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ON THE APPROXIMATION OF FIXED POINTS OF LOCALLY NONEXPANSIVE MAPPINGS

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ABSTRACT. It is shown that techniques of Browder and Ishikawa for approximating fixed points of nonexpansive mappings extend in a more restricted sense to the locally nonexpansive case.

1. **Introduction.** In this note we call attention to the fact that two fundamental techniques for approximating fixed points of nonexpansive mappings (Theorem 1 of Browder [1] and Theorem 1 of Ishikawa [5]) extend under suitable additional assumptions to the locally nonexpansive case. To obtain these extensions we basically use ideas developed elsewhere, in conjunction with a condition which is known to assure existence of fixed points for locally nonexpansive mappings in uniformly convex spaces (see Kirk [6]). While this condition precludes domains with empty interior, aside from boundedness (in Theorem 1) we place no further restrictions on our domains.

To fix our terminology and notation, let X be a Banach space and D a subset of X. A mapping $T: D \to X$ is said to be *locally nonexpansive* on D if each point $x \in D$ has a neighborhood U such that $\|T(u) - T(v)\| \leq \|u - v\|$ for all u, $v \in U$. Throughout the paper we use $\bar{D}$ and ∂D to denote respectively the closure and the boundary of D, and for u, $v \in X$ we use $S[u, v]$ to denote the segment $\{tu + (1-t)v : t \in [0, 1]\}$.

We remark that if D is open and $T: D \to X$ has a Gateaux derivative T'_x at each point $x \in D$ with $\|T'_x\| \leq 1$, then it is a simple matter to show that the restriction of T to any convex neighborhood of x is nonexpansive; thus T is locally nonexpansive on D.

Our basic observations are the following:

THEOREM 1. *Let H be a Hilbert space, D a bounded open subset of H, and $T: \bar{D} \to H$ a continuous mapping which is locally nonexpansive on D. Suppose z is a point of D for which*

$$\|z - T(z)\| < \|x - T(x)\| \quad \textit{for all} \quad x \in \partial D.$$

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Then there is a unique continuous path $t \mapsto x_t \in D$, $t \in [0, 1)$, *satisfying*

$$x_t = tT(x_t) + (1-t)z. \tag{1}$$

Moreover, $\lim_{t\to 1^-} x_t = x \in D$, *where* x *is a fixed point of* T.

THEOREM 2. *Let* X *be an arbitrary Banach space,* D *an open subset of* X, *and* $T:\bar{D}\to X$ *a continuous mapping which is locally nonexpansive on* D *and for which* $T(\bar{D})$ *is precompact. Suppose* x_1 *is a point of* D *for which*

$$\|x_1 - T(x_1)\| < \|x - T(x)\| \quad \textit{for all} \quad x \in \partial D, \tag{2}$$

and let $\{t_n\} \subset \mathbb{R}$ *satisfy*

$$\sum_{n=1}^{\infty} t_n = \infty \quad \textit{and} \quad 0 \le t_n \le b < 1, \qquad n = 1, 2, \ldots.$$

Then the sequence $\{x_n\}$ *defined by*

$$x_{n+1} = (1-t_n)x_n + t_n T(x_n), \qquad n = 1, 2, \ldots,$$

lies in D *and converges to a fixed point of* T *in* D.

While the existence part of Theorem 2 appears to be new, the existence part of Theorem 1 is a very special case of Theorem 1 of [6].

2. **Proof of Theorem 1.** In proving Theorem 1 we shall utilize two propositions which are valid in much more general settings. These will then enable us to complete the proof following a general line of argument due to Halpern [4]. Our first proposition is implicit in [8].

PROPOSITION 1. *Let* X *be a Banach space,* D *a bounded open subset of* X *with* $0 \in D$, *and* $T:\bar{D}\to X$ *a continuous mapping which is locally nonexpansive on* D. *Suppose*

$$\|T(0)\| < \|x - T(x)\| \quad \textit{for all} \quad x \in \partial D.$$

Then there is a unique continuous path $t \mapsto x_t$, $t \in [0, 1)$, *satisfying* $tT(x_t) = x_t$. *Moreover for this path,* $\|x_t - T(x_t)\| \downarrow 0$ *as* $t \uparrow 1$.

Proof. Since T is continuous there exists $\delta \in (0, 1)$ such that $tT(B) \subset B$ for some ball $B \subset D$ centered at the origin and for all $t \in (0, \delta)$. Since tT is a contraction mapping on B, it follows that for such t there exist unique points $x_t \in B$ such that $tT(x_t) = x_t$. By Lemma 1 of [8] there is a largest number $r \in (0, 1]$ for which there exists a unique continuous path $t \mapsto x_t$, $t \in [0, r)$ with $tT(x_t) = x_t$ $(x_0 = 0)$. If $r < 1$, then Lemma 2(iv) of [8] implies $rT(\bar{x}) = \bar{x} \in \bar{D}$, where $\bar{x} = \lim_{t\to r^-} x_t$. Moreover, Proposition 1 of [6] (also see [7]) implies $\bar{x} \notin D$. But by Lemma 2(ii) of [8] the map $t \mapsto \|x_t - T(x_t)\|$ is nonincreasing for $t \in [0, r)$; hence

$$\|\bar{x} - T(\bar{x})\| \le \|T(0)\| < \|x - T(x)\|$$

for all $x \in \partial D$, yielding $\bar{x} \notin \partial D$. This contradiction implies $r = 1$. Therefore the path $\{x_t\}$ is defined for all $t \in [0, 1)$, and since D is bounded the observation

$$x_t - T(x_t) = (1 - t^{-1})x_t \to 0 \quad \text{as} \quad t \to 1^-$$

completes the proof.

The following rather well-known fact (cf. [2], [3]) is needed for our second proposition.

LEMMA 1. *Let X be a uniformly convex Banach space and B a bounded subset of X. Then for each $\varepsilon > 0$ there exists a (largest) number $\xi = \xi(\varepsilon, B) \in (0, \varepsilon]$ such that if $u, v \in B$, $x \in X$, $\lambda \in [0, 1]$, and $m = \lambda u + (1 - \lambda)v$ satisfy*

$$\|x - u\| \le \|m - u\| + \xi \quad \text{and} \quad \|x - v\| \le \|m - v\| + \xi,$$

then $\|x - m\| \le \varepsilon$.

PROPOSITION 2. *If in addition to the assumptions of Proposition 1 the space X is uniformly convex, then the segments $S[x_t, x_s]$ lie in D for all $s, t \in (0, 1)$ sufficiently near 1.*

Proof (cf. [6]). Select $\rho > 0$, $\rho < \inf\{\|x - T(x)\| : x \in \partial D\}$. Since $\|x_t - T(x_t)\| \to 0$ as $t \to 1^-$ there exists $\bar{t} \in (0, 1)$ such that $t \in [\bar{t}, 1)$ implies $\|x_t - T(x_t)\| < \xi(\rho)$. Now fix $t \in [\bar{t}, 1)$ and let

$$H = \{s \in [t, 1) : S[x_t, x_s] \subset D\}.$$

If $s \in H$ and $m = (1 - \lambda)x_t + \lambda x_s$, $\lambda \in (0, 1)$, then since T is nonexpansive on $S[x_t, x_s]$,

$$\begin{aligned} \|T(m) - x_t\| &\le \|T(m) - T(x_t)\| + \|T(x_t) - x_t\| \\ &\le \|m - x_t\| + \|x_t - T(x_t)\| \\ &\le \|m - x_t\| + \xi(\rho). \end{aligned}$$

Similarly,

$$\|T(m) - x_s\| \le \|m - x_s\| + \xi(\rho).$$

By the preceding lemma, $\|m - T(m)\| \le \rho$, and in particular $m \in D$. Since D is open and $\|x_s - T(x_s)\| \le \|x_t - T(x_t)\|$ for $s > t$ (Proposition 1) it easily follows that H is open in $[t, 1)$. To see that H is also closed in $[t, 1)$, let $\{s_i\} \subset H$ and suppose $s_i \to s_0 \in [t, 1)$. Since $S[x_t, x_{s_i}] \subset D$ for each i and since each point $m_0 \in S[x_t, x_{s_0}]$ is the limit of a sequence $\{m_i\}$ with $m_i \in S[x_t, x_{s_i}]$ (hence $\|m_i - T(m_i)\| \le \rho$), it follows that $\|m_0 - T(m_0)\| \le \rho$; thus $m_0 \in D$ and $s_0 \in H$. Therefore $H = [t, 1)$ proving $S[x_t, x_s] \subset D$ for all $s, t \in [\bar{t}, 1)$.

Proof of Theorem 1. For simplicity (and without loss of generality) we take $z = 0$. Existence of a path $t \mapsto x_t$, $t \in [0, 1)$, with the desired properties is assured by Proposition 1, so we need only show that $\{x_t\}$ converges to a fixed point of T as $t \to 1^-$. By Proposition 2 there exists $\bar{t} \in (0, 1)$ such that $S[x_t, x_s] \subset D$ for all $s, t \in (\bar{t}, 1)$. Fix such s, t with $s < t$. Since $\|T(x_t) - T(x_s)\| \leq \|x_t - x_s\|$, it is possible to follow precisely the proof of Theorem 1 of Halpern [4] to conclude that

$$\|x_t\|^2 \geq \|x_s\|^2 + \|x_t - x_s\|^2.$$

Thus $\{\|x_t\|\}_{\bar{t} \leq t < 1}$ is monotonic, hence convergent, and it follows that $\|x_t - x_s\|^2 \leq \|x_t\|^2 - \|x_s\|^2 \to 0$ as $s, t \to 1^-$. Therefore $x_t \to x \in \bar{D}$ as $t \to 1^-$ and continuity implies $x = T(x)$ (hence $x \in D$).

3. **Proof of Theorem 2.** Here we merely observe that Ishikawa's proof of Theorem 1 of [5] is really sufficient. This will be evident from the following:

LEMMA 2. *Let X be a Banach space, D an open subset of X, and $T : \bar{D} \to X$ a continuous mapping which is locally nonexpansive on D. Suppose for $z \in D$,*

$$\|z - T(z)\| < \|x - T(x)\| \quad \textit{for all} \quad x \in \partial D.$$

Then (1) $S[z, T(z)] \subset D$, *and*
(2) *if* $m \in S[z, T(z)]$, *then* $\|m - T(m)\| \leq \|z - T(z)\|$.

Proof. Let $K = \{m \in S[z, T(z)] : S[z, m] \subset D\}$. Suppose $\{m_i\} \subset K$ satisfies $m_i \to \bar{m}$ as $i \to \infty$. Then $\|T(z) - T(m_i)\| \leq \|z - m_i\|$ for each i, and by continuity, $\|T(z) - T(\bar{m})\| \leq \|z - \bar{m}\|$. Therefore

$$\begin{aligned} \|\bar{m} - T(\bar{m})\| &\leq \|\bar{m} - T(z)\| + \|T(z) - T(\bar{m})\| \\ &\leq \|\bar{m} - T(z)\| + \|z - \bar{m}\| \\ &= \|z - T(z)\|. \end{aligned}$$

Thus $\|\bar{m} - T(\bar{m})\| < \|x - T(x)\|$ for all $x \in \partial D$, proving $\bar{m} \in D$, hence $\bar{m} \in K$. This proves K is closed in $S[z, T(z)]$. Since K is obviously open in $S[z, T(z)]$ (because D is open), this proves (1). The above inequalities (with m replacing $\bar{m}$) prove (2).

We proceed now with Ishikawa's Lemma 2.

LEMMA 3. *Under the assumptions of Theorem 2 (and with boundedness replacing precompactness of $T(\bar{D})$) the sequence $\{x_n\}$ lies in D and $x_n - T(x_n) \to 0$ as $n \to \infty$.*

Proof. The fact that $\{x_n\}$ lies in D (and hence can be defined inductively by (2)) follows from Lemma 2(1). The proof of Lemma 2 of [5] now carries over. Nonexpansiveness of T is invoked there only in the opening step and in (6). In each of these instances the desired inequality follows from our Lemma 2(2).

Proof of Theorem 2. Since the sequence $\{x_n\}$ lies in the compact set $\{x_1\} \cup \overline{\text{conv}}(T(\bar{D}))$, some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges to a point $u \in \bar{D}$.

By Lemma 3, $x_{n_i} - T(x_{n_i}) \to 0$ as $i \to \infty$ and by continuity of T on $\bar{D}$, $T(x_{n_i}) \to T(u)$. It follows that $T(u) = u$. Also $\|x - T(x)\| > 0$ for $x \in \partial D$, so $u \in D$. Therefore for i_0 sufficiently large all the points x_{n_i}, $i \geq i_0$, lie in a ball B centered at u with $B \subset D$. Using the fact that T is nonexpansive on B, if $x_n \in B$ for some $n \geq i_0$ we have

$$\begin{aligned}\|x_{n+1} - u\| &= \|(1 - t_n)x_n + t_n T(x_n) - u\| \\ &= \|(1 - t_n)(x_n - u) + t_n (T(x_n) - T(u))\| \\ &\leq \|x_n - u\|.\end{aligned}$$

This together with $x_{n_i} \to u$ implies $x_n \to u$ as $n \to \infty$.

Added in proof: S. Reich has extended the *global* version of Theorem 1 to Banach spaces having a Gateaux differentiable norm and possessing a weakly sequentially continuous duality map (see J. Math. Anal. Appl. 44 (1973), 57–70). In a forthcoming paper (Strong convergence theorems for resolvents of accretive operators in Banach spaces, J. Math. Anal. Appl. (to appear)), he shows also that this result holds in all uniformly smooth Banach spaces. In connection with theorem 2, Reich has shown [J. Math. Anal. Appl. 67 (1979), 274–276] that if C is a closed convex subset of a uniformly convex Banach space X with a Fréchet differentiable norm, if $T : C \to C$ is nonexpansive and has a fixed point, and if $\{c_n\} \subset [0, 1]$ satisfies $\sum_{n=1}^{\infty} c_n(1 - c_n) = \infty$, then the sequence defined by: $x_1 \in C$, $x_{n+1} = c_n T(x_n) + (1 + c_n)x_n$, converges weakly to a fixed point of T.

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