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FOURIER-TYPE TRANSFORMS ON REARRANGEMENT-INVARIANT QUASI-BANACH FUNCTION SPACES

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Abstract. We establish the mapping properties of Fourier-type transforms on rearrangement-invariant quasi-Banach function spaces. In particular, we have the mapping properties of the Laplace transform, the Hankel transforms, the Kontorovich-Lebedev transform and some oscillatory integral operators. We achieve these mapping properties by using an interpolation functor that can explicitly generate a given rearrangement-invariant quasi-Banach function space via Lebesgue spaces.

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1. Introduction. The main theme of this paper is the mapping properties of the Fourier-type transforms on rearrangement-invariant quasi-Banach function spaces.

We call a linear operator T a Fourier-type transform if there exist two pairs (p_i, q_i), $i = 0, 1$, such that T is of strong type (p_i, q_i), $i = 0, 1$ with

$$\frac{a}{p_i} + \frac{b}{q_i} = c,$$

for some $a, b, c > 0$.

It is easy to see that the above definition is motivated by the Hausdorff–Young inequality of the Fourier transform, which states that

$$\|\hat{f}\|_{L^{p'}} \leq C\|f\|_p,$$

where $\hat{f}$ is the Fourier transform of f and $1 \leq p \leq 2$.

Other than the Fourier transform, there are a number of interesting examples of Fourier-type transforms.

Recently, the Hausdorff–Young type inequalities for the Hankel transforms are obtained in [**3**]. Moreover, the Hausdorff–Young inequalities for the Kontorovich–Lebedev transform are established in [**30**]. For the Hausdorff–Young inequalities of the oscillatory integral operators, see [**19**, Theorem 1.1].

Therefore, the Hankel transforms, the Kontorovich–Lebedev transform and oscillatory integrals are Fourier-type transforms.

In view of [**13**, Theorem 6.1], the Hausdorff–Young inequality for the Fourier transform can be extended to rearrangement-invariant quasi-Banach function spaces (r.i.q.B.f.s.). Notice that the r.i.q.B.f.s. includes the Lorentz spaces, the Orlicz spaces, the Lorentz–Zygmund spaces and the Lorentz–Karamata spaces. Therefore, [**13**, Theorem 6.1] gives extensions of Hausdorff–Young inequalities to the above-mentioned function spaces. In addition, the study of r.i.q.B.f.s. has been further extended to ball quasi-Banach function spaces, see [**27**].

Moreover, the mapping properties of the generalized Hankel conjugate transformations, which are operators defined via the Hankel transform, on rearrangement-invariant Banach function spaces are established in [**21**].

Therefore, it motivates us to study the mapping properties of Fourier-type transforms on r.i.q.B.f.s.

In this paper, we employ the interpolation functor used in [**13**] to obtain our main results. As shown in [**13**], this interpolation functor can generate the function spaces involved in the mapping properties on Fourier transform. The main result in this paper shows that the function spaces needed for the mapping properties of the Laplace transform, the Hankel transform, the Kontorovich–Lebedev transform and the oscillatory integral operator can also be generated by this interpolation functor.

This paper is organized as follows. Some preliminary results for r.i.q.B.f.s. are presented in Section 2. In Section 3, we introduce the interpolation functor used to prove our main result. A new family of function spaces used to characterize the mapping properties of the Laplace transform, the Hankel transform, the Kontorovich–Lebedev transform and the oscillatory integral operator is also introduced in Section 3. The main result for interpolation of operators is presented and proved in Section 4. The mapping properties of the Laplace transform, the Hankel transform, the Kontorovich–Lebedev transform and the oscillatory integral operator are established in Section 5. We present the application of our main results on Lorentz–Karamata spaces in Section 6.

2. Preliminaries. For any σ-finite measure μ, let $\mathcal{M}(\mu)$ be the set of μ-measurable functions. Write $\mathcal{M}(0,\infty)$ and $\mathcal{M}(\mathbb{R}^n)$ to be the classes of Lebesgue measurable functions on $(0,\infty)$ and $\mathbb{R}^n$, respectively. For any $1 \le p \le \infty$, let $L^p(\mu)$ be the Lebesgue spaces with respect to μ. In particular, let $L^p(0,\infty)$ and $L^p(\mathbb{R}^n)$ denote the Lebesgue spaces on $(0,\infty)$ and $\mathbb{R}^n$, respectively.

For any $f \in \mathcal{M}(\mu)$ and $s > 0$, write

$$d_f(s) = \mu(\{x \in \mathbb{R}^n : |f(x)| > s\}),$$

and

$$f^*_\mu(t) = \inf\{s > 0 : d_f(s) \le t\}, \quad t > 0.$$

We say that f and g are equimeasurable, if $d_f(s) = d_g(s)$ for all $s > 0$. We write $f \approx g$ if

$$Bf \le g \le Cf,$$

for some constants $B, C > 0$ independent of appropriate quantities involved in the expressions of f and g.

We recall the definition of rearrangement-invariant quasi-Banach function space (r.i.q.B.f.s.) from [**12**, Definition 4.1].

DEFINITION 2.1. Let μ be a σ-finite measure. A quasi-Banach space $X \subset \mathcal{M}(\mu)$ is called a rearrangement-invariant quasi-Banach function space on μ if there exists a quasi-norm $\rho_X : \mathcal{M}(0, \infty) \to [0, \infty]$ satisfying

(1) $\rho_X(f) = 0 \Leftrightarrow f = 0$ a.e.,
(2) $|g| \leq |f|$ a.e. $\Rightarrow \rho_X(g) \leq \rho_X(f)$,
(3) $0 \leq f_n \uparrow f$ a.e. $\Rightarrow \rho_X(f_n) \uparrow \rho_X(f)$,
(4) $\chi_E \in \mathcal{M}(0, \infty)$ and $|E| < \infty \Rightarrow \rho_X(\chi_E) < \infty$,
so that

$$\|f\|_X = \rho_X(f^*_\mu), \quad \forall f \in X. \tag{1}$$

Write

$$\bar{X} = \{g \in \mathcal{M}(0, \infty) : \rho_X(g) < \infty\}.$$

It is obvious that $\bar{X}$ is a quasi-Banach space.

Furthermore, a Banach space $X \subset \mathcal{M}(\mu)$ is a Banach function space if $\|\cdot\|_X$ is a norm and satisfies Items (1)–(3),

$$\chi_E \in \mathcal{M}(\mu) \quad \text{and} \quad \mu(E) < \infty \Rightarrow \chi_E \in X \tag{2.2}$$

and

$$\chi_E \in \mathcal{M}(\mu) \quad \text{and} \quad \mu(E) < \infty \Rightarrow \int_E f \, d\mu \leq C_E \|f\|_X, \tag{2.3}$$

for some $C_E > 0$.

Moreover, X is a rearrangement-invariant Banach function space (r.i.B.f.s) if X is a Banach function space and for any equimeasurable functions f and g, $\|f\|_X = \|g\|_X$.

Whenever X is a r.i.B.f.s. on μ, the Luxemburg representation theorem [**1**, Chapter 2, Theorem 4.10] assures the existence of ρ_X for X. For the uniqueness of ρ_X, the reader is referred to [**1**, p. 64].

For any $s \geq 0$ and $f \in \mathcal{M}(0, \infty)$, define $(D_s f)(t) = f(st)$, $t \in (0, \infty)$. Let $\|D_s\|_{\bar{X} \to \bar{X}}$ be the operator norm of D_s on $\bar{X}$. We recall the definition of Boyd's indices for r.i.q.B.f.s. from [**23**].

DEFINITION 2.2. Let X be a r.i.q.B.f.s. on μ. Define the lower Boyd index of X, p_X, and the upper Boyd index of X, q_X, by

$$p_X = \sup\{p > 0 : \exists C > 0 \text{ such that } \forall\, 0 \leq s < 1, \ \|D_s\|_{\bar{X} \to \bar{X}} \leq C s^{-1/p}\},$$
$$q_X = \inf\{q > 0 : \exists C > 0 \text{ such that } \forall\, 1 \leq s, \ \|D_s\|_{\bar{X} \to \bar{X}} \leq C s^{-1/q}\},$$

respectively.

The Boyd indices of the Lorentz space $L_{p,q}$ are $p_{L_{p,q}} = q_{L_{p,q}} = p$. In addition, the reader is referred to [**1**] for the Boyd indices of the Orlicz space.

Let X be a r.i.q.B.f.s. on μ. For any $0 < p < \infty$, the p-convexification of X, X^p is defined by

$$X^p = \{f : |f|^p \in X\}.$$

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We equip X^p with the quasi-norm $\|f\|_{X^p} = \||f|^p\|_X^{1/p}$. For a complete account of the p-convexification, the reader may consult [**22**, Volumne II, p. 53–53] for the case $1 \leq p < \infty$ and [**25**, Section 2.2] for the general case. Notice that in [**25**], the p-convexification of X is called as the $\frac{1}{p}$th power of X.

Next, we recall another index for r.i.q.B.f.s. It is used in the proof of the interpolation theorem.

For any r.i.q.B.f.s. X, according to Aoki–Rolewicz theorem [**20**, Theorem 1.3], there exists a $\kappa_X \in (0, 1]$ such that $\rho_X^{\kappa_X}$ is sub-additive. That is,

$$\rho_X^{\kappa_X}(f+g) \leq \rho_X^{\kappa_X}(f) + \rho_X^{\kappa_X}(g).$$

3. Interpolation functor. In this section, we first introduce the interpolation functor used in Section 4 to establish the mapping properties of the Fourier-type transforms.

The definition of this interpolation functor requires the notion of category and compatible couples. For brevity, we refer the reader to [**29**, Section 1.2] for details of category and compatible couples.

We recall the definition of K-functional from [**1**, Section 3.1] and [**29**, Section 1.3.1].

DEFINITION 3.1. Let (X_0, X_1) be a compatible couple of quasi-normed spaces. For any $f \in X_0 + X_1$, the K-functional is defined as

$$K(f, t, X_0, X_1) = \inf\{\|f_0\|_{X_0} + t\|f_1\|_{X_1} : f = f_0 + f_1\},$$

where the infimum is taken over all $f = f_0 + f_1$ for which $f_i \in X_i$, $i = 0, 1$.

We introduce the interpolation functor used in [**13**].

DEFINITION 3.2. Let μ be a σ-finite measure. Let $0 < \theta, r < \infty$ and X be a r.i.q.B.f.s. on μ. Let (X_0, X_1) be a compatible couple of quasi-normed spaces. The space $(X_0, X_1)_{\theta,r,X}$ consists of all f in $X_0 + X_1$ such that

$$\|f\|_{(X_0,X_1)_{\theta,r,X}} = \rho_X(t^{-\frac{1}{r}}K(f, t^{\frac{1}{\theta}}, X_0, X_1)) < \infty \tag{3.1}$$

where ρ_X is the quasi-norm given in (1).

We now present one of main results from [**13**], it guarantees that $(X_0, X_1)_{\theta,r,X}$ is indeed an interpolation functor.

THEOREM 3.1. *Let μ be a σ-finite measure. Let $0 < \theta, r < \infty$ and X be a r.i.q.B.f.s. on μ with $0 < p_X \leq q_X < \infty$. If $\frac{1}{q_X} + \frac{1}{\theta} > \frac{1}{r}$ and $r < p_X$, then $(\cdot, \cdot)_{\theta,r,X}$ is an interpolation functor.*

In addition, if (X_0, X_1) and (Y_0, Y_1) are compatible couples of quasi-normed spaces and T is a linear operator such that

$$\|Tf\|_{Y_i} \leq M_i\|f\|_{X_i}, \quad i = 0, 1. \tag{3.2}$$

Then, for any ϵ, there exists a constant $C_\epsilon > 0$ independent of M_i, $i = 0, 1$ such that

$$\|Tf\|_{(Y_0,Y_1)_{\theta,r,X}} \leq C_\epsilon M\|f\|_{(X_0,X_1)_{\theta,r,X}}, \tag{3.3}$$

where

$$M = \Big(\frac{M_1}{M_0}\Big)^{\frac{\theta}{r}} M_0 \max\Big(\Big(\frac{M_1}{M_0}\Big)^{-\frac{\theta}{p_X}+\epsilon}, \Big(\frac{M_1}{M_0}\Big)^{-\frac{\theta}{q_X}-\epsilon}\Big). \tag{3.4}$$

For the proof of the above theorem, the reader is referred to [**13**, Theorem 4.1].

The subsequent result assures that X can be generated from Lebesgue spaces and the functor $(\cdot,\cdot)_{\theta,r,X}$.

PROPOSITION 3.2. *Let* $0 < p_0 < p_1 < \infty$, μ *be a* σ*-finite measure and* X *be a r.i.q.B.f.s. on* μ *with* $0 < p_X \leq q_X < \infty$. *Suppose that* p_0 *and* p_1 *satisfy* $p_1 > q_X$, $p_0 < p_X$ *and* $\frac{1}{\theta} = \frac{1}{p_0} - \frac{1}{p_1}$. *Then*

$$(L^{p_0}(\mu), L^{p_1}(\mu))_{\theta,p_0,X} = X.$$

For the proof of the above proposition, the reader is referred to [**13**, Corollary 4.3]. Even though the proof in [**13**] is presented for Lebesgue measure, it is still valid for function spaces defined on the σ-finite measure μ.

We now introduce the function spaces used to characterize the mapping properties of the Laplace transform, the Hankel transform, the Kontorovich-Lebedev transform and some oscillatory integral operators.

DEFINITION 3.3. Let $\beta > 0$, $\gamma \geq 0$ and μ, ν be σ-finite measures. For any r.i.q.B.f.s. X on μ, the set $\hat{X}_{\beta,\gamma}(\nu)$ consists of all $f \in \mathcal{M}(\nu)$ such that

$$\|f\|_{\hat{X}_{\beta,\gamma}(\nu)} = \rho_X(t^{-\gamma} f^*_\nu(t^{-\beta})) < \infty.$$

When μ and ν are Lebesgue measures and $X = L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, we find that

$$\|f\|_{L^p(\mathbb{R}^n)} = \Big(\int_0^\infty f^*(t)^p\, dt\Big)^{1/p}.$$

Therefore,

$$\|f\|_{\hat{L^p}_{\beta,\gamma}(\mathbb{R}^n)} = \Big(\int_0^\infty (t^{-\gamma} f^*(t^{-\beta}))^p\, dt\Big)^{1/p}.$$

By using the change of variable $t = s^{-1/\beta}$, we obtain

$$\begin{aligned}\|f\|_{\hat{L^p}_{\beta,\gamma}(\mathbb{R}^n)} &= \Big(\frac{1}{\beta}\int_0^\infty s^{\gamma p/\beta} (f^*(s))^p s^{-\frac{1}{\beta}-1}\, ds\Big)^{1/p}\\ &= \Big(\frac{1}{\beta}\int_0^\infty s^{\frac{\gamma p-1}{\beta}} (f^*(s))^p \frac{ds}{s}\Big)^{1/p}.\end{aligned}$$

That is, when $\gamma > \frac{1}{p}$, $\hat{L^p}_{\beta,\gamma}(\mathbb{R}^n)$ is the Lorentz space $L_{q,p}(\mathbb{R}^n)$ where $q = \frac{\gamma p-1}{\beta p}$. Therefore, $\hat{X}_{\beta,\gamma}(\mathbb{R}^n)$ can be considered as the Lorentz space associated with X.

Moreover, for $\gamma = 1$, we have $\hat{X}_{\beta,1}(\mathbb{R}^n) = \hat{X}_\beta$ where $\hat{X}_\beta$ is introduced in [**13**, Definition 3.2]. In [**13**, Section 6], the function space $\hat{X}_\beta$ is used to characterize the

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mapping properties of the Fourier transform, the oscillatory integral operator and the restriction theorem on r.i.q.B.f.s. In particular, we find that

$$\|\hat{f}\|_{\hat{X}_1} \le C\|f\|_X, \tag{3.5}$$

if the Boyd's indices of the r.i.q.B.f.s. X satisfy

$$1 < p_X \le q_X < 2, \tag{3.6}$$

see [**13**, Theorem 6.1].

In particular, we can obtain the mapping properties of the Fourier transform for Orlicz spaces whenever the Boyd's indices of the Orlicz spaces are strictly in between 1 and 2. Notice that there exists a sharp estimate of the mapping properties of the Fourier transform for Orlicz spaces in the terms of the Paley–Titchmarsh inequalities [**11**, Theorem 2.6].

The interpolation method given in this paper cannot regenerate this sharp result. On the other hand, in [**11**], the sharp estimate is obtained by an interpolation method for quasi-linear operator of weak type (a, a) and weak type (b, b), $1 \le a < b < \infty$, see [**11**, Theorem 2.1].

Therefore, the method given in [**11**] cannot be applied to linear operators T of strong type (p_i, q_i), $i = 0, 1$ with

$$\frac{a}{p_i} + \frac{b}{q_i} = c,$$

for some $a, b, c > 0$, which are the mapping properties satisfied by the Laplace transform, the Hankel transforms, the Kontorovich–Lebedev transform and the oscillatory integral operator.

The following theorem shows that whenever $\gamma > \frac{1}{p_X}$, $\hat{X}_{\beta,\gamma}(\nu)$ is a r.i.q.B.f.s.

THEOREM 3.3. *Let $\beta > 0$, $\gamma > 0$ and μ, ν be σ-finite measures. If X is a r.i.q.B.f.s on μ with $1 < p_X \le q_X < \infty$ and*

$$\gamma > \frac{1}{p_X},$$

then $\hat{X}_{\beta,\gamma}(\nu)$ is a r.i.q.B.f.s. on ν.

Proof. Items (1)–(3) of Definition 2.1 are obviously fulfilled.

It suffices to show that $\|\cdot\|_{\hat{X}_{\beta,\gamma}(\nu)}$ is a quasi-norm and it satisfies Item (4) of Definition 2.1.

Since

$$(f+g)^*_\nu(t) \le f^*_\nu(t/2) + g^*_\nu(t/2), \quad \forall t > 0,$$

we find that

$$t^{-\gamma}(f+g)^*_\nu(t^{-\beta}) \le t^{-\gamma} f^*_\nu(t^{-\beta}/2) + t^{-\gamma} g^*_\nu(t^{-\beta}/2).$$

Consequently, that $\|\cdot\|_{\hat{X}_{\beta,\gamma}(\nu)}$ is a quasi-norm follows from the assumption $q_X < \infty$.

For the proof of Item (4) of Definition 2.1, it suffices to consider $E = (0, b)$, $b > 0$. Let $c = b^{-\beta}$. We find that

$$\begin{aligned}\rho_X^{\kappa_X}(t^{-\gamma}\chi_{(0,b)}(t^{-\beta})) &= \rho_X^{\kappa_X}(t^{-\gamma}\chi_{(c,\infty)}(t))\\ &\leq \sum_{k=0}^{\infty}\rho_X^{\kappa_X}(t^{-\gamma}\chi_{(2^k c, 2^{k+1}c]}(t))\\ &\leq C\sum_{k=0}^{\infty}2^{-k\gamma\kappa_X}\rho_X^{\kappa_X}((D_{2^{-k}}\chi_{(c,2c]})(t))\\ &\leq C\sum_{k=0}^{\infty}2^{-k\gamma\kappa_X}2^{\kappa_X k(\frac{1}{p_X}-\epsilon)}\rho_X^{\kappa_X}(\chi_{(c,2c]}(t)).\end{aligned}$$

Therefore, the assumption $\frac{1}{p_X} < \gamma$ yields an $\epsilon > 0$ such that $\frac{1}{p_X} - \epsilon < \gamma$. Consequently, $\|\cdot\|_{\hat{X}_{\beta,\gamma}(\nu)}$ fulfils Item (4) of Definition 2.1. □

4. Interpolation of operators. We present a general interpolation theory that is tailored for the Fourier-type transforms in this section.

We first establish an interpolation result of Lebesgue spaces by using the functor $(\cdot,\cdot)_{\theta,r,X}$. The following theorem shows that the interpolation functor $(\cdot,\cdot)_{\theta,r,X}$ is not introduced to offer some abstract existing results. The following theorem shows that $\hat{X}_{\beta,\gamma}(\nu)$ can be generated from Lebesgue spaces via the functor $(\cdot,\cdot)_{\theta,r,X}$.

THEOREM 4.1. *Let μ, ν be σ-finite measures. Let $0 < \theta, r < \infty$, $0 < u_0 < u_1 < \infty$ and X be a r.i.q.B.f.s. on μ. Let η satisfies*

$$\frac{1}{\eta} = \frac{1}{u_0} - \frac{1}{u_1}. \tag{4.1}$$

Suppose that

$$\frac{1}{r} > \frac{1}{p_X} \geq \frac{1}{q_X} > \frac{1}{r} - \frac{1}{\theta}. \tag{4.2}$$

If

$$\frac{1}{r} - \frac{1}{\theta} + \frac{\eta}{\theta u_0} > \frac{1}{p_X}, \tag{4.3}$$

then

$$(L^{u_1}(\nu), L^{u_0}(\nu))_{\theta,r,X} = \hat{X}_{\beta,\gamma}(\nu), \tag{4.4}$$

where

$$\beta = \frac{\eta}{\theta} \quad \text{and} \quad \gamma = \frac{1}{r} - \frac{1}{\theta} + \frac{\eta}{\theta u_0}.$$

Proof. As

$$\gamma = \frac{1}{r} - \frac{1}{\theta} + \frac{\eta}{\theta u_0} > \frac{1}{p_X},$$

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Theorem 3.3 guarantees that $\hat{X}_{\beta,\gamma}(\nu)$ is a r.i.q.B.f.s.

Since

$$K(f, t, L^{u_1}(\nu), L^{u_0}(\nu)) = tK(f, t^{-1}, L^{u_0}(\nu), L^{u_1}(\nu)),$$

the Holmstedt formulas for Lebesgue spaces [**17**] gives

$$\begin{aligned} t^{-\frac{1}{r}} K(f, t^{\frac{1}{\theta}}, L^{u_1}(\nu), L^{u_0}(\nu)) &= t^{-\frac{1}{r}} t^{\frac{1}{\theta}} K(f, t^{-\frac{1}{\theta}}, L^{u_0}(\nu), L^{u_1}(\nu)) \\ &\approx t^{-\frac{1}{r}+\frac{1}{\theta}} \left(\int_0^{t^{-\frac{\eta}{\theta}}} (f_\nu^*(s))^{u_0} ds\right)^{\frac{1}{u_0}} + t^{-\frac{1}{r}} \left(\int_{t^{-\frac{\eta}{\theta}}}^\infty (f_\nu^*(s))^{u_1} ds\right)^{\frac{1}{u_1}}. \end{aligned}$$

We find that

$$\begin{aligned} &\rho_X(t^{-\frac{1}{r}} K(f, t^{\frac{1}{\theta}}, L^{u_1}(\nu), L^{u_0}(\nu))) \\ &\le C\rho_X\Big(t^{-\frac{1}{r}+\frac{1}{\theta}}\Big(\int_0^{t^{-\frac{\eta}{\theta}}} (f_\nu^*(s))^{u_0} ds\Big)^{\frac{1}{u_0}}\Big) + C\rho_X\Big(t^{-\frac{1}{r}}\Big(\int_{t^{-\frac{\eta}{\theta}}}^\infty (f_\nu^*(s))^{u_1} ds\Big)^{\frac{1}{u_1}}\Big) \\ &= C\rho_X\Big(t^{-\frac{1}{r}+\frac{1}{\theta}-\frac{\eta}{\theta u_0}}\Big(\int_0^1 (f_\nu^*(st^{-\frac{\eta}{\theta}}))^{u_0} ds\Big)^{\frac{1}{u_0}}\Big) \\ &\quad + C\rho_X\Big(t^{-\frac{1}{r}-\frac{\eta}{\theta u_1}}\Big(\int_1^\infty (f_\nu^*(st^{-\frac{\eta}{\theta}}))^{u_1} ds\Big)^{\frac{1}{u_1}}\Big) \\ &= I + II \end{aligned}$$

for some $C > 0$.

We first deal with I. Since f_ν^* is non-increasing, we obtain

$$\begin{aligned} I &\le C\rho_{X^{\frac{1}{u_0}}}^{\frac{1}{u_0}}\Big(t^{-\frac{u_0}{r}+\frac{u_0}{\theta}-\frac{\eta}{\theta}} \int_0^1 (f_\nu^*(st^{-\frac{\eta}{\theta}}))^{u_0} ds\Big) \\ &\le C\rho_{X^{\frac{1}{u_0}}}^{\frac{1}{u_0}}\Big(\sum_{j=-\infty}^0 t^{-\frac{u_0}{r}+\frac{u_0}{\theta}-\frac{\eta}{\theta}} 2^{j-1} (f_\nu^*(2^{j-1}t^{-\frac{\eta}{\theta}}))^{u_0}\Big) \\ &\le C\rho_{X^{\frac{1}{u_0}}}^{\frac{1}{u_0}}\Big(\sum_{j=-\infty}^0 2^{(j-1)\frac{\theta}{\eta}(-\frac{u_0}{r}+\frac{u_0}{\theta})} (D_{2^{(-j+1)\theta/\eta}}F)(t)^{u_0}\Big), \end{aligned}$$

where $F(t) = t^{-\frac{1}{r}+\frac{1}{\theta}-\frac{\eta}{\theta u_0}} f_\nu^*(t^{-\frac{\eta}{\theta}})$.

According to the Aoki–Rolewicz theorem, we have a $0 < \kappa_0 \le 1$ such that $\rho_{X^{\frac{1}{u_0}}}^{\kappa_0}$ is sub-additive. Consequently,

$$\begin{aligned} I^{u_0\kappa_0} &\le C\rho_{X^{\frac{1}{u_0}}}^{\kappa_0}\Big(\sum_{j=-\infty}^0 2^{(j-1)\frac{\theta}{\eta}(-\frac{u_0}{r}+\frac{u_0}{\theta})} (D_{2^{(-j+1)\theta/\eta}}F)(t)^{u_0}\Big) \\ &\le C\sum_{j=0}^\infty 2^{(-j-1)\frac{\theta}{\eta}(-\frac{u_0}{r}+\frac{u_0}{\theta})\kappa_0} (\rho_X(D_{2^{(j+1)\theta/\eta}}F))^{u_0\kappa_0}. \end{aligned}$$

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For any $\epsilon > 0$, the definition of Boyd's index q_X yields

$$I^{u_0\kappa_0} \leq C \sum_{j=0}^{\infty} 2^{(-j-1)\frac{\theta}{\eta}(-\frac{u_0}{r}+\frac{u_0}{\theta})\kappa_0} 2^{-\frac{j\theta u_0 \kappa_0}{\eta(q_X+\epsilon)}} \rho_X(F)^{u_0\kappa_0}.$$

Thus, (4.2) guarantees that there exists an $\epsilon > 0$ such that $-\frac{1}{r}+\frac{1}{\theta}+\frac{1}{q_X+\epsilon} > 0$. Consequently,

$$I^{u_0\kappa_0} \leq C\rho_X(F)^{u_0\kappa_0}. \tag{4.5}$$

We consider II. The Aoki–Rolewicz theorem gives a $0 < \kappa_1 \leq 1$ such that $\rho^{\kappa_1}_{X^{\frac{1}{u_1}}}$ is sub-additive. Consequently,

$$\begin{aligned} II^{u_1\kappa_1} &\leq C\rho^{\kappa_1}_{X^{\frac{1}{u_1}}} \Big(\sum_{j=0}^{\infty} 2^{-j\frac{\theta}{\eta}\frac{u_1}{r}} (D_{2^{-j\theta/\eta}}F)(t)^{u_1} \Big) \\ &\leq C \sum_{j=0}^{\infty} 2^{-j\frac{\theta}{\eta}\frac{u_1\kappa_1}{r}} \rho^{\kappa_1}_{X^{\frac{1}{u_1}}} \Big((D_{2^{-j\theta/\eta}}F)(t)^{u_1} \Big), \end{aligned}$$

because (4.1) gives

$$F(t) = t^{-\frac{1}{r}+\frac{1}{\theta}-\frac{\eta}{\theta u_0}} f^*_\nu(t^{-\frac{\eta}{\theta}}) = t^{-\frac{1}{r}-\frac{\eta}{\theta u_1}} f^*_\nu(t^{-\frac{\eta}{\theta}}).$$

The definition of Boyd's index p_X assures that for any $p_X > \epsilon > 0$, we have

$$\begin{aligned} II^{u_1\kappa_1} &\leq C \sum_{j=-\infty}^{0} 2^{j\frac{\theta}{\eta}\frac{u_1\kappa_1}{r}} \rho_X(D_{2^{j\theta/\eta}}F)^{u_1\kappa_1} \\ &\leq C \sum_{j=-\infty}^{0} 2^{j\frac{\theta}{\eta}\frac{u_1\kappa_1}{r}} 2^{-j\frac{\theta u_1\kappa_1}{\eta(p_X-\epsilon)}} \rho_X(F)^{u_1\kappa_1}. \end{aligned}$$

Therefore, (4.2) provides an $\epsilon > 0$ such that $\frac{1}{r} > \frac{1}{p_X-\epsilon}$. Consequently, we find that

$$II^{u_1\kappa_1} \leq \rho_X(F)^{u_1\kappa_1}. \tag{4.6}$$

Since $\rho_X(F) = \rho_X(t^{-\frac{1}{r}+\frac{1}{\theta}-\frac{\eta}{\theta u_0}} f^*_\nu(t^{-\frac{\eta}{\theta}})) = \|f\|_{\hat{X}_{\beta,\gamma}(\nu)}$, (4.5) and (4.6) yield the embedding $\hat{X}_{\beta,\gamma}(\nu) \hookrightarrow (L^{u_1}(\nu), L^{u_0}(\nu))_{\theta,r,X}$.

Next, since f^*_ν is non-increasing, we get

$$\begin{aligned} \rho_X(t^{-\frac{1}{r}} K(f, t^{\frac{1}{\theta}}, L^{u_1}(\nu), L^{u_0}(\nu))) &\geq C\rho_X\Big(t^{-\frac{1}{r}+\frac{1}{\theta}} \Big(\int_0^{t^{-\frac{\eta}{\theta}}} (f^*_\nu(s))^{u_0} ds \Big)^{\frac{1}{u_0}} \Big) \\ &\geq C\rho_X\Big(t^{-\frac{1}{r}+\frac{1}{\theta}-\frac{\eta}{\theta u_0}} f^*_\nu(t^{-\frac{\eta}{\theta}}) \Big). \end{aligned}$$

Thus, the embedding $(L^{u_1}(\nu), L^{u_0}(\nu))_{\theta,r,X} \hookrightarrow \hat{X}_{\beta,\gamma}(\nu)$ is also valid. Hence, we establish (4.4). □

The above theorem is a complementary result of [**13**, Theorem 4.2].

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We are now to apply the above theorem to obtain the mapping properties for the Fourier type transforms on r.i.q.B.f.s.

THEOREM 4.2. *Let μ, ν be σ-finite measures. Let $a, b, c, p_0, p_1 > 0$ and X be a r.i.q.B.f.s. on μ with $p_0 < p_X \leq q_X < p_1$. Suppose that the linear operator $T : L^{p_i}(\mu) \to L^{q_i}(\nu)$, $i = 0, 1$, are bounded where*

$$\frac{a}{p_i} + \frac{b}{q_i} = c, \quad i = 0, 1. \tag{4.7}$$

Then, T can be extended to be a bounded linear operator from X to $\hat{X}_{\frac{b}{a},\frac{c}{a}}(\nu)$.

Proof. Let $\frac{1}{\theta} = \frac{1}{p_0} - \frac{1}{p_1}$ and $r = p_0$. Corollary 3.2 yields that

$$(L^{p_0}(\mu), L^{p_1}(\mu))_{\theta,p_0,X} = X.$$

In view of Theorem 3.1, it remains to show that

$$(L^{q_0}(\nu), L^{q_1}(\nu))_{\theta,p_0,X} = \hat{X}_{\frac{b}{a},\frac{c}{a}}(\nu).$$

Notice that $q_1 < q_0$. We are going to apply Theorem 4.1 with $u_0 = q_1$ and $u_1 = q_0$. Thus, $\frac{1}{\eta} = \frac{1}{q_1} - \frac{1}{q_0}$. Since

$$\frac{a}{p_0} + \frac{b}{q_0} = \frac{a}{p_1} + \frac{b}{q_1},$$

we find that $\frac{a}{\theta} = \frac{b}{\eta}$. That is, $\frac{\eta}{\theta} = \frac{b}{a}$.

Therefore,

$$\frac{1}{r} = \frac{1}{p_0} > \frac{1}{p_X}$$

and

$$\frac{1}{q_X} > \frac{1}{p_1} = \frac{1}{p_0} - \left(\frac{1}{p_0} - \frac{1}{p_1}\right).$$

Furthermore, (4.7) with $i = 0$ gives $\frac{c}{a} > \frac{1}{p_0}$. Consequently, we have

$$\frac{1}{r} - \frac{1}{\theta} + \frac{\eta}{\theta u_0} = \frac{1}{p_1} + \frac{b}{aq_1} = \frac{1}{a}\left(\frac{a}{p_1} + \frac{b}{q_1}\right) = \frac{c}{a} > \frac{1}{p_0} > \frac{1}{p_X}.$$

Therefore, the conditions in Theorem 4.1 are fulfilled. Since

$$\beta = \frac{\eta}{\theta} = \frac{b}{a} \quad \text{and} \quad \frac{1}{r} - \frac{1}{\theta} + \frac{\eta}{\theta u_0} = \frac{c}{a},$$

Theorems 3.1 and 4.1 yield the boundedness of $T : X \to \hat{X}_{\frac{b}{a},\frac{c}{a}}(\nu)$. □

Notice that the Kontorovich–Lebedev transform maps $L^p(0, \infty)$ to

$$L^p(dt/t) = \left\{ f \in \mathcal{M}(0, \infty) : \int_0^\infty |f(t)|^p dt/t < \infty \right\},$$

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see Section 5.2. This is the main motivation why we consider the operators that maps $L^p(\mu)$ to $L^q(\nu)$ with different σ-finite measures μ and ν in Theorem 4.2.

In Theorem 4.2, we just consider the case (4.7). The idea of the proof can be applied to those bounded linear operators $T : L^p \to L^q$ with $\frac{a}{p} = \frac{b}{q} + c$, $a, b > 0$, see [**16**]. Moreover, the above idea can be further extended to the linear operator on Hardy spaces, see [**13**, **14**].

5. Fourier-type transforms.

In this section, as applications of Theorem 4.2, we obtain the mapping properties for some concrete transforms on r.i.q.B.f.s.

5.1. Laplace transform.

For any $f \in \mathcal{M}(0, \infty)$, the Laplace transform of f is given by

$$\mathcal{L}f(s) = \int_0^\infty e^{-st} f(t)dt.$$

It is easy to see that $\mathcal{L} : L^1(0, \infty) \to L^\infty(0, \infty)$ is bounded. According to [**8**, p.189], $\mathcal{L}$ is bounded on $L^2(0, \infty)$.

Notice that the pairs $(p_0, q_0) = (1, \infty)$ and $(p_1, q_1) = (2, 2)$ satisfy

$$\frac{1}{p_i} + \frac{1}{q_i} = 1, \quad i = 0, 1.$$

Therefore, Theorem 4.2 gives the following result.

THEOREM 5.1. *Let X be a r.i.q.B.f.s. on $(0, \infty)$ with $1 < p_X \le q_X < 2$. The Laplace transform $\mathcal{L}$ is bounded from X to $\hat{X}_{1,1}(0, \infty)$.*

5.2. Kontorovich–Lebedev transform.

The mapping properties of the Kontorovich–Lebedev transform give us an example for which the σ-finite measures μ and ν are different.

Let $K_\nu(z)$ be the modified Bessel function of the second kind. The Kontorovich–Lebedev transform is defined as

$$KLf(x) = 2\int_0^\infty K_{i\tau}(2\sqrt{x}) f(\tau)\tau\, d\tau.$$

According to [**30**, Section 2, Theorem 1], we have the following L_p boundedness of the Kontorovich–Lebedev transform. Let dt denote the Lebesgue measure on $(0, \infty)$.

THEOREM 5.2. *Let $1 \le p \le 2$. The Kontorovich–Lebedev transform is bounded from $L^p(0, \infty)$ to $L^{p'}(dt/t)$.*

Similar to the Fourier transform and the Laplace transform, we have the mapping properties of the Kontorovich–Lebedev transform on r.i.q.B.f.s.

By applying Theorem 4.2 with $d\mu = dt$ and $d\nu = dt/t$, we obtain the following theorem.

THEOREM 5.3. *Let X be a r.i.q.B.f.s. on $(0, \infty)$ with $1 < p_X \le q_X < 2$. The Kontorovich–Lebedev transform KL is bounded from X to $\hat{X}_{1,1}(dt/t)$.*

5.3. Hankel transforms. The studies of the Hankel transforms offer an example for which the constant $c \neq 1$ in Theorem 4.2.

Let $\alpha \geq -\frac{1}{2}$ and $v, u \in \mathbb{R}$. The operator $\mathcal{L}^{\alpha}_{v,u}$ is defined by

$$\mathcal{L}^{\alpha}_{v,u} f(y) = y^{u} \int_{0}^{\infty} (xy)^{v} f(x) J_{\alpha}(xy) dy. \tag{5.1}$$

where $J_{\alpha}(r)$ is the Bessel function of the first kind. The family of operators $\{\mathcal{L}^{\alpha}_{v,u}\}$ includes a number of operators used in analysis. For instance, if we denote the Fourier transform of $f(|x|)$ by $\hat{f}(|\xi|)$, then

$$\hat{f}(|\xi|) = (2\pi)^{n/2} \mathcal{L}^{\frac{n}{2}-1}_{\frac{n}{2},1-n} f(|\xi|). \tag{5.2}$$

According to [**10**], for any $\alpha > -1$, the operator $\mathcal{L}^{\alpha}_{\alpha+1,-2\alpha-1}$ is named as the Hankel transform. Noticed that $\mathcal{L}^{\alpha}_{\alpha+1,-2\alpha-1} = \tilde{\mathcal{H}}_{\alpha}$ is also called as the Fourier-Bessel transform of order α in [**4**]. Moreover,

$$\mathcal{H}_{\alpha} f = \mathcal{L}^{\alpha}_{\frac{1}{2},0} f = \int_{0}^{\infty} f(t)(xt)^{\frac{1}{2}} J_{\alpha}(xt) dt,$$

is the so-called Hankel transform of order α.

Therefore, the family of operators $\{\mathcal{L}^{\alpha}_{v,u}\}$ contains those operators related to the Hankel transform.

We now state the mapping properties of the operator $\mathcal{L}^{\alpha}_{v,u}$ from [**3**, Theorem 1.1].

THEOREM 5.4. *Let $u, v \in \mathbb{R}$, $\alpha \geq -\frac{1}{2}$ and $1 \leq p \leq q \leq \infty$. The operator $\mathcal{L}^{\alpha}_{v,u}$ is bounded from $L^{p}(0, \infty)$ to $L^{q}(0, \infty)$ if and only if*

$$u = 1 - \frac{1}{p} - \frac{1}{q}, \quad \text{and} \quad -\alpha - 1 + \frac{1}{p} < v \leq \frac{1}{2} - \max\{u, 0\}. \tag{5.3}$$

Since $1 \leq p \leq q$ and

$$\frac{1}{p} + \frac{1}{q} = 1 - u,$$

we find that the conditions in the above theorem impose a restriction on u. Precisely, u satisfies $-1 \leq u \leq 1$.

Additionally, for any fixed u, we have

$$\frac{1}{p} \leq \frac{1}{p} + \frac{1}{q} = 1 - u.$$

Since $1 \leq p \leq q$, we also have

$$\frac{2}{p} \geq \frac{1}{p} + \frac{1}{q} = 1 - u.$$

Therefore, the conditions in Theorem 5.4 show that u and p fulfil

$$\frac{1 - u}{2} \leq \frac{1}{p} \leq 1 - u. \tag{5.4}$$

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With the above preparations, we now ready to present the mapping properties of $\mathcal{L}^{\alpha}_{v,u}$ on r.i.q.B.f.s.

THEOREM 5.5. *Let* $-1 < u < 1$, $v \in \mathbb{R}$ *and* $\alpha \geq -\frac{1}{2}$. *Let* X *be a r.i.q.B.f.s. on* $(0, \infty)$ *with*

$$\frac{1-u}{2} < \frac{1}{q_X} \leq \frac{1}{p_X} < 1-u \tag{5.5}$$

$$-\alpha - 1 + \frac{1}{p_X} < v \leq \frac{1}{2} - \max\{u, 0\}. \tag{5.6}$$

The operator $\mathcal{L}^{\alpha}_{v,u}$ *is bounded from* X *to* $\hat{X}_{1,1-u}(0, \infty)$.

Proof. As $1 - u > \frac{1}{p_X}$, Theorem 3.3 assures that $\hat{X}_{1,1-u}$ is a r.i.q.B.f.s. Take $1 < p_0 < p_1 < \infty$ such that $p_0 < p_X \leq q_X < p_1$, $\frac{1-u}{2} < \frac{1}{p_1} < \frac{1}{p_0} < 1 - u$ and

$$-\alpha - 1 + \frac{1}{p_0} < v \leq \frac{1}{2} - \max\{u, 0\}.$$

Let q_0, q_1 satisfy

$$\frac{1}{q_0} = 1 - \frac{1}{p_0} - u, \quad \text{and} \quad \frac{1}{q_1} = 1 - \frac{1}{p_1} - u. \tag{5.7}$$

Notice that $q_0 > q_1$.

Since $\frac{1-u}{2} < \frac{1}{p_1}$, we find that $\frac{2}{p_1} > 1 - u$ and hence,

$$\frac{1}{p_1} > 1 - \frac{1}{p_1} - u = \frac{1}{q_1}.$$

That is, $q_1 > p_1$. Similarly, as $\frac{1-u}{2} < \frac{1}{p_0}$, we also have

$$\frac{1}{p_0} > 1 - \frac{1}{p_0} - u = \frac{1}{q_0}.$$

That is, $q_0 > p_0$.

Therefore, Theorem 5.4 assures that $\mathcal{L}^{\alpha}_{v,u} : L^{p_0} \to L^{q_0}$ and $\mathcal{L}^{\alpha}_{v,u} : L^{p_1} \to L^{q_1}$ are bounded.

Since $\mathcal{L}^{\alpha}_{v,u}$ is of strong type (p_i, q_i), $i = 0, 1$ where

$$\frac{1}{p_i} + \frac{1}{q_i} = 1 - u, \qquad i = 0, 1.$$

With $a = b = 1$ and $c = 1 - u$, Theorem 4.2 yield the boundedness of $\mathcal{L}^{\alpha}_{v,u} : X \to \hat{X}_{1,1-u}(0, \infty)$. □

When X is the Lebesgue space $L^p(0, \infty)$, aside from the boundary points $\frac{1}{p} = 1 - u$ and $\frac{1}{p} = \frac{1-u}{2}$, (5.5) becomes (5.4).

We give an application of Theorem 5.5 on the Hankel transform of order α, $\mathcal{H}_\alpha$. According to Theorem 5.5, for any r.i.q.B.f.s. on $(0, \infty)$, X satisfying

$$\frac{1}{2} < \frac{1}{q_X} \leq \frac{1}{p_X} < 1, \tag{5.8}$$

we have

$$\|\mathcal{H}_\alpha f\|_{\hat{X}_{1,1}(0,\infty)} \leq \|f\|_X. \tag{5.9}$$

We find that the mapping properties for the Fourier transform is the same as the Hankel transform of order α (5.8), (5.9).

Furthermore, for $\alpha > -\frac{1}{2}$, Theorem 5.5 also yields the mapping properties of the Fourier-Bessel transform of order α, $\tilde{\mathcal{H}}_\alpha = \mathcal{L}^\alpha_{\alpha+1,-2\alpha-1}$. Precisely, whenever a r.i.q.B.f.s. on $(0, \infty)$, X satisfies

$$1 + \alpha < \frac{1}{q_X} \leq \frac{1}{p_X} < 2 + 2\alpha,$$

then

$$\|\tilde{\mathcal{H}}_\alpha f\|_{X_{1,2+2\alpha}(0,\infty)} \leq C\|f\|_X,$$

for some $C > 0$.

For the modular inequalities for Hankel transforms, the reader is referred to [**15**].

5.4. Oscillatory integrals. In [**13**, Section 6], we investigate the boundedness of some oscillatory integral operators on r.i.q.B.f.s. In this section, we study another class of oscillatory integral operators with singularities different from the one appeared in [**13**]. This class of oscillatory integral operators gives us an application of Theorem 4.2 with $b \neq 1$.

Let $a(x, y, z) \in C_0^\infty(\mathbb{R}^3)$. Let $\phi(x, y, z) \in C^\infty(\mathbb{R}^3)$ be real valued and the Jacobian $D(\frac{\partial\phi}{\partial z}, \frac{\partial^2\phi}{\partial z^2})/D(x, y)$ is non-zero in supp$a$. For any $f \in C_0^\infty(\mathbb{R})$ and $\lambda > 0$, the oscillatory integral operator T_λ is defined as

$$T_\lambda f(x, y) = \int e^{i\lambda\phi(x,y,z)} a(x, y, z) f(z) dz.$$

We have the L^p boundedness of T_λ from [**19**, Theorem 1.2].

THEOREM 5.6. *Let* $1 \leq p < 4$. *The oscillatory integral operator* $T_\lambda : L^p(\mathbb{R}) \to L^q(\mathbb{R}^2)$ *is bounded where*

$$\frac{1}{p} + \frac{3}{q} = 1.$$

By using Theorem 4.2, we can extend the above result to r.i.q.B.f.s.

THEOREM 5.7. *Let* $\lambda > 0$ *and* X *be a r.i.q.B.f.s. on* $\mathbb{R}$ *with* $1 < p_X \leq q_X < 4$. *The oscillatory integral operator* T_λ *is bounded from* X *to* $\hat{X}_{3,1}(\mathbb{R}^2)$.

Proof. Let $p_0 = 1$. According to Theorem 5.6, $T_\lambda : L^1(\mathbb{R}) \to L^\infty(\mathbb{R}^2)$. Next, pick p_1 satisfying $q_X < p_1 < 4$. Theorem 5.6 assures that $T_\lambda : L^{p_1}(\mathbb{R}) \to L^{q_1}(\mathbb{R}^2)$ is bounded where

$$\frac{1}{p_1} + \frac{3}{q_1} = 1.$$

Therefore, by applying Theorem 4.2 with $a = c = 1$ and $b = 3$, we establish the boundedness of $T_\lambda : X \to \hat{X}_{3,1}(\mathbb{R}^2)$. □

6. Lorentz–Karamata spaces. Finally, we apply the above results to the Lorentz–Karamata space. The family of Lorentz–Karamata spaces includes Lebesgue spaces, Lorentz spaces, Lorentz–Zygmund spaces and generalized Lorentz–Zygmund. Therefore, the followings give the mapping properties of $\mathcal{L}^\alpha_{v,u}$ on the above-mentioned function spaces.

To give the definition of Lorentz–Karamata spaces, we recall the notion of slowly varying function from [**7**, Definition 3.4.32].

We say that the function $f : [1, \infty) \to (0, \infty)$ is equivalent to a function $g : [1, \infty) \to (0, \infty)$ if there exist constants B, $C > 0$ such that

$$Cg(t) \leq f(t) \leq Bg(t), \quad t \geq 1.$$

Definition 6.1. A Lebesgue measurable function $b : [1, \infty) \to (0, \infty)$ is called as a slowly varying function if for any given $\epsilon > 0$, $t^\epsilon b(t)$ and $t^{-\epsilon} b(t)$ are equivalent to a non-decreasing function and a non-increasing function, respectively.

Let $a > 0$. Whenever b is a slowly varying function, $b^a(t) = b(t^a)$ is also a slowing varying function [**7**, Proposition 3.4.33 (viii)].

For any slowly varying function b, define $\gamma_b : (0, \infty) \to (0, \infty)$ by

$$\gamma_b(t) = b(\max\{t, 1/t\}), \quad t > 0.$$

We are now ready to define the Lorentz–Karamata space [**7**, Definition 3.4.38].

Definition 6.2. Let μ be a σ-finite measure. Let $1 < p, q < \infty$ and b be a slowly varying function. The Lorentz–Karamata space $L_{p,q,b}(\mu)$ consists of those Lebesgue measurable function f satisfying

$$\|f\|_{L_{p,q,b}(\mu)} = \left(\int_0^\infty (t^{1/p} \gamma_b(t) f^*_\mu(t))^q \frac{dt}{t} \right)^{1/q} < \infty.$$

The Lorentz–Karamata space $L_{p,q,b}(\mu)$ is a r.i.B.f.s., see [**7**, Theorem 3.4.41]. For the studies of interpolation and the Hausdorff–Young inequality on Lorentz–Karamata spaces on $\mathbb{R}^n$, the reader is referred to [**9**]. Furthermore, we refer the reader to [**7**, **6**, **24**] for some comprehensive accounts and discussions on Lorentz–Karamata spaces.

Next, we compute the Boyd's indices of Lorentz–Karamata spaces.

Proposition 6.1. *Let $1 < p, q < \infty$ and b be a slowly varying function. The lower and upper Boyd's indices of $L_{p,q,b}(\mu)$ are p.*

Proof. It suffices to consider the case where μ is the Lebesgue measure on $(0, \infty)$.

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For any $s > 1$, we find that

$$\left(\int_0^\infty (t^{1/p}\gamma_b(t)f_\mu^*(st))^q \frac{dt}{t}\right)^{1/q} = \left(\int_0^\infty (t^{1/p}\gamma_b(t)f_\mu^*(st))^q \frac{dt}{t}\right)^{1/q}$$
$$= \left(\int_0^\infty ((ys^{-1})^{1/p}\gamma_b(ys^{-1})f_\mu^*(y))^q \frac{dy}{y}\right)^{1/q},$$

where we use the change of variable $y = st$.

In view of [7, Proposition 3.4.33 (ii)], for any $\epsilon > 0$, $t^\epsilon \gamma_b(t)$ is equivalent to a non-decreasing function. Hence,

$$\left(\int_0^\infty (t^{1/p}\gamma_b(t)f_\mu^*(st))^q \frac{dt}{t}\right)^{1/q} = \left(\int_0^\infty ((ys^{-1})^{\frac{1}{p}-\epsilon}(ys^{-1})^\epsilon\gamma_b(ys^{-1})f_\mu^*(y))^q \frac{dy}{y}\right)^{1/q}$$
$$\leq C\left(\int_0^\infty ((ys^{-1})^{\frac{1}{p}-\epsilon}y^\epsilon\gamma_b(y)f_\mu^*(y))^q \frac{dy}{y}\right)^{1/q}$$
$$= Cs^{-\frac{1}{p}+\epsilon}\left(\int_0^\infty y^{\frac{1}{p}}\gamma_b(y)f_\mu^*(y))^q \frac{dy}{y}\right)^{1/q}.$$

That is, for any $\epsilon > 0$,

$$\|D_s\|_{L_{p,q,b}(0,\infty)\to L_{p,q,b}(0,\infty)} \leq Cs^{-\frac{1}{p}+\epsilon}.$$

Hence, $q_{L_{p,q,b}} \leq p$.

Similarly, [7, Proposition 3.4.33 (ii)] guarantees that for any $\epsilon > 0$, $t^{-\epsilon}\gamma_b(t)$ is equivalent to a non-increasing function. We obtain

$$\left(\int_0^\infty (t^{1/p}\gamma_b(t)f_\mu^*(st))^q \frac{dt}{t}\right)^{1/q}$$
$$= \left(\int_0^\infty ((ys^{-1})^{\frac{1}{p}+\epsilon}(ys^{-1})^{-\epsilon}\gamma_b(ys^{-1})f_\mu^*(y))^q \frac{dy}{y}\right)^{1/q}$$
$$\geq C\left(\int_0^\infty ((ys^{-1})^{\frac{1}{p}+\epsilon}y^{-\epsilon}\gamma_b(y)f_\mu^*(y))^q \frac{dy}{y}\right)^{1/q}$$
$$= Cs^{-\frac{1}{p}-\epsilon}\left(\int_0^\infty y^{\frac{1}{p}}\gamma_b(y)f_\mu^*(y))^q \frac{dy}{y}\right)^{1/q}.$$

The above inequalities yield $\|D_s\|_{L_{p,q,b}(0,\infty)\to L_{p,q,b}(0,\infty)} \geq Cs^{-\frac{1}{p}-\epsilon}$ and, hence, $q_{L_{p,q,b}} \geq p$.

Therefore, $q_{L_{p,q,b}(0,\infty)} = p$. Since the proof of $p_{L_{p,q,b}(0,\infty)} = p$ is similar to the proof of $q_{L_{p,q,b}(0,\infty)} = p$, for brevity, we leave the detail of the proof of $p_{L_{p,q,b}(0,\infty)} = p$ to the reader. □

For any $\beta > 0$, we have

$$\gamma_b(t^{-\frac{1}{\beta}}) = b(\max\{t, 1/t\}^{\frac{1}{\beta}}) = \gamma_{b^{1/\beta}}(t).$$

Consequently, when $\lambda > \frac{1}{p}$,

$$\|f\|_{(\hat{L}_{p,q,b})_{\beta,\lambda}(\mu)} = \left(\int_0^\infty (t^{1/p}\gamma_b(t)t^{-\lambda}f_\mu^*(t^{-\beta}))^q \frac{dt}{t}\right)^{1/q}$$
$$= \left(\int_0^\infty s^{-\frac{1}{p\beta}}\gamma_b(s^{-\frac{1}{\beta}})s^{\frac{\lambda}{\beta}}f_\mu^*(s)\frac{ds}{s}\right)^{1/q},$$

where we use the change of variable $s = t^{-\beta}$.

Therefore, we find that $(\hat{L}_{p,q,b})_{\beta,\lambda}(\mu) = L_{r,q,b^{1/\beta}}(\mu)$ where $\frac{1}{r} = \lambda - \frac{1}{p}$.

We have the following results for the Laplace transform and the Kontorovich–Lebedev transform on Lorentz–Karamata spaces.

COROLLARY 6.2. *Let $1 < p < 2$, $1 < q < \infty$ and b be a slowly varying function. The Laplace transform is bounded from $L_{p,q,b}(0,\infty)$ to $L_{p',q,b}(0,\infty)$ and the Kontorovich–Lebedev transform is bounded from $L_{p,q,b}(0,\infty)$ to $L_{p',q,b}(dt/t)$.*

By using Theorem 5.5, we have the following mapping properties for the operator $\mathcal{L}^\alpha_{v,u}$ on Lorentz–Karamata spaces.

COROLLARY 6.3. *Let $1 < p, q < \infty$ and b be a slowly varying function. Let $-1 < u < 1$, $v \in \mathbb{R}$ and $\alpha \geq -\frac{1}{2}$. If*

$$\frac{1-u}{2} < \frac{1}{p} < 1-u$$
$$-\alpha - 1 + \frac{1}{p} < v \leq \frac{1}{2} - \max\{u, 0\}.$$

The operator $\mathcal{L}^\alpha_{v,u}$ is bounded from $L_{p,q,b}(0,\infty)$ to $L_{r,q,b}(0,\infty)$ where $\frac{1}{r} = 1 - u - \frac{1}{p}$.

The above result shows that when $1 < p < 2$, the Hankel transform of order α, $\mathcal{H}_\alpha$ is bounded from $L_{p,q,b}(0,\infty)$ to $L_{p',q,b}(0,\infty)$. Furthermore, for any $v \leq \frac{1}{2}$, if $1+\alpha < \frac{1}{p} < 2+2\alpha$, then the Fourier-Bessel transform of order α, $\tilde{\mathcal{H}}_\alpha$, is bounded from $L_{p,q,b}(0,\infty)$ to $L_{r,q,b}(0,\infty)$, where $\frac{1}{r} = 2+2\alpha - \frac{1}{p}$.

Similarly, we have the mapping properties for the oscillatory integral operator on Lorentz-Karamata spaces.

COROLLARY 6.4. *Let $\lambda > 0$, $1 < p < 4$, $1 < q < \infty$ and b be a slowly varying function. The oscillatory integral operators T_λ is bounded from $L_{p,q,b}(\mathbb{R})$ to $L_{p',q,b^{1/3}}(\mathbb{R}^2)$.*

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