le \lambda_n{}^2$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned}\sup_{x\in S(X)} \frac{\|\widehat{x} + \lambda_n F\| + \|\widehat{x} - \lambda_n F\| - 2}{\lambda_n} &\ge \sup_{x\in S(X)} \frac{(f_n + g_n)(x) + \lambda_n F(f_n - g_n) - 2}{\lambda_n}\\ &\ge r - \lambda_n > 0 \quad \text{for sufficiently large } n.\end{aligned}$$

But this contradicts the norm of X^{**} being UG on $S(\widehat{X})$. □

The duality between WUR space X and the UG of the norm of its dual X^* provides the proof of Proposition 1.1(iv).

It is well known that a Banach space X where the norm satisfies Proposition 1.1(iii) on $S(X)$ is an Asplund space [8, page 106]. This along with Theorem 2.1(i) ⇔ (ii) and Hájek's Asplund result reveals even more structure for a Banach space with WUR dual.

Corollary 2.2. *A Banach space X with WUR dual X^* is an Asplund space with Asplund space dual.*

3. Implications of *WUR* for natural embeddings

Given a Banach space X, for each $n = 0, 1, 2, \ldots$ we denote by Q_n the natural embedding of the nth dual space $X^{(n)}$ into the $(n + 2)$th dual space $X^{(n+2)}$.

We need to recall the following fundamental properties:

- for $n = 0, 1, 2, \ldots$, we have $Q_{n-1}{}^* Q_n = I_n$, the identity mapping on $X^{(n)}$;
- $P_n = Q_n Q_{n-1}{}^*$ is the norm-one projection of $X^{(n+2)}$ onto $\widehat{X^{(n)}}$ and $I_{n+3} - P_n$ is the projection of $X^{(n+3)}$ onto $X^{(n)\perp}$.

Lemma 3.1 [2, page 352]. *Given a Banach space X:*

(i) $\|I - P_0\| = \|Q_2 - Q_0{}^{**}\|$;
(ii) $\|(Q_2 - Q_0{}^{**})(F)\| \ge d(F, \widehat{X})$ *for all* $F \in X^{**}$;
(iii) $\|(Q_2 - Q_0{}^{**})\| = 1$ *if and only if* $\|(Q_2 - Q_0{}^{**})(F)\| = d(F, \widehat{X})$ *for all* $F \in X^{**}$.

Given a nonreflexive Banach space X, for $f \in S(X^*)$ not attaining its norm, we have $Q_2 F_f \ne Q_0{}^{**} F_f$, where $F_f \in D(f)$ [3, page 70], and from Lemma 3.1 it is clear that $\|Q_2 - Q_0{}^{**}\| \ge 1$. Brown [2] demonstrated that the Banach space c_0 has $\|Q_2 - Q_0{}^{**}\| = 1$ and that $\|Q_3 - Q_1{}^{**}\| = 2$.

We now show that a Banach space X, where $\|Q_2 - Q_0{}^{**}\| = 1$, has some regular structure related to WUR. But to explore natural embedding relations associated with WUR we need to note the effect UG of the norm has on higher duals.

Lemma 3.2 [10, page 325]. *Given a Banach space X with UG norm, for each $x \in S(X)$ all elements of $D(\widehat{x})$ have the form $\widehat{f_x} + y^\perp$, where $f_x \in D(x)$ and $y^\perp \in X^\perp$.*

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Proof. We show that if the norm of X is UG, then the norm of X^{**} is Gâteaux differentiable at every $F \in S(X^{**})$ in $S(\widehat{X})$ directions. Suppose that the norm of X^{**} is not Gâteaux differentiable at some $F \in S(X^{**})$ in the direction $\widehat{x} \in S(\widehat{X})$. Then there exist $r > 0$ and a sequence of positive numbers $\{\lambda_n\}$, where $\lambda_n \to 0$, such that

$$\frac{\|F + \lambda_n\widehat{x}\| + \|F - \lambda_n\widehat{x}\| - 2}{\lambda_n} > r$$

and sequences $\{f_n\}$ and $\{g_n\}$ in $S(X^*)$ such that

$$(F + \lambda_n\widehat{x})(f_n) > \|F + \lambda_n\widehat{x}\| - {\lambda_n}^2 \quad \text{and} \quad (F - \lambda_n\widehat{x})(g_n) > \|F - \lambda_n\widehat{x}\| - {\lambda_n}^2.$$

Then

$$\frac{F(f_n + g_n) + \lambda_n\widehat{x}(f_n - g_n) - 2 + 2{\lambda_n}^2}{\lambda_n} > r,$$

so $\widehat{x}(f_n - g_n) + 2\lambda_n > r$. Now, as $n \to \infty$, $\|f_n + g_n\| \geq |F(f_n + g_n)| \to 2$ but $\widehat{x}(f_n - g_n) \nrightarrow 0$; that is, X^* is not W^*UR and so X does not have UG norm. If the norm of X^{**} is Gâteaux differentiable at $F \in S(X^{**})$ in the direction $\widehat{x} \in S(\widehat{X})$, then

$$\lim_{\lambda \to 0} \frac{\|F + \lambda\widehat{x}\| - \|F\|}{\lambda} = \mathfrak{F}_F(\widehat{x}).$$

So, for $\mathfrak{F}_F \in D(F)$, $\mathfrak{F}_F|_{\widehat{X}}$ is a unique limit, which implies that $D(\widehat{x})$ consists of elements of the form $\widehat{f_x} + y^\perp$. □

Theorem 3.3. *A nonreflexive Banach space X, where there exists $F \in S(X^{**})$ such that $d(F, \widehat{X}) = 1$ and where $\|Q_2 - {Q_0}^{**}\| = 1$, cannot be WUR.*

Proof. By the Hahn–Banach theorem, there exists $x^\perp \in S(X^\perp)$ such that $x^\perp(F) = 1$ and $\|x^\perp\| = 1/d(F, \widehat{X})$. But $F \in S(X^{**})$ was chosen such that $d(F, \widehat{X}) = 1$, so $\|x^\perp\| = 1$, $x^\perp \in D(F)$ and $\widehat{F} \in D(x^\perp)$.

Consider $Q_2(F)$ and $(Q_2 - {Q_0}^{**})(F)$ in X^{****}. Now $\|Q_2(F)\| = 1$ and, by Lemma 3.1(iii), $\|(Q_2 - {Q_0}^{**})(F)\| = 1$. However, ${Q_0}^{**}(F)(x^\perp) = F{Q_0}^*(x^\perp) = 0$ for all $x^\perp \in X^\perp$. So, both $Q_2(F)$ and $(Q_2 - {Q_0}^{**})(F) \in D(x^\perp)$. But ${Q_0}^{**}(F)Q_1(g) = F(g) \neq 0$ for some $g \in X^*$. So, ${Q_0}^{**}(F) \notin X^{*\perp}$.

By Lemma 3.2, the dual X^* cannot have UG norm and consequently X cannot be WUR. □

For the dual X^*, one of the assumptions of Theorem 3.3 is automatically fulfilled.

Lemma 3.4. *Given a nonreflexive Banach space X, where $\|Q_2 - {Q_0}^{**}\| = 1$, every $x^\perp \in S(X^\perp)$ has the property that*

$$\|x^\perp\| = d(x^\perp, \widehat{X^*}) = 1.$$

Proof. We have $\|x^\perp\| = \|(I - P_0)(x^\perp - \widehat{f})\| \leq \|I - P_0\|\,\|x^\perp - \widehat{f}\|$ for all $f \in X^*$. But Lemma 3.1(i) gives us that $\|I - P_0\| = 1$, so $\|x^\perp\| \leq d(x^\perp, \widehat{X^*})$. However, $\|x^\perp\| \geq d(x^\perp, \widehat{X^*})$, so $\|x^\perp\| = d(x^\perp, \widehat{X^*}) = 1$. □

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We now see that a nonreflexive Banach space X with WUR dual has its effect on the natural embedding relations.

Theorem 3.5. *A nonreflexive Banach space X, satisfying both* $\|Q_2 - Q_0{}^{**}\| = 1$ *and* $\|Q_3 - Q_1{}^{**}\| = 1$*, cannot have* WUR *dual* X^*.

Proof. From Lemma 3.4, we see that $\|Q_2 - Q_0{}^{**}\| = 1$ provides the required assumption to apply Theorem 3.3 to the dual space X^*. So, we have both $Q_3(x^{\perp})$ and $(Q_3 - Q_1{}^{**})(x^{\perp}) \in D(\phi)$, where $\phi \in D(x^{\perp})$ and $\widehat{x^{\perp}} \in D(\phi)$. But $Q_1{}^{**}(x^{\perp})Q_2(F) = x^{\perp}(F)$ for some $F \in X^{**}$, so $Q_1{}^{**}(x^{\perp}) \notin X^{**\perp}$. By Lemma 3.2, this second dual X^{**} cannot have UG norm and consequently the dual X^* cannot be WUR. □

This theorem follows an unpublished argument due to Mark Smith, which can be found in [11, page 82].

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