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A note on a paper of E.R. Love

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Applying a new and very elegant method of proof of the Schur-Hardy inequality, given by E.R. Love at the Oberwolfach conference on Linear Spaces and Approximation (1977), norm estimates of integral operators with homogeneous kernels are established in the setting of abstract function norms. Applications to Flett's inequality, to integral means, and to fractional integrals are given.

Let ρ denote a function norm on the set $P\big((0, \infty)\big)$ of all non-negative, Lebesgue measurable functions on $(0, \infty)$, that is, a mapping $\rho : P\big((0, \infty)\big) \to [0, \infty]$ with the properties that, for all $f, g \in P$,

(i) $\rho(f) = 0$ if and only if $f = 0$ almost everywhere,
$\rho(\lambda f) = \lambda\rho(f)$ $(\lambda > 0)$; $\rho(f+g) \le \rho(f) + \rho(g)$;

(ii) $f \le g$ almost everywhere only if $\rho(f) \le \rho(g)$.

The space L^ρ of all measurable, real valued functions f with $\|f\|_\rho := \rho(|f|) < \infty$, functions which coincide almost everywhere being identified, is called the *normed Köthe space* generated by ρ, see [9, p. 42]. The operator norm on L^ρ of the dilation operator E_s, defined by $\big(E_s f\big)(t) = f(st)$, $s, t > 0$, is called the indicator function $h(s)$ of L^ρ, that is, $h(s) := \|E_s\|_{[L^\rho]}$. If $\|\cdot\|_\rho$ is, in particular, the Lebesgue norm $\|\cdot\|_q$, $1 \le q < \infty$, then $h(s) = s^{-1/q}$.

In the sequel let $K(t, s)$ denote a nonnegative, measurable function

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of $t, s > 0$, and let K be the operator defined by

(1) $$(Kf)(t) = \int_0^\infty K(t, s)f(s)ds \quad (t > 0,\ f \in L^\rho) .$$

THEOREM 1. *Let* $K(t, s)$ *be homogeneous of degree* γ *, where* $\gamma \geq -1$ *, such that*

$$A := \int_0^\infty K(1, s)s^{-(1+\gamma)}h(s)ds < \infty$$

and $(\cdot)^{1+\gamma}f \in L^\rho$. *Then, if* L^ρ *is complete,* $Kf \in L^\rho$ *, and*

(2) $$\|Kf\|_\rho \leq A\|(\cdot)^{1+\gamma}f\|_\rho .$$

Proof. By the homogeneity of the kernel $K(t, s)$ one has, with $s = tu$,

$$\begin{aligned} |(Kf)(t)| &\leq \int_0^\infty K(t, tu)|f(tu)|tdu \\ &= \int_0^\infty K(1, u)t^{1+\gamma}|(E_uf)(t)|du \\ &= \int_0^\infty K(1, u)u^{-(1+\gamma)}\left|\left(E_u(\cdot)^{1+\gamma}f\right)(t)\right|du . \end{aligned}$$

Now (2) follows on account of the monotonicity of ρ and the completeness of L^ρ .

This is essentially the method of proof of Love [8, Theorem 1.2]. That theorem follows from our Theorem 1 by taking $\gamma = -1$ and $\|\cdot\|_\rho = \|\cdot\|_q$, $1 \leq q < \infty$, (the Lebesgue norm), namely

COROLLARY 2 (Schur-Hardy inequality). *Let* $K(t, s)$ *be homogeneous of degree* -1 *such that*

$$A_q = \int_0^\infty K(1, s)s^{-1/q}ds < \infty ,$$

and $f \in L_q$. *Then* $Kf \in L_q$ *and*

(3) $$\|Kf\|_q \leq A_q\|f\|_q .$$

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A slightly modified version of Theorem 1 is possible if one proceeds as follows: if f^* denotes the nonincreasing rearrangement of f (for definition, see for example [4]), and $f^{**}(t) := (1/t)\int_0^t f^*(s)ds$, then $f^*(t) \le f^{**}(t)$, and $(E_u f)^*(t) = (E_u f^*)(t)$. Therefore

$$\begin{aligned}|(Kf)(t)| &\le t^{1+\gamma}\int_0^\infty K(1,\, u)|(E_u f)(t)|du \\ &\le t^{1+\gamma}\int_0^\infty K^*(1,\, u)(E_u f^*)(t)du \\ &= \int_0^\infty K^*(1,\, u)u^{-(1+\gamma)}\left\{E_u(\cdot)^{1+\gamma}f^*\right\}(t)du ,\end{aligned}$$

yielding

THEOREM 1*. *If* $K(t,\, s)$ *is homogeneous of degree* γ, *where* $\gamma \ge -1$, *such that*

$$A^* := \int_0^\infty K^*(1,\, s)s^{-(1+\gamma)}h(s)ds < \infty ,$$

and f *a measurable function on* $(0,\, \infty)$ *with* $(\cdot)^{1+\gamma}f^{**} \in L^\rho$, *then* $Kf \in L^\rho$ *and*

$$\|Kf\|_\rho \le A^*\|(\cdot)^{1+\gamma}f^{**}\|_\rho . \tag{2*}$$

Note that Theorem 1* as well as Theorem 1 include in particular the cases when $\|\cdot\|_\rho$ is the Lebesgue, Lorentz, or Orlicz norm, respectively. In the Lebesgue case, for instance, Theorem 1* reduces to

COROLLARY 2*. *Let* $\gamma \ge -1$, $1 \le q < \infty$, *and* $p := q/\big(1+q(1+\gamma)\big)$. *If* $K(t,\, s)$ *is homogeneous of degree* γ *such that*

$$A_q^* := \int_0^\infty K^*(1,\, s)s^{-(1+\gamma)}s^{-1/q}ds < \infty ,$$

and $f \in L_{pq}$ *(equals the Lorentz space,* see [4]*), then* $Kf \in L_q$ *and*

$$\|Kf\|_q \le (p/q)^{1/q}A_q^*\|f\|_{pq} . \tag{3*}$$

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Evidently Corollary 2* implies [8, Theorem 2.1] by replacing γ by $-r^{-1}$, and observing that $\|\cdot\|_{pq} \le (p/(p-1))\|\cdot\|_p$ for $1 < p \le q$. Our constant A_q^* is different from that of [8], since on the right hand side of (3*) the Lorentz norm is used instead of the Lebesgue norm. A simple computation shows, however, that the constant C of [8] can be estimated by $C \le A_q^*/r$. Indeed (note that $r \ge 1$) :

$$C :\equiv \|K(1, s)s^{-1/q}\|_r \le \|K^*(1, s)s^{-1/q}\|_r = \|K^*(1, s)s^{-1/q}\|_{rr}^*$$
$$\le \|K^*(1, s)s^{-1/q}\|_{r1}^* = (1/r)A_q^* ,$$

where $\|\cdot\|_{pq}^*$ denotes the modified Lorentz norm with f^{**} replaced by f^*.

REMARK. It is well known that the $\underline{\underline{K}}$-functional of Peetre with respect to the spaces L_1 and L_∞, namely

$$\underline{\underline{K}}(t, f; L_1, L_\infty) := \inf\{\|f_1\|_1 + t\|f_2\|_\infty : f = f_1+f_2,\ f_1 \in L_1,\ f_2 \in L_\infty\} ,$$

can be expressed in terms of f^*. Indeed,

$$\underline{\underline{K}}(t, f; L_1, L_\infty) = \int_0^t f^*(s)ds = tf^{**}(t) .$$

Therefore the assertion of Theorem 1* remains valid if (2*) is replaced by

$$\|Kf\|_\rho \le A^*\left\|t^\gamma \underline{\underline{K}}(t, f; L_1, L_\infty)\right\|_\rho . \tag{4*}$$

With the notations of [5], in particular $\gamma = -\theta$, $\theta \le 1$, and

$$\|f\|_{\theta,\rho;\underline{\underline{K}}} := \rho\left[t^{-\theta}\underline{\underline{K}}(t, f; L_1, L_\infty)\right] ,$$

Theorem 1* can be reformulated in the language of interpolation theory as follows.

COROLLARY 3*. *Under the assumptions of Theorem* 1* *the operator* K *is a bounded operator from the interpolation space* $(L_1, L_\infty)_{-\gamma,\rho;\underline{\underline{K}}}$ *of* L_1 *and* L_∞ *into the space* L^ρ *such that* (4*) *holds.*

Here recall that

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$$(L_1, L_\infty)_{-\gamma,\rho;\underline{\underline{K}}} := \left\{f \in L_1+L_\infty : t^\gamma \underline{\underline{K}}(t, f; L_1, L_\infty) \in L^\rho\right\} .$$

Finally let us discuss some applications of the above theorems to special kernels. As a first example, consider the average operator P_θ, $\theta > 0$ (see Boyd [2]), defined by

$$(P_\theta f)(t) := t^{-\theta} \int_0^t s^{\theta-1} f(s) ds \quad (t > 0) .$$

This operator is obviously of type (1) with kernel

$$K(t, s) = t^{-\theta} s^{\theta-1} \chi_{(0,t)}(s) ,$$

$\chi_{(0,t)}$ denoting the characteristic function of the interval $(0, t)$. Since this kernel is homogeneous of degree $\gamma = -1$, Theorem 1 yields that

$$\|P_\theta f\|_\rho \le A_\theta \|f\|_\rho \quad (f \in L^\rho) , \tag{5}$$

if

$$A_\theta := \int_0^1 s^{\theta-1} h(s) ds < \infty . \tag{6}$$

In particular, if $\theta = 1$ and $\|\cdot\|_\rho = \|\cdot\|_q$, $q > 1$, then (5) reduces to the classical *Hardy inequality* (see [7, p. 240] or [3] and the literature quoted there). For the case of Lorentz norms see also [10]. More generally, if $\|\cdot\|_\rho$ is a rearrangement-invariant norm - for definition see, for example, [2], [5] - then (5) is precisely the *generalized Hardy inequality* in the setting of rearrangement-invariant norms, as established in Butzer and Fehér [3]. Let us also mention that the constant A_θ of (6) is identical with that of [3]; therefore (6) is equivalent to the index condition $\alpha < \theta$, where α denotes the upper index of ρ, that is to say,

$$\alpha = \inf_{0<s<1} \left(-\log h(s)/\log s\right)$$

(compare [5]).

As a second example one might consider the modified operator of fractional integration M_λ, $\lambda > 0$, defined by (see [8])

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$$\left(M_\lambda f\right)(t) := t^{-\lambda} \int_0^t \frac{(t-s)^{\lambda-1}}{\Gamma(\lambda)} f(s)ds \quad (t > 0) .$$

As the kernel of this operator is again homogeneous of degree -1, Theorems 1 and 1*, as well as the above corollaries, immediately apply to this operator.

If, in particular, $\|\cdot\|_\rho = \|\cdot\|_{pq}$, $1 \le p, q < \infty$, then $h(s) = s^{-1/p}$ and Theorem 1, for example, asserts that for $p > 1$,

(7) $$\|M_\lambda f\|_{pq} \le \frac{\Gamma(1-1/p)}{\Gamma(1-1/p+\lambda)} \|f\|_{pq} \quad \left(f \in L_{pq}\right) .$$

Similarly, *Flett's inequality* [6], namely,

(8) $$\left\|(\cdot)^{-\beta-1/q} M_\lambda f\right\|_q \le \text{const.}\|(\cdot)^{-\beta-1/p} f\|_p \quad (\beta > -1) ,$$

is easily obtained from Corollary 2* (if $A_q^* < \infty$, $p > 1$) by taking

$$K(t, s) = \chi_{(0,t)}(s) t^{-\lambda-\beta-1/q}(t-s)^{\lambda-1} s^{\beta+1/p}/\Gamma(\lambda)$$

and replacing $f(s)$ by $s^{-\beta-1/p} f(s)$ (see [8]).

If one chooses $\|\cdot\|_\rho = \|\cdot\|_{pq}$, then Corollary 2* implies a generalized version of Flett's inequality with respect to Lorentz norms (compare [11]).

A very important example of an integral operator the kernel of which is homogeneous of degree $\gamma > -1$ is the operator I_λ of fractional integration of order $\lambda > 0$, in the sense of Riemann-Liouville, defined by

$$\left(I_\lambda f\right)(t) := \frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} f(s)ds \quad (t > 0) .$$

Since the kernel of I_λ is homogeneous of degree $\gamma = \lambda - 1$, Corollary 2* yields

COROLLARY 4*. *Assume that* $1 < q < \infty$, $0 < \lambda < 1-1/q$, *and* $p := q/(1+\lambda q)$. *If* $f \in L_p$, *then* $I_\lambda f \in L_q$ *and*

(9) $$\|I_\lambda f\|_q \le \frac{p}{p-1} \frac{\Gamma(1-\lambda-1/q)}{\Gamma(1-1/q)} \|f\|_p .$$

This corollary is in accordance with [7, Theorem 383]. The proof in

[7] does not furnish any value of the constant in (9), although rather deep results are used in the proof. In contrast to the very elegant method of proof of Love, the usual way of establishing (9) makes appeal to interpolation theory (in particular to the Marcinkievicz interpolation theorem), a fact which now seems understandable, if one recalls Corollary 3*.

In case of Lorentz norms there exist quite different proofs for the boundedness of the operator I_λ (compare [13]) using theorems of multiplication and of convolution for Lorentz norms. This boundedness result, namely (compare [13, Proposition 4 with $n = 1$])

$$\left\|(\cdot)^{-\beta}I_\lambda f\right\|_{pq} \le \text{const.}\|(\cdot)^\alpha f\|_{p_0q_0} , \tag{10}$$

with $1/p_0 = 1/p + \lambda - (\alpha+\beta)$ and $q_0 \le q$, can be deduced from Theorem 1 by taking $\|\cdot\|_\rho = \|\cdot\|_{pq}$, K the operator (1) with kernel $K(t, s) = \chi_{(0,t)}(s)t^{-\beta}(t-s)^{\lambda-1}s^\beta/\Gamma(\lambda)$, and replacing $f(s)$ by $s^{-\beta}f(s)$. Indeed, the left side of (10) is equal to

$$\left\|(\cdot)^{-\beta}I_\lambda f\right\|_{pq} = \|K((\cdot)^{-\beta}f)\|_{pq} \le A\|(\cdot)^{\lambda-\beta}f\|_{pq} =$$

$$= A\|(\cdot)^{1/p_0-1/p}(\cdot)^\alpha f\|_{pq} \le AB\|(\cdot)^\alpha f\|_{p_0q} \le AB\,\frac{p_0}{p_0-1}\,\|(\cdot)^\alpha f\|_{p_0q_0} ,$$

with $A = \Gamma(1-\lambda-1/q+\beta)/\Gamma(1-1/q+\beta)$. Here use was made of the multiplication theorem for Lorentz norms (compare [1]), namely

$$\|fg\|_{pq} \le B\|f\|_{p_0q}\|g\|_{p_1\infty} , \tag{11}$$

where $1/p = 1/p_0 + 1/p_1$, and B is a suitable constant, as for example $B = (p/p_0)^{1/p}2^{1/p}p/(p-1)$. For the particular case $p = q$ of (10), that is the Lebesgue case, see for example [12].

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