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P-AMPLE BUNDLES AND THEIR CHERN CLASSES

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Introduction

In [9], Hartshorne extended the concept of ampleness from line bundles to vector bundles. At that time, he conjectured that the appropriate Chern classes of an ample vector bundle were positive, and it was hoped that there would be some criterion for ampleness of vector bundles similar to Nakai's criterion for line bundles. In the same paper, Hartshorne also introduced the notion of p-ample when the ground field had characteristic p, proved that a p-ample bundle was ample and asked if the converse were true.

In the first chapter of this paper, we will show that a p-ample bundle has positive Chern classes when the characteristic of the ground field is $p \neq 0$, and that a quotient bundle of a direct sum of ample line bundles also has positive Chern classes in any characteristic. We also give a series of polynomials in the Chern classes of a bundle E which are positive if E is ample.

The second chapter will be devoted to some criteria for a vector bundle to be ample. The final chapter gives two examples of ample vector bundles on $\boldsymbol{P}^2$ when the characteristic of k is $p \neq 0$. The first example will be ample, but not p-ample. The second bundle E will be p-ample, but $H^1(\boldsymbol{P}^2, \Gamma^{p^n}(E) \otimes F)$ will be non-zero if F is a bundle and n is large.

We fix our notation. k will denote an algebraically closed field. A variety X will be a reduced, irreducible scheme of finite type over k. If X is non-singular, $A(X)$ will denote the Chow ring modulo numerical equivalence. If X is n-dimensional and complete, we have $A^n(X)$ canonically identified with $\boldsymbol{Z}$. We will often call a locally free sheaf over X a bundle. The bundle E is defined to be ample if for any coherent sheaf F on $X, F \otimes S^n(E)$ is generated by global sections for n large. E is ample if

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and only if the tautological bundle $\mathcal{O}(1)$ on $\boldsymbol{P}(E)$ is ample. For line bundles, this definition is equivalent to the usual definition. If X is projective, E is ample if and only if for any coherent sheaf F, $H^i(X, S^n(E)\otimes F) = 0$ for n large and $i > 0$. Again with X projective, we have that the extension of two ample bundles is ample and that any quotient bundle of an ample bundle is ample.

If the characteristic of k is $p \neq 0$, then we can find a new notion of p-ample. Let f be the Frobenius endomorphism from X to X. Then E^p is defined to be $f^*(E)$. E^{p^2} is $(E^p)^p$ and so forth. E is defined to be p-ample if for any coherent sheaf $F, F\otimes E^{p^n}$ is generated by global sections for n large. E is cohomologically p-ample if for any coherent sheaf $F, H^i(X, F\otimes E^{p^n}) = 0$ for $i > 0$ and n large.

Finally, we have a notion introduced by Griffiths under the name of ample [5, § 4.4].

Definition. A bundle E is *strongly ample* if E is generated by its global sections and if for every closed point x with sheaf of ideals $m, E\otimes m$ is generated by global sections.

We will show that a strongly ample sheaf is ample in Chapter III. The second condition in this definition was phrased by Griffiths in the following way: the natural map from $E\otimes m$ to $\Omega^1_X\otimes E/m$ is surjective. However, since $\Omega^1_X\otimes E/m$ is canonically identified with $E\otimes m/m^2$, Nakayama's lemma show the two conditions are equivalent.

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CHAPTER I. Positivity of Bundles and Their Chern Classes

In this chapter, we will be studying the problem of the positivity of the Chern classes of a bundle E under various assumptions on the positivity of E. Let X be a nonsingular variety of dimension n. A cycle Z of dimension r on X is said to be positive if $Z.Y > 0$ for every effective cycle of codimension r.

We will prove that the appropriate Chern classes of a strongly ample bundle are positive by showing that these Chern classes are represented by effective cycles. If E is p-ample, we will see that the Chern classes of E are positive by applying the previous result to the strongly ample bundle E^{p^n}, where n is some large integer. Finally, we establish the positivity of the Chern classes of a quotient of a direct sum of ample line bundles by a specialization argument.

Many of the results of this chapter were first established by Griffiths in characteristic zero by analytic arguments. Thus he proved Lemma 1.1 and Corollary 1.1 using homology theory and Schubert cycles (6, Prop. 4.13). Also Bloch and Gieseker [4] have established that $c_n(E)>0$ when E is an ample bundle of rank greater than n-1 on a non-singular variety of dimension n, providing the strong Lefschetz theorem holds for all varieties over the ground field.

Now let E be a bundle on a variety X and V a finite dimensional subspace of $H^0(X,E)$. V itself is a variety with the Zariski topology. If Y is a subscheme of X, we will say the generic section in V vanishes on Y at a set of dimension l if there is an open subset U of V so that if s is a section in U, then the subset of Y where s vanishes is of dimension l.

LEMMA 1.1. *Let E be a bundle of rank r on an n-dimensional space Y and let V be a finite dimensional subspace of $H^0(Y,E)$ so that*

i) *for every closed point $y\in Y$, the stalk of E at y is generated by the sections in V;*

ii) *there is some simple closed point $x\in Y$ with sheaf of ideals m so that the stalk of $E\otimes m$ is generated by the sections in $V\cap H^0(Y,E\otimes m)$.*

Then the generic section in V vanishes on a subset of codimension r if $r\leq n$ and does not vanish otherwise.

Proof. We have an exact sequence

$$0\longrightarrow J\longrightarrow \mathcal{O}_Y^N\longrightarrow E\longrightarrow 0$$

where N is the dimension of V and J is locally free. Hence we get a diagram

$$\begin{tikzcd} \boldsymbol{P}(J^{\vee}\otimes\mathcal{O}_Y) \arrow[r, "j"] \arrow[dr, "l"'] & \boldsymbol{P}(\mathcal{O}_Y^N)=Y\times\boldsymbol{P}^{N-1} \arrow[d] \\ & \boldsymbol{P}^{N-1} \end{tikzcd}$$

Now let $[a_1,\cdots,a_N]$ denote the point in $\boldsymbol{P}^{N-1}$ with homogeneous coordinates $(a_1,\cdots,a_N)$ and let e_i denote the image of the i^{th} canonical section of $\mathcal{O}_Y^N$

in $H^0(Y, E)$. Then $(x, [a_1, \cdots, a_N])$ is in the image of j if and only if

$$\sum a_i e_i(x) = 0$$

Hence for $p \in \boldsymbol{P}^{N-1}$, $\not{\ell}^{-1}(p)$ is identified with the subset of Y on which the section of E corresponding to p vanishes. We compute

$$\dim (P(J^\vee)) = \dim Y + \operatorname{rank} J - 1 = (N-1) + (r-1).$$

If $r > n$, then $\not{\ell}$ cannot be dominating, so the generic section of E does not vanish anywhere on Y. So we may assume $r \leq n$. We can find $\bar{f}_1, \cdots, \bar{f}_r$ in m/m^2 so that $\bar{f}_1, \cdots, \bar{f}_r$ are independent. Now locally about y, we may assume that E is free and has a basis $h_1, \cdots, h_r$. Now in $E \otimes m/m^2$, we can consider $\bar{s} = \sum \bar{h}_i \otimes \bar{f}_i$ where $\bar{h}_i$ is the image of h_i. Let s be a global section in V which maps onto $\bar{s}$. Then locally about y, s can be written as $\sum h_i \otimes f_i$ where the image of f_i is $\bar{f}_i$. Now if Z_i denotes the zeros of f_i, we have that locally about y, the set Z of zeros of s is just $Z_1 \cap \cdots \cap Z_r$. But

$$\dim Z = \dim \mathcal{O}_{Y,y}/(f_1, \cdots, f_r) = n - r$$

Hence Z has a component of dimension $n-r$. Now let p be the point in $\boldsymbol{P}^{N-1}$ corresponding to s. Then $\not{\ell}^{-1}(p)$ has a component of dimension $n-r$. Hence $\not{\ell}$ must be dominating since if it is mapped onto a set of dimension less than $N-1$, the dimension of every component of every fiber of $\not{\ell}$ would have to be greater than $n-r$. Since $\not{\ell}$ is dominating, almost every fiber has pure dimension $n-r$. So the zeros of the generic section s on Y have codimension r in X. q.e.d.

Let us introduce some notation. If $I = (i_1, \cdots, i_r)$ is an r-tuple of non-negative integers and E is a bundle, we define

$$c^I(E) = c_1^{i_1}(E) \cdot c_2^{i_2}(E) \cdots c_r^{i_r}(E).$$

Let $|I| = i_1 + 2i_2 + \cdots + ri_r$.

THEOREM 1.2. *Let E be a strongly ample bundle of rank r on a non-singular quasi-projective variety of dimension n. Let Y be an effective l-cycle. If $j \leq r$, $c_j(E)$ can be represented by an effective cycle Z such that*

$$\begin{aligned} \dim (\operatorname{Supp}(Z \cap Y)) &= l - j \qquad \text{if} \quad l - j \geq 0 \\ \operatorname{Supp}(Z \cap Y) &= \phi \qquad\quad \text{if} \quad l - j < 0. \end{aligned}$$

Hence $c^I.Y$ can be represented by an effective cycle if $|I| \leq r$ and $|I| \leq l$.

COROLLARY 1.1. *If X is a projective variety of dimension n and E is a strongly ample bundle of rank r on X, then $c^I(E)$ is positive if $|I| \leq r$ and $|I| \leq n$.*

Proof of Corollary. If Y is an effective cycle of dimension $n - |I|$, then $Y . c^I(E)$ can be represented by an effective cycle of points.

Proof of Theorem 1.2. If the rank of E is greater than n, we can find a nowhere zero section of E, so

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E \longrightarrow E' \longrightarrow 0$$

is exact, where E' is again strongly ample. Since $c^I(E) = c^I(E')$, we can reduce to the case in which $r \leq n$. We now work by induction on r. Now let V' be a finite dimensional subspace of $H^0(X, E)$ so that the image V of V' in $H^0(Y, E)$ satisfies conditions i) and ii) of Lemma 1.1 and so that E is generated by the sections in V at every point $x \in X$. Let s be a section in V and Z the cycle of zeros of s. By Lemma 1.1, we can choose s so that

$$\operatorname{codim}_X Z = r$$
$$\dim Z \cap Y = n - r - l$$

Now $c_r(E)$ is represented by Z [7, Corollary to Theorem 2]. Hence

$$c_r . Y = Z . Y$$

is represented by an effective cycle if $n - r - l \geq 0$. Now consider

$$X' = X - \operatorname{Supp} Z.$$

Let E' denote E restricted to X'. Then we have an exact sequence of locally free sheaves over X',

$$0 \longrightarrow \mathcal{O}_{X'} \longrightarrow E' \longrightarrow F \longrightarrow 0.$$

So $c_j(E') = c_j(F)$. By induction, $c_j(F)$ is represented by an effective cycle Z' so that

$$\dim (\operatorname{Supp} Z' \cap (Y \cap X')) = l - j$$

for $j < r$.

Now if Y is a non-singular quasi-projective variety and Z is a subscheme of codimension r, then the map from $A^k(Y)$ to $A^k(Y - Z)$ is an isomorphism if $k < r$. Now let Z'' be the closure of Z' in X. Then $c_j(E)$ is represented by Z'' since they both map to Z' in $A(X')$. Also

$$\dim(\operatorname{Supp} Z'' \cap Y) = l - j.$$

Lemma 1.3. *Let X be quasi-projective. Then there is a strongly ample line bundle on X.*

Proof. Let L be an ample bundle on X. Then $p_1^*L \otimes p_2^*L$ is ample on $X \times X$, where p_1 and p_2 are the projections of $X \times X$ onto X. Hence if I is the ideal sheaf of the diagonal in $X \times X$,

$$p_1^*L^{\otimes n} \otimes p_2^*L^{\otimes n} \otimes I$$

is generated by global sections for n large. For each closed point x in X, we have

$$p_1^*L^{\otimes n} \otimes p_2^*L^{\otimes n} \otimes I_{|p_2^{-1}\{x\}} \simeq L^{\otimes n} \otimes m_X$$

is generated by global sections.

Lemma 1.4. *If E is p-ample on X, then E^{p^n} is strongly ample for n large.*

Proof. Let M be a strongly ample bundle on X. Then for n large and $q = p^n$, $M \otimes E^q$ is generated by global sections. Hence we have a surjection

$$\oplus M \longrightarrow E^q \longrightarrow 0.$$

Since a quotient of a strongly ample bundle is strongly ample, we see E^q is strongly ample.

Lemma 1.5. $c_n(E^p) = p^n c_n(E)$.

Proof. We may assume that E has a filtration by line bundles L_i by the usual trick of passing to the flag manifold of E. Now there is a homogeneous polynomial Φ of degree n so that

$$c_n(E) = \Phi_n(c_1(L_1), \cdots, c_1(L_r))$$

and

$$c_n(E^p) = \Phi_n(p(c_1(L_1)), \cdots, p(c_1(L_r))$$

since $L_1^p, \cdots, L_r^p$ is a filtration of E^p. Hence $c_n(E^p) = p^n c_n(E)$.

Theorem 1.3. *If E is a p-ample bundle of rank r on a non-singular projective variety of dimension n, then $c^I(E)$ is positive if $|I|$ is less than $n+1$ and $r+1$.*

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Proof. E^q is strongly ample for some q of the form p^n. Let y be an effective cycle of dimension $|I|$. Then

$$y.c^I(E^q) > 0.$$

Since

$$y.c^I(E^q) = (q)^{|I|} y.c^I(E)$$

we deduce

$$y.c^I(E) > 0.$$

Next, we can extend our results to the quotient F of a direct sum of ample line bundles. In characteristic p, the result follows immediately since F is then p-ample. We deduce our general result by specialization. We need a knowledge of the behavior of Chern classes under specialization. If X is a variety over a field k and E a coherent sheaf on X, then $\bar{X}$ and $\bar{E}$ will denote the corresponding objects over $\bar{k}$.

LEMMA 1.6. *Let $Y \xrightarrow{f} Z$ be a smooth projective morphism of noetherian integral schemes and X a subscheme of Y which is flat over Z and so that all the fibers X_z of $g: X \longrightarrow Z$ are geometrically reduced and k-dimensional. Let E be a locally free sheaf of rank r on Y. Then for each z in Z, consider $\bar{E}_z$ on the fiber over $\bar{X}_z$ over $\overline{k(z)}$. Then given I so that $|I| = n - k$,*

$$\bar{X}_z.c^I(\bar{E}_z)$$

is an integer independent of z.

Proof. We actually prove that if $L_1, \cdots, L_l$ are line bundles, then

$$\bar{X}_z.c^I(\bar{E}_z).\bar{L}_1 \cdots \bar{L}_l$$

is constant for $|I| = n - k - l$. We work by induction on r. If $r = 1$, then

$$\bar{X}_z.c_1(\bar{E}_z).\bar{L}_1 \cdots \bar{L}_l$$

can be computed as the coefficient of $n_1 n_2 \cdots n_{l+1}$ in

$$p(n, \cdots, n_{l+1}) = \chi(\bar{L}_1^{\otimes n_1} \otimes \cdots \otimes \bar{L}_l^{\otimes n_l} \otimes \bar{E}^{\otimes n_{l+1}} \otimes \mathcal{O}_{\bar{X}_z})$$

[Kleiman 12]. However, we may omit the bars, since the dimension of cohomology groups is invariant under field extension. But this Euler

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characteristic is constant. Now if we have proved our claim for $r-1$ we can consider the projection $p: \boldsymbol{P}(E) \longrightarrow Y$. Then if Q denotes the tautological bundle on $\boldsymbol{P}(E)$, we have

$$\bar{X}_z . c^I(\bar{E}) . \bar{L}_1 \cdots \bar{L}_l = \bar{Q}^{r-1} . p^*\bar{L}_1 \cdots c^I(p^*\bar{E}_z) . p^*(\bar{X}_z).$$

However, we have an exact sequence

$$0 \longrightarrow E' \longrightarrow p^*E \longrightarrow Q \longrightarrow 0.$$

Hence $c^I(p^*\bar{E}_z)$ is a sum of $c_1(\bar{Q})^j . c^J(\bar{E}'_z)$ for various $(r-1)$-tuples J. Since rank $E' = r-1$, we are done by induction.

Theorem 1.4. *Let X be an n-dimensional projective variety and E a bundle of rank r which is a quotient of a direct sum of ample line bundles $\oplus L_i$. Then $c^I(E)$ is positive for $|I|$ less than $r+1$ and $n+1$.*

Proof. Let Z be a subvariety of dimension l in X. Then there is a subring A of k with the following properties: A is of finite type over $\boldsymbol{Z}$, there is a scheme X' and a subscheme Z' of X' and a smooth projective morphism $f: X' \longrightarrow \operatorname{Spec} A$ so that Z' is flat over Spec A and all the fibers Z'_a are varieties for a in Spec A. Furthermore, there are ample invertible sheaves L'_i and a locally free sheaf E' on X' so that E' is a quotient of a direct sum of the L'_i's. Finally, if we tensor the whole situation with k, X' becomes X, E' becomes E, etc. [cf. 8, IV. 8.5, 8.10]. Since intersection numbers are invariant under field extension,

$$c^I(E) . Z = c^I(\bar{E}'_g) . \bar{Z}_g$$

where g is the generic point of Spec A. But by Lemma 1.5,

$$c^I(\bar{E}'_g) . \bar{Z}'_g = c^I(\bar{E}_y) . \bar{Z}'_y$$

where y' is any point of Spec A. In particular, we may choose y so that the residue field of y has non-zero characteristic. Then $\bar{E}'_y$ is p-ample, so $c^I(\bar{E}'_y)$ is positive.

Finally, we introduce a sequence of polynomials in the Chern classes of E which are positive if E is ample. These were studied by Griffiths in [5], who proved Theorem 1.5 in characteristic zero by analytic arguments.

Lemma 1.7. *Let $Y_1, \cdots, Y_n$ be indeterminates over $\boldsymbol{Z}$. Let $L_0 = 1$. Then for each k there is a unique symmetric polynomial L_k in the Y_i satisfying*

$$* \qquad L_k - s_1 L_{k-1} + s_2 L_{k-2} - \cdots = 0$$

where s_i *is the i-th symmetric polynomial. We have*

$$** \qquad L_k(Y_1, \cdots, Y_n) = \sum_{\Sigma i_j = k} Y_1^{i_1} \cdots Y_n^{i_n}.$$

Proof. Let L_k be defined by $**$. We show the L_k satisfy $*$ by induction on n. We abbreviate $(Y_1, \cdots, Y_{n-1})$ to Y.

$$s_{n-k}(Y, Y_n) = s_{n-k}(Y) + s_{n-k-1}(Y).Y_n$$

$$L_k(Y, Y_n) = L_k(Y) + L_{k-1}(Y)Y_n + L_{k-2}(Y)Y_n^2 \cdots.$$

Hence we get

$$L_k(Y, Y_n)s_{n-k}(Y, Y_n) = L_k(Y)s_{n-k}(Y) + A_k + A_{k+1}$$

where

$$A_k = \sum_{i=1}^{k} s_{n-k}(Y)L_{k-i}(Y)Y_n^{k-i}.$$

So

$$\sum_{k=0}^{n}(-1)^k L_k(Y, Y_k)s_{n-k}(Y_k) = \sum_{k=0}^{n} L_k(Y)s_{n-k}(Y) = 0.$$

Since L_k is symmetric, it can be written as a polynomial in the symmetric functions:

$$L_k(Y_1, \cdots, Y_r) = \Phi_k(s_1, \cdots, s_r)$$

For a vector bundle of rank r, we set

$$\Phi_k(E) = \Phi_k(c_1(E), \cdots, c_r(E)).$$

For instance,

$$\Phi_1(E) = c_1$$

$$\Phi_2(E) = c_1^2 - c_2$$

$$\Phi_3(E) = c_1^3 - 2c_1c_2 + c_3.$$

We have

$$\Phi_k(E) - c_1(E)\Phi_{k-1}(E) + \cdots = 0.$$

LEMMA 1.8. *Let Z be a non-singular projective variety of dimension n and E a bundle of rank r. Let ξ denote the class of $\mathcal{O}(1)$ in $A^1(\boldsymbol{P}(E))$. Then for $k \leq r$ and any cycle y of dimension k on Z,*

$$(y.\Phi_k(E))_Z = (\pi^* y.\xi^{k+r-1})_{\boldsymbol{P}(E)}.$$

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(Note: $A^n(X)=A^{n+r-1}(\boldsymbol{P}(E))=\boldsymbol{Z}$; $\pi:\boldsymbol{P}(E)\longrightarrow Z$.)

Proof. We work by induction on k and denote π^*y by y'. If $k=0$, then we can take y to be a point and then it is well known that

$$1=y'.\xi^{r-1}.$$

Now suppose we have proved the theorem for all $k'<k$. Now we have

$$\xi^r-c_1\xi^{r-1}+c_2\xi^{r-2}-\cdots=0.$$

Hence multiplying by $y'.\xi^{k-1}$, we get

$$y'.\xi^{k+r-1}-y'.c_1.\xi^{k+r-2}+\cdots=0.$$

We also have

$$y.\Phi_k-y.c_1.\Phi_{k-1}+\cdots=0.$$

But by our induction hypothesis,

$$y.c_i.\Phi_{k-i}=y'.c_i.\xi^{k+r-i-1}$$

for $i>0$. Hence we see

$$y'.\xi^{k+r-1}=y.\Phi_k.$$

THEOREM 1.5. *Let Z be a non-singular projective n-dimensional variety and E an ample bundle on Z. Then $\Phi_k(E)$ is positive for $k\leq n$.*

Proof. If y is a k-cycle,

$$\Phi_k(E).y=\xi^{k+r-1}.y'>0$$

since ξ is the class of an ample line bundle.

We note that Griffiths has a cone Π of "positive" polynomials so that if $P\in\Pi$, then $P(c_1,\cdots,c_r)>0$ if $c_1,\cdots,c_r$ are the Chern classes of a strongly ample bundle. Π contains monomials c^I and the Φ_k [6, Theorem *D*].

CHAPTER II. Ample Bundles on Curves

A major deficiency in the theory of ample vector bundles is the lack of an adequate test for the ampleness of a bundle E on a variety X. Barton [2] has given a test similar to Kleiman's criterion for ampleness of a line bundle. If X is a non-singular curve over $\boldsymbol{C}$, a necessary and sufficient condition for a bundle E on X to be ample is that the degree of every quotient

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bundle be positive [11]. If E has rank two, and X is a non-singular curve of genus g over a field of characteristic p, then E is ample if every quotient bundle of E is ample and if $\deg E > \frac{2}{p}(g-1)$ [9].

First we will give a test for a vector bundle E on any variety to be ample if E is generated by global sections. We need a lemma.

LEMMA 2.1. *Suppose X is a curve, E is a bundle on X, and s a section of E which does not vanish at any singular point of x. Then E has a sub-bundle L so that s is a section of L. If s vanishes at any point, L is ample.*

Proof. Let $x_1, \cdots, x_n$ be the points at which s vanishes. Now s determines a section of $\boldsymbol{P}(E^\vee)$ over $Y - \{x_1, \cdots, x_n\}$ and hence a section of $\boldsymbol{P}(E^\vee)$ over all Y since $x_1, \cdots, x_n$ are non-singular. Such a section is equivalent to a sub-line bundle L of E and s is actually a section of L. If s actually vanishes, then $\deg L > 0$, so L is ample.

PROPOSITION 2.1. *Suppose Y is proper over k and E is a bundle over Y generated by its global sections. Then E is ample if and only if every quotient line bundle of $E_{|C}$ is ample for every curve C in Y.*

Proof. First suppose E is a line bundle. Then E gives a map f to projective space so that $E = f^*(\mathcal{O}(1))$. But f is a quasi-finite map. For if f collapsed a curve C to a point, we would have $E_{|C} \simeq \mathcal{O}_C$, and so $E_{|C}$ would not be ample. Since f is proper, f is finite and so $f^*(\mathcal{O}(1))$ is ample.

Now suppose Y is a curve. We work by induction on the rank of E so we may suppose every quotient bundle F of E is ample if $F \neq E$. Let S be the set of singular points of Y. We will establish that E has a section which does not vanish at any point of S, but which does vanish. By Lemma 2.1, E will have an ample sub-line bundle L and so will be ample as the extension of E and E/L.

Let n be the dimension of $\Gamma(E)$. Then there is a surjective map from $\mathcal{O}_Y^n$ to E, and we may suppose that E is a quotient of $\mathcal{O}_Y^n$. Now the Grassmannian G of all $n-r$ planes in k^n represents the functor which assigns to any variety Z the set of all locally free quotients of rank r of $\mathcal{O}_Z^n$. So we get a map φ from Y to G so that E is the pull back of the universal bundle U on G. Now $\varphi(Y)$ is not a point, since then E would be trivial as the pull back of U restricted to $\varphi(Y)$. So we can pick a closed point

$y \in Y$ so that $f(x) \notin f(S)$. Now if z and w are two closed points of G, then the universal mapping property of G shows that $z = w$ if and only if

$$\Gamma(U \otimes m_z) = \Gamma(U \otimes m_w).$$

So $\Gamma(U \otimes m_{f(y)}) \cap \Gamma(U \otimes m_{f(s)})$ is a proper subspace of $\Gamma(U \otimes m_{f(y)})$ for all $s \in S$. Since k is infinite and S is finite, we can choose a section of U which vanishes at $f(y)$, but not at any point $f(s)$. Pulling back this section to $E = \varphi^*(U)$, we get a section of E with the required property.

Now if E has rank greater than one, on an arbitrary complete variety Y, consider the tautological bundle $\mathcal{O}(1)$ on $\boldsymbol{P}(E)$. We denote the projection from $\boldsymbol{P}(E)$ to Y by p. Then to show E is ample, we need only show $\mathcal{O}(1)$ is ample. Now let C be any curve in $\boldsymbol{P}(E)$. If C is contained in a fiber of p, then $\mathcal{O}(1)$ is ample on C. If C is not contained in a fiber, the map from C to $p(C)$ is finite. Hence p^*E is ample on C. Since $\mathcal{O}(1)$ is a quotient of p^*E, we see $\mathcal{O}(1)$ is ample on C. Since $\mathcal{O}(1)$ is generated by global sections, it is ample.

Before proceeding to our discussion of curves, we prove a strongly ample bundle is ample. See the introduction for the definition of strongly ample. This was proved by Griffiths [6] in characteristic zero.

THEOREM 2.1. *A strongly ample bundle E on a proper variety Y is ample.*

Proof. We need only show that E is ample on each curve C in Y. Let x be a non-singular point of C with sheaf of ideals $I \subseteq \mathcal{O}_C$. Then $E \otimes I$ is generated by global sections, so $E_{|C}$ can be written as a quotient of a direct sum of line bundles $I^{\vee}$. Hence $E_{|C}$ is ample.

The following proposition gives a class of bundles on a curve so that all quotients have positive degree.

PROPOSITION 2.2. *Let X be a non-singular curve and suppose F an ample bundle on X and that we have a non-trivial extension*

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E \longrightarrow F \longrightarrow 0$$

Then every quotient bundle G of E has positive degree.

Proof. We have a map s from $\mathcal{O}_X$ to G which factors through E. s factors through a sub-line bundle L of G. If s is zero, G is a quotient of F and so has positive degree. If $s \neq 0$, then $\deg L \geq 0$ since L has a section. Now if G is a line bundle, $L = G$. In this case $\deg L$ must be greater than

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zero, since if it were zero, L would be isomorphic to $\mathcal{O}_X$ and $\mathcal{O}_X$ would be a direct factor of E. If $\operatorname{rank} G > 1$, then G/L is ample as a quotient of F and so

$$\deg G = \deg G/L + \deg L > 0.$$

The following lemma is sometimes useful as a test for ampleness.

LEMMA 2.2. *Let X be a non-singular curve of genus g and E a bundle of rank r. Suppose $\deg E > rg$ and that every quotient bundle F of E is ample if $F \neq E$. Then E is ample.*

Proof. Let I_x be the ideal sheaf of a point $x \in X$. Then by Riemann-Roch,

$$\dim \Gamma(E \otimes I_x) \geq \deg E - r + r(1-g) > 0$$

Hence E has a section which vanishes at some point, and hence an ample sub-line bundle L. Hence E is ample as the extension of L and E/L.

We will now study bundles E on the curve X which are non-trivial extensions of the form

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E \longrightarrow F \longrightarrow 0$$

where F is ample. We will show E is ample if the characteristic of k is zero and give an example of Serre which shows E need not be ample even if F is a line bundle when char $k = 3$.

LEMMA 2.3 (char $k = 0$). *Let f be a finite, flat map from a variety X to a variety Y, and let E be a vector bundle on Y. Then the natural map from $H^i(Y, E)$ to $H^i(X, f^*(E))$ is injective.*

Proof. We have a natural map from $\mathcal{O}_Y$ to $f_*\mathcal{O}_X$ and the trace map gives an $\mathcal{O}_Y$-linear map from $f_*\mathcal{O}_X$ to $\mathcal{O}_Y$. Since the characteristic of k is zero, it follows that $\mathcal{O}_Y$ is a direct summand of $f_*\mathcal{O}_X$. So E is a direct summand of $f_*(\mathcal{O}_X) \otimes E$. But $H^i(X, f^*(E))$ is canonically isomorphic to $H^i(Y, f_*(f^*(E))$ and $f_*(f^*(E))$ is isomorphic to $E \otimes f_*(\mathcal{O}_X)$. So the map from $H^i(Y, E)$ to $H^i(X, f^*(E))$ is injective.

THEOREM 2.2 (char $k = 0$). *Suppose that X is a complete non-singular curve, that F is an ample bundle on X, and that we have a non-trivial extension*

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E \longrightarrow F \longrightarrow 0$$

Then E is ample.

Proof. We must show $\mathcal{O}(1)$ is ample on $\boldsymbol{P}(E)$. We may regard $\boldsymbol{P}(F)$ as a subvariety of $\boldsymbol{P}(E)$. Let s be the section of E given by the map from $\mathcal{O}_X$ to E. Then s gives a section of $p^*(E)$ and hence a section s' of $\mathcal{O}(1)$ on $\boldsymbol{P}(E)$. The divisor consisting of the zeros of s' is just $\boldsymbol{P}(F)$. Further, $\mathcal{O}_{\boldsymbol{P}(E)}(1)$ restricted to $\boldsymbol{P}(F)$ is just $\mathcal{O}_{\boldsymbol{P}(F)}(1)$. Now to prove that $\mathcal{O}_{\boldsymbol{P}(E)}(1)$ is ample, it suffices to prove that if Y is any sub-variety of $\boldsymbol{P}(E)$, then there is a section of $\mathcal{O}(n)$ for some n which vanishes at some point of Y, but does not vanish identically [Theorem 1, Chapter III, 12]. If Y is contained in $\boldsymbol{P}(F)$, this condition certainly obtains, since $\mathcal{O}_Y(1)$ is ample and if Y meets $\boldsymbol{P}(F)$, and is not contained in $\boldsymbol{P}(F)$, the condition also obtains since s' does not vanish identically, but does vanish on $Y \cap \boldsymbol{P}(F)$. So it suffices to prove that every Y on $\boldsymbol{P}(E)$ meets $\boldsymbol{P}(F)$. Suppose it does not. Clearly we may assume Y is a curve. If Y is contained in a fiber of the map from $\boldsymbol{P}(E)$ to X, it certainly meets F. If Y is not contained in a fiber, we have a finite map f from Y to X and $f^*(E)$ has $\mathcal{O}_Y$ as a quotient, since $\mathcal{O}_Y(1)$ has a nowhere zero section. So by Lemma 2.3, the extension

$$\mathcal{O} \longrightarrow \mathcal{O}_{X'} \longrightarrow f^*E \longrightarrow f^*F \longrightarrow 0$$

would be trivial. But then the map from $H^1(X, F^\vee)$ to $H^1(X, f^*F^\vee)$ would not be injective. So Y must meet $\boldsymbol{P}(F)$ and $\mathcal{O}(1)$ is ample.

The following is a special case of a recent result of Hartshorne's [11], who used the theory of semi-stable bundles and unitary representations.

COROLLARY 2.1 (char $k=0$). *Let E be a bundle on a complete non-singular curve X and suppose every quotient bundle F of E is ample if $F \neq E$. Then E is ample if $\Gamma(E) \neq 0$.*

Proof. The section s of E factors through a subline bundle L of E. If s vanishes at some point of X, L is ample and E is ample as the extension of E/L and L. If s vanishes nowhere, E is ample since it is a non-trivial extension

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E \longrightarrow E/\mathcal{O}_X \longrightarrow 0.$$

The following example of Serre shows that a bundle E may not be ample even though the degree of every quotient is positive and the rank of E is two.

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Suppose the characteristic of k is three and let C be the non-singular curve of genus 3 in $\boldsymbol{P}^2$ given by

$$0 = f(X, Y, Z) = X^4 - Y^3Z - Z^3Y.$$

LEMMA 2.4. *The Frobenius endomorphism p^* of $H^1(C, \mathcal{O}_C)$ is identically zero.*

Proof. $U_1 = \{(X, Y, Z) | Y \neq 0\}$ and $U_2 = \{(X, Y, Z) | Z \neq 0\}$ will denote affine subvarieties of $\boldsymbol{P}^2$, which has homogeneous coordinates (X, Y, Z). Then $C \subseteq U_1 \cap U_2$. Now let $\alpha \in H^1(\mathcal{O}_C)$. Since $C \cap U_1$ and $C \cap U_2$ are an affine covering of C, we can realize α as a function $\bar{h}$ on $C \cap U_1 \cap U_2$. This function extends to a function h on $U_1 \cap U_2$, i.e., in the ring $k\left[\frac{X}{Y}, \frac{Y}{Z}, \frac{Z}{Y}\right]$. Now $\bar{h}^3$ represents $p^*(\alpha)$, so we wish to show $\bar{h}^3$ is a coboundary. Since h is the sum of monomials $X^i/(Y^jZ^{i-j}) = g_{i,j}$, we need only show each monomial is a coboundary, that is, that there are functions $h_{i,j}$ and $\ell_{i,j}$ in $k\left[\frac{X}{Y}, \frac{Z}{Y}\right]$ and $k\left[\frac{X}{Z}, \frac{Y}{Z}\right]$ respectively so that

$$g_{i,j} \equiv h_{i,j} - \ell_{i,j} \pmod f$$

Now if $i \geq 4$, we can write

$$g_{i,j} = \frac{X^i}{Y^jZ^{i-j}} \equiv \frac{X^{i-4}}{Y^{j-3}Z^{i-j-1}} + \frac{X^{i-4}}{Y^{j-1}Z^{i-j-3}} \pmod f$$

So to show $\bar{g}^3_{i,j}$ is a coboundary, we assume $i < 4$. Clearly, we may also assume $0 < j < i$. So the only cases are X^2/YZ, X^3/Y^2Z and X^3/Z^2Y. But

$$\frac{X^6}{Y^3Z^3} \equiv \frac{X^2}{Z^2} + \frac{X^2}{Y^2} \pmod f$$

and

$$\frac{X^9}{Y^6Z^3} \equiv \frac{X}{Z} + \left(\frac{2XZ}{Y^2} + \frac{XZ^3}{Y^4}\right) \pmod f$$

and by the symmetry of Y and Z, X^9/Y^3Z^6 is also a coboundary. So p^* is zero. q.e.d.

Now let P be a point on C. Then the Frobenius map f from $H^1(\mathcal{O}(-P))$ to $H^1(\mathcal{O}(-3P))$ cannot be injective. For we have an exact diagram

$$\begin{array}{ccccccccc} & & & & & & H^1(\mathcal{O}(-P)) & \longrightarrow & H^1(\mathcal{O}) \\ & & & & & & \downarrow f & & \downarrow p^* \\ 0 & \longrightarrow & H^0(\mathcal{O}) & \longrightarrow & H^0(K) & \longrightarrow & H^1(\mathcal{O}(-3P)) & \longrightarrow & H^1(\mathcal{O}) \end{array}$$

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where K is the cokernel of the map $\mathcal{O}(-3P) \longrightarrow \mathcal{O}$. Since the map from $H^1(\mathcal{O})$ to $H^1(\mathcal{O})$ is zero, $H^1(\mathcal{O}(-P))$ must be mapped to the image of $H^0(K)$ in $H^1(\mathcal{O}(-3P))$. However, since $\dim H^0(K)$ is 3, this image has dimension 2; since $H^1(\mathcal{O}(-P))$ has dimension 3, f is p-linear, and k is perfect, we see f cannot be injective.

Now we can construct the example. Let α be a non-zero element of the kernel of f. Thus α determines a non-trivial extension

$$0 \longrightarrow \mathcal{O} \longrightarrow E \longrightarrow \mathcal{O}(P) \longrightarrow 0$$

The degree of every quotient bundle of E is positive, but the extension

$$0 \longrightarrow \mathcal{O} \longrightarrow E^3 \longrightarrow \mathcal{O}(3P) \longrightarrow 0$$

splits, so E^3 is not ample and hence neither is E.

We will now prove that a vector bundle over an elliptic curve X is ample if and only if every direct summand has positive degree. This result has been proved independently by Hartshorne [11]. The idea of our proof is the following: We use induction on the rank of E and Lemma 2.5 to reduce our problem to the case of a non-trivial extension E of an ample bundle by the structure sheaf. We construct another elliptic curve X' and a map of high degree f from X' to X so the extension remains non-trivial. Then Proposition 2.2 and Lemma 2.2 show $f^*(E)$ is ample, and hence E is ample.

LEMMA 2.5. *Let E be an indecomposable bundle over an elliptic curve X and F an indecomposable quotient of E. If $\deg E > 0$, then $\deg F > 0$.*

Proof. For any line bundle L of degree zero, $\Gamma(E^\vee \otimes L) = 0$, since

$$\dim \Gamma(E^\vee \otimes L) = \dim \Gamma(E \otimes L^\vee) - \deg E = 0$$

by the Riemann-Roch Theorem and the fact that if F is indecomposable and $\deg F > 0$, then $\dim \Gamma(F) = \deg F$ [1, Lemma 15]. Hence $\Gamma(F^\vee \otimes L) = 0$. But if $\deg F < 0$, then $\Gamma(F^\vee \otimes L) \neq 0$. If $\deg F = 0$, then $F = F_r \otimes L$ for one of Atiyah's canonical bundles F_r and some L of degree zero. It again follows that $\Gamma(F \otimes L^\vee) \neq 0$ (cf. Theorem 5 [1]), since $\Gamma(F_r) \neq 0$.

LEMMA 2.6. *Let X be an elliptic curve, E an ample bundle on X and n a positive integer. Then there is another elliptic curve X' and a map f from X' to X so that $\deg f > n$ and the map from $H^1(X, E^\vee)$ to $H^1(X', f^*(E^\vee))$ is injective.*

Proof. Let X be elliptic and f any map of degree 2 from another complete non-singular curve X'. Then there is an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow f_*\mathcal{O}_{X'} \longrightarrow L \longrightarrow 0$$

where L is a line bundle on X. *I* claim $\deg L \leq 0$. For there is an exact sequence

$$0 \longrightarrow L^{\vee} \longrightarrow L^{\vee} \otimes f_*\mathcal{O}_{X'} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

and if $\deg L > 0$, we would have an exact sequence

$$0 \longrightarrow H^0(X, \mathcal{O}_X) \longrightarrow H^1(X, L^{\vee}) \longrightarrow H^1(X^1, f^*(L^{\vee}))$$

since $H^i(L^{\vee} \otimes f_*\mathcal{O}_X)$ is naturally isomorphic to $H^i(f^*L^{\vee})$ and $H^0(f^*(L^{\vee})) = 0$ since $f^*(L^{\vee})$ is negative. But then we would have a non-trivial extension

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E \longrightarrow L \longrightarrow 0,$$

but the extension

$$0 \longrightarrow \mathcal{O}_{X'} \longrightarrow f^*E \longrightarrow f^*L \longrightarrow 0$$

would be trivial. This is impossible since E and f^*E are ample by Corollary 7.8 of [9].

Now let E be any ample bundle. Then the map from $H^1(X, E^{\vee})$ to $H^1(X', f^*(E^{\vee}))$ has kernel $H^0(X, E^{\vee} \otimes L)$. Since $(E^{\vee} \otimes L)^{\vee}$ is ample, $E^{\vee} \otimes L$ has no sections, so our map is injective.

If X is an elliptic curve, we can find another elliptic curve X' and a map f of degree two from X' to X. We can consider X as a group by choosing an identity for the group law, and then multiplication by two in X is a map of degree four. Thus if $K(X)$ denotes the function field of X, multiplication by two defines $K(X)$ as a sub-field of degree 4 in $K(X)$. Then there is a sub-field K' of degree 2 between $K(X)$ and a non-singular model of K' gives us our curve X'.

Finally, we can construct the curve X' and the map f of the Lemma of large degree by letting f be the composition of morphisms of degree two.

THEOREM 2.3. *Let X be an elliptic curve and E a bundle on X. Then E is ample if and only if every direct summand of E has positive degree.*

Proof. We use induction on the rank of E. The induction hypothesis and Lemma 2.5 show we may assume every quotient bundle F of E is ample if $F \neq E$. Since $\deg E > 0$, there is a non-zero map f from $\mathcal{O}_X$ to E. This map factors through a sub-line bundle L of E. Now either $\mathcal{O}_X$ is mapped isomorphically to L or L is ample. If L is ample, E is ample since it is an extension of two ample bundles, E/L and L. If L is isomorphic to $\mathcal{O}_X$, we have a non-trivial extension

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E \longrightarrow E' \longrightarrow 0$$

where E' is ample. Now let X' be an elliptic curve and f a map from X' to X so that $\deg f > \operatorname{rank} E$ and the map from $H^1(X, E^\vee)$ to $H^1(X, f^*(E^\vee))$ is injective. Then the extension

$$0 \longrightarrow \mathcal{O}_{X'} \longrightarrow f^*(E) \longrightarrow f^*(E') \longrightarrow 0$$

is non-trivial. By Lemma 2.5 and the induction hypothesis, every quotient bundle F of f^*E is ample if $F \neq f^*E$. Also $\deg f^*E = \deg f^* \deg E > \operatorname{rank} f^*E$. So Lemma 2.2 shows f^*E is ample, and hence E is ample.

CHAPTER III. Ample Bundles on $\boldsymbol{P}^2$

Suppose the characteristic of k is $p \neq 0$. Then a line bundle is ample if and only if it is p-ample. Furthermore, a bundle over a non-singular curve is p-ample if and only if it is ample. The first purpose of this chapter is to construct an ample bundle on $\boldsymbol{P}^2$ which is not p-ample. Using this bundle we will construct a series of ample bundles F_n on $\boldsymbol{P}^2$ so that $F_n(k)$ has no sections if $k \leq n$. Such a sequence exists even if the characteristic of k is zero. We note a result of Barton [3] which says in this case that there is a k_n so that $E(k)$ is generated by global sections if E is ample, $c_1(E) = nH$ and $k \geq k_n$.

Our second main purpose is to study the cohomology of $\Gamma^n(E) \otimes F$ where E is an ample bundle and F is coherent. In characteristic zero, we have $\Gamma^n(E)$ is isomorphic to $S^n(E)$, so the higher cohomology groups vanish for n large. But in characteristic p the higher cohomology groups no longer vanish.

The non-vanishing of these cohomology groups has implications for relative cohomology. To illustrate this, we let Y be $\boldsymbol{P}^2$ and E a vector bundle over $\boldsymbol{P}^2$ with the property that for any locally free F, then

$$H^1(Y, \Gamma^{p^n}(E)\otimes F)) \neq 0$$

for n large. We will exhibit such a p-ample E later. Let X be $\boldsymbol{P}(E^\vee \oplus \mathcal{O}_X)$ and consider the natural embedding of Y in X. The normal bundle of Y in X is E, and we let I denote the sheaf of ideals of Y. Then $H^1(X_{/Y}, \hat{F})$ is infinite dimensional since

$$H^1(X_{/Y}, \hat{F}) = \sum_n H^1(Y, I^n/I^{n+1}\otimes F)$$

and since

$$H^1(Y, I^n/I^{n+1}\otimes F)$$

is dual to

$$H^1(Y, F^\vee \otimes \Gamma^n(E)(-3)).$$

$(\Gamma^n(E) = (I^n/I^{n+1})^\vee)$. Now by [10, 4.4], $H^2(X-Y, F)$ is infinite dimensional, and taking into account the long exact sequence of local cohomology, $H^3_Y(F)$ is also infinite dimensional. Hence the situation in characteristic p is completely different from that in characteristic zero, where $H^2(W-Z, F)$ is finite dimentional if Z is a non-singular projective surface with ample normal bundle in a non-singular projective fourfold [10].

THEOREM 3.1. *In any characteristic, there is on $\boldsymbol{P}^2$ an exact sequence of locally free sheaves*

$$0 \longrightarrow \mathcal{O}(-7)^2 \longrightarrow \mathcal{O}^4 \longrightarrow E \longrightarrow 0$$

with E ample. E is not p-ample when char $k = p$.

The idea of the proof is to construct a surjection from $\mathcal{O}^4$ to $\mathcal{O}(7)^2$ of a sufficiently general nature. Then we will show E restricted to every curve is ample. The integer seven above may be replaced by any larger integer.

Let X denote $\boldsymbol{P}_k^2$ and H denote a line in X.

LEMMA 3.1. *The generic linear system of dimension 3 in $|7H|$ has no base points and contains no divisor with multiple components.*

Proof. Using the formula,

$$\dim|mH| = \frac{(m+1)(m+2)}{2} - 1$$

we readily verify that

$$\dim |7H| \geq 4 + \dim |(7-i)H| + \dim |iH|,$$

for $0 < i < 7$. On the other hand,

$$\dim |(7-i)H| + \dim |iH|$$

is the dimension of the subset G_i of $|7H|$ consisting of all divisors D which can be written as $D_1 + D_2$ where D_1 is in $|(7-i)H|$ and D_2 is in $|iH|$. Since codim $G_i \leq 4$, it follows that the generic linear system of dimension 3 does not meet G_i and so all divisors in this system are prime.

It is also clear the generic linear system of dimension 3 has no base points since four divisors usually do not have a point in common.

Now given a linear map D of $\boldsymbol{P}^3$ into $|7H|$, we can get a map from $\mathcal{O}^4$ to $\mathcal{O}(7)$. Indeed any linear map between projective spaces comes from a linear map on the corresponding vector spaces, in this case k^4 and $H^0(X, \mathcal{O}(7))$. Hence given two maps D and D'', we get a map

$$\mathcal{O}^4 \longrightarrow \mathcal{O}(7)^2.$$

LEMMA 3.2. *It is possible to choose D and D'' so that the above map is surjective. Furthermore, $D_t \cap D''_t$ is a finite collection of points for all t in $\boldsymbol{P}^3$.*

Proof. Let L and L' be disjoint linear systems of dimension 3 in $|7H|$ which have no base points and which contain no divisors with multiple components, and let D and D' be linear maps of $\boldsymbol{P}^3$ onto L and L' respectively. We will alter D' by an automorphism of $\boldsymbol{P}^3$ to obtain D''. For x in $\boldsymbol{P}^2$ and σ in $PGL(3)$, consider the planes

$$F(x) = \{t \mid x \in D_t\} \subseteq \boldsymbol{P}^3$$

$$F'(x) = \{t \mid x \in D'_t\} \subseteq \boldsymbol{P}^3$$

and the subset B of $PGL(3) \times \boldsymbol{P}^2$

$$B = \{(\sigma, x) \mid \sigma F(x) = F'(x)\}.$$

Let p_1 and p_2 denote the projections of B into $PGL(3)$ and $\boldsymbol{P}^2$. Now if $B_x = p_2^{-1}\{x\}$, then

$$\operatorname{codim}_{PGL(3)}(B_x) = 3$$

since B_x is just the set of σ which map $F(x)$ onto $F'(x)$. So $\dim B < \dim PGL(3)$. Hence there is a σ in $PGL(3)$ so that σ is not in $p_2(B)$. Now define

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$$D'' : \boldsymbol{P}^3 \longrightarrow |7H|$$

by

$$D''_t = D'_{\sigma(t)}$$

But then for each x, there are s and t in $\boldsymbol{P}^3$ so that

$$x \in D''_s;\ x \notin D_s$$
$$x \notin D''_t;\ x \in D_t.$$

Hence the map $f : \mathcal{O}^4 \longrightarrow \mathcal{O}(7)^2$ is surjective. For let w be a section of $\mathcal{O}^4$ corresponding to s and let p_1 and p_2 be the projections of $\mathcal{O}(7)^2$ onto $\mathcal{O}(7)$. Then $p_1 f$ corresponds to D and $p_2 f$ to D''. Then $p_2(f(w))$ vanishes at x, but $p_1(f(w))$ does not. Since L and L' are disjoint and contain only irreducible divisors, $D''_t \cap D_t$ is a finite set of points for all t.

We are now ready to construct our ample bundle. We take a surjective map of the type constructed above and let $E^\vee$ be the kernel.

$$0 \longrightarrow E^\vee \longrightarrow \mathcal{O}^4 \longrightarrow \mathcal{O}(7)^2 \longrightarrow 0$$
$$0 \longrightarrow \mathcal{O}(-7)^2 \longrightarrow \mathcal{O}^4 \longrightarrow E \longrightarrow 0.$$

Now E is not p-ample since $H^0(E^{p^n} \otimes \mathcal{O}(-1)) = 0$ for all n. Indeed, we have the exact sequence

$$0 = H^0(\mathcal{O}(-1)^4) \longrightarrow H^0(E^{p^n}(-1)) \longrightarrow H^1(\mathcal{O}(-7p^n - 1)^2) = 0.$$

To show E is ample, we use the criterion of Proposition 2.1. Let C be an irreducible curve on $\boldsymbol{P}^2$. We must show every quotient line bundle of $E_{|C}$ is ample. But if L were not ample, we would have L isomorphic to $\mathcal{O}_C$ since L is generated by global sections. This in turn would give a nowhere zero section of $E^\vee_{|C}$. Hence we would get a section of $\mathcal{O}^4_C$. This section would extend to a section $\mathcal{O}^4_X$. Finally, the image of this section in $\mathcal{O}(7)^2$ would vanish on C. But this section would correspond to a point t in $\boldsymbol{P}^3$ and then

$$C \subseteq D_t \cap D''_t.$$

But this intersection has only a finite number of points. Hence L is ample and so E is ample.

We now construct a series of ample bundles F_n so that $F_n(k)$ has no sections if $k < n$. First, suppose the characteristic of k is $p \neq 0$. A theorem of Barton states that if E is ample and F is a bundle, then $E^{p^t} \otimes F$ is ample

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for l large [3]. Hence we choose F_n to be $E^{p^l}\otimes\mathcal{O}(-n)$ where l is large enough so that F_n is ample. We have an exact sequence

$$0 \longrightarrow \mathcal{O}(-7p^l - n)^2 \longrightarrow \mathcal{O}(-n)^4 \longrightarrow F_n \longrightarrow 0$$

Using the long exact sequence of cohomology, we see $F_n(k)$ has no sections for $k < n$.

Now we can also construct such bundles in characteristic zero. Let A be a local integral domain so that the residue field k has characteristic p and whose quotient field K has characteristic zero. Then we can extend the two maps from $\mathcal{O}^4_{P^2_k}(n)$ to $\mathcal{O}_{P^2_k}(7p^l + n)$ to maps from $\mathcal{O}^4_{P^2_A}(n)$ to $\mathcal{O}_{P^2_A}(n+7p^l)$. So we can construct a new locally free sheaf E'' over $\boldsymbol{P}^2_A$

$$0 \longrightarrow \mathcal{O}_{P^2_A}(-7p^l - n)^2 \longrightarrow \mathcal{O}_{P^2_A}(-n)^4 \longrightarrow E'' \longrightarrow 0$$

Now letting $F_n = E''\otimes K$, we see that

$$0 \longrightarrow \mathcal{O}_{P_K}(-7p^l - n)^2 \longrightarrow \mathcal{O}_{P^2_K}(-n)^4 \longrightarrow F_n \longrightarrow 0.$$

Furthermore, F_n is ample since the set of $x\in\operatorname{Spec} A$ such that E'' is ample on the fiber over x is open and non-empty.

As we promised in the introduction, we now give an example of a p-ample bundle E on $\boldsymbol{P}^2$ so that for any bundle F,

$$H^1(\Gamma^{p^n}(E)\otimes F) \neq 0$$

for all n large. We need the following lemma which will enable us to relate the cohomology of $\Gamma^{p^n}(E)$ and E^{p^n}.

LEMMA 3.3 (char $k = p$). *Let E be a vector bundle of rank 2. Then for each $n > 1$, there is an exact sequence*

$$* \quad 0 \longrightarrow S^{p^{n-1}}(E^p) \longrightarrow S^{p^n}(E) \longrightarrow \Gamma^{p^n}(E) \longrightarrow \Gamma^{p^{n-1}}(E^p) \longrightarrow 0.$$

Proof. Let A be a ring of characteristic p. Let P denote the representation of $Gl(2, A)$ on A^2 defined by

$$P\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} a^p & b^p \\ c^p & d^p\end{pmatrix}$$

S^k will denote the symmetric power representation of $Gl(2. A)$ on A^{k+1}. Then we define a new representation Γ^n by

$$\Gamma^n(F) = (((S^n(F^{-1})^t)^{-1})^t).$$

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However, since S^n is actually a group homomorphism and since transpose and inverse commute, we have

$$\Gamma^n(F) = (S^n(F^t))^t.$$

We let $q = p^n$ and $r = p^{n-1}$. So to prove $*$, we merely have to show there is an exact sequence of representations

$$** \quad 0 \longrightarrow S^r P \longrightarrow S^q \longrightarrow \Gamma^q \longrightarrow \Gamma^r P \longrightarrow 0.$$

Now let l be the largest integer such that p^l divides $k!(q-k)!$ for all k. Denote p^l by s and the $(q+1)\times(q+1)$ diagonal matrix given by

$$b_{kk} = \frac{k!(q-k)!}{s}$$

by B. Now let $g_0, \cdots, g_q$ be the usual basis for A^{q+1} and $f_0, \cdots, f_r$ the usual basis for A^{r+1}. Let C be the linear map from A^{r+1} to A^{q+1} which sends f_k to g_{pk} and D the map of A^{q+1} to A^{r+1} which sends g_{pk} to f_k and g_l to zero if p does not divide l. Now p divides b_{kk} if and only if p divides k. I claim that for any F in $Gl(2, A)$ we have a commutative diagram with exact rows,

$$\begin{tikzcd} 0 \arrow[r] & A^{r+1} \arrow[r, "C"] \arrow[d, "S^r P(F)"] & A^{q+1} \arrow[r, "B"] \arrow[d, "S^q(F)"] & A^{q+1} \arrow[r, "D"] \arrow[d, "\Gamma^q(F)"] & A^{r+1} \arrow[r] \arrow[d, "\Gamma^r P(F)"] & 0 \\ 0 \arrow[r] & A^{r+1} \arrow[r, "C"'] & A^{q+1} \arrow[r, "B"'] & A^{q+1} \arrow[r, "D"'] & A^{r+1} \arrow[r] & 0. \end{tikzcd}$$

This will prove $**$ and hence $*$.

We need only check commutativity of the diagram for F of the form

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

As a sample of these checks, we prove commutativity of the middle square when

$$F = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

$$S^q(F)(g_k) = \sum_{j=k}^{q} \binom{q-k}{q-j} g_j$$

$$S^q(F^t)(g_k) = \sum_{j=0}^{k} \binom{k}{j} g_j$$

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So we have

$$\Gamma^q(F)(g_k) = \sum_{j=k}^{n} \binom{j}{k} g_j .$$

So

$$\Gamma^q(F)(B(g_k)) = \sum_{j=k}^{n} \binom{j}{k} \frac{k!(q-k)!}{s} g_j$$

$$BS^q(F)(g_k) = \sum_{j=k}^{n} \binom{q-k}{q-j} \frac{j!(q-j)!}{s} g_j .$$

However the reader may easily verify that in $\boldsymbol{Z}$,

$$\binom{j}{k} \frac{k!(q-k)!}{s} = \binom{q-k}{q-j} \frac{j!(q-j)!}{s}$$

The above lemma gives us a hold on the cohomology of $\Gamma^{p^n}(E)$.

THEOREM 3.2. *Let F be a projective surface, E a p-ample bundle of rank 2 on F, and G a coherent sheaf on F. Then for large n,*

$$\dim H^1(\Gamma^{p^n}(E) \otimes G) \geq \dim H^1(E^{p^n} \otimes G).$$

Proof. First, there are m_0 and n_0 so that

$$* \qquad H^2(S^{p^n}(E^{p^m}) \otimes G) = 0$$

if $m \geq m_0$ or $n \geq n_0$. Indeed, let $\mathcal{O}(1)$ be an ample line bundle so that

$$H^2(G \otimes \mathcal{O}(k)) = 0$$

for all $k > 0$. Now choose m_0 so that if $m \geq m_0$ then E^{p^m} is a quotient of a direct sum of $\mathcal{O}(1)$'s. Then $S^{p^n}(E^{p^m}) \otimes G$ is a quotient of a direct sum of $\mathcal{O}(k) \otimes G$'s for $k > 0$. Hence $*$ holds if $m \geq m_0$. On the other hand, the bundles $E, E^p, \cdots, E^{p^{m_0}}$ are all ample. So there is an n_0 so that if $n \geq n_0$, then

$$H^2(S^{p^n}(E^{p^m}) \otimes G) = 0$$

if $m \leq m_0$. So $*$ is established if $m \geq m_0$ or $n \geq n_0$. Now from Lemma 3.3, we have exact sequences

$$0 \longrightarrow S^{p^{n-1}}(E^{p^{m+1}}) \otimes G \longrightarrow S^{p^n}(E^{p^m}) \otimes G \longrightarrow D_{m,n} \longrightarrow 0$$

$$0 \longrightarrow D_{m,n} \longrightarrow \Gamma^{p^n}(E^{p^m}) \otimes G \longrightarrow \Gamma^{p^{n-1}}(E^{p^{m+1}}) \otimes G \longrightarrow 0.$$

Now $*$ and the fact that H^2 is right exact show that

$$H^2(D_{m,n}) = 0$$

if $m \geq n_0$ or $n \geq n_0$. So we have an exact sequence

$$H^1(\Gamma^{p^n}(E^{p^m}) \otimes G) \longrightarrow H^1(\Gamma^{p^{n-1}}(E^{p^{m+1}}) \otimes G) \longrightarrow 0$$

So if n is greater than $2m_0$ and $2n_0$, we have exact sequences

$$\begin{gathered} H^1(\Gamma^{p^n}(E) \otimes G) \longrightarrow H^1(\Gamma^{p^{n-1}}(E^p) \otimes G) \longrightarrow 0 \\ H^1(\Gamma^{p^{n-1}}(E^p) \otimes G) \longrightarrow H^1(\Gamma^{p^{n-2}}(E^{p^2}) \otimes G) \longrightarrow 0 \\ \vdots \\ H^1(\Gamma^p(E^{p^{n-1}}) \otimes G) \longrightarrow H^1(E^{p^n} \otimes G) \longrightarrow 0. \end{gathered}$$

So our theorem follows.

Now we can give our example, which is one of a series constructed by Kleiman [13]. Fix a surjection on P^2

$$\mathcal{O}^3 \longrightarrow \mathcal{O}(2) \longrightarrow 0.$$

Dualizing and twisting by $\mathcal{O}(1)$, we get an exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O}(1)^3 \longrightarrow E \longrightarrow 0.$$

Then E is p-ample since it is a quotient of a direct sum of ample line bundles. Now let G be a bundle on P^2. Then

$$H^1(G \otimes E^{p^n}) \neq 0,$$

for n large. For if n is large, $H^1(G(p^n))$ and $H^2(G(p^n))$ are both zero, so $H^1(G \otimes E^{p^n})$ and $H^2(G(-p^n))$ are isomorphic. However,

$$\dim H^2(G(-p^n)) = \dim H^0(G^\vee \otimes \mathcal{O}(p^n - 3)).$$

Hence by our theorem,

$$H^1(\Gamma^{p^n}(E) \otimes G) \neq 0$$

for n large.

We give the following theorem which clarifies somewhat the relations between the various properties we have been discussing.

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THEOREM 3.3 (char $k=p$). *Let F be a projective surface and E a p-ample bundle of rank 2 on F. Then if for any coherent sheaf G,*

$$H^1(\Gamma^{p^n}(E)\otimes G)=0$$

for n large, then E is cohomologically p-ample.

Proof. From Theorem 3.2, we see $H^1(E^{p^n}\otimes G)$ vanishes for large n. Now if L is a line bundle so that $H^2(L)=0$, we write $G\otimes E^{p^n}$ as a quotient of copies of L. Then $H^2(G\otimes E^{p^n})$ vanishes.

BIBLIOGRAPHY

[1] M.F. Atiyah, Vector Bundles Over an Elliptic Curve, Proc. Lond. Math. Soc., 3, VII (1957), 414–452.

[2] C. Barton, Tensor Products of Ample Bundles in Characteristic p (to appear).

[3] ———, Contributions to the Theory of Ample Vector Bundles, Thesis, Columbia University, 1968.

[4] S. Bloch and D. Gieseker, The Positivity of the Chern Classes of an Ample Vector Bundle, Invent. math. **12** (1971), 112–117.

[5] P.A. Griffiths, The Extension Problem in Complex Analysis II. Embeddings with Positive Normal Bundle, Amer. J. Math. **88** (2) (1965), 366–446.

[6] ———, Hermitian differential geometry, Chern classes, and positive vector bundles, in Global Analysis, edited by D.C. Spencer and S. Iyanaga, Princeton Mathematical Series, No. 29, Tokyo, 1969.

[7] A. Grothendieck, La Théorie des Classes de Chern, Bull. Soc. Math. France **86** (1958), 137–154.

[8] ——— and J. Dieudonné, Éléments de Géométrie Algébrique, Publ. Math. IHES 1960 ff.

[9] R. Hartshorne, Ample Vector Bundles, Publ. Math. IHES **29** (1966), 63–94.

[10] ———, Cohomological Dimension of Algebraic Varieties, Ann. of Math. **88** (3), 1968, 403–450.

[11] ———, Ample Vector Bundles on Curves, Nagoya Math. J. Vol. **43** (this volume).

[12] S.L. Kleiman, Towards a Numerical Theory of Ampleness, Ann. of Math. **84** (2) (1966), 291–344.

[13] ———, Ample Vector Bundles on Surfaces, Proc. Amer. Math. Soc. 21, No. 3, June 1969, 673–676.

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