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THE DECOMPOSITION OF THE MODULE OF n-th ORDER DIFFERENTIALS IN ARBITRARY CHARACTERISTIC

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1. Introduction.

Throughout this paper, it is assumed that A is the complete, equicharacteristic, local ring of an algebraic curve at a one-branch singularity whose residue field k is algebraically closed and contained in A. Hence, the domain A is dominated by only one valuation ring in its quotient field F, and if t is a uniformizing parameter, then the integral closure of A in F, denoted by $\bar{A}$, is $k[[t]]$.

The kernel of the multiplication map $\pi(\sum_{i=1}^{s} (x_i \otimes y_i)) = \sum_{i=1}^{s} x_i y_i$

$$\bar{A} \otimes_A \bar{A} \xrightarrow{\pi} \bar{A} \to 0$$

is denoted by $I(\bar{A}/A)$, and $D^n(\bar{A}/A) = I(\bar{A}/A)/I^{n+1}(\bar{A}/A)$ is called the *module of nth order differentials*. It was shown in [**2**] that $I(\bar{A}/A)$ is nilpotent. This follows essentially from the finiteness of $\bar{A}$ over A. It was also shown in [**2**] under the assumption that k is of characteristic zero, that the decomposition of $I(\bar{A}/A)$ over the P.I.D. $\bar{A}$ is uniquely determined by the multiplicity sequence of A, and an explicit formula for the decomposition was listed [**2**, p. 123]. The abstract of that paper incorrectly stated the result to have been shown for any characteristic of k when in fact it was proved for char $k = 0$ only.

The purpose of this note is to prove that this decomposition and the subsequent formula do indeed hold for arbitrary characteristic of k. The arguments are, in part, improvements of those used in [**2**] and the reader is referred there for terminology and background. Recently, William C. Brown studied these and related results in the more general non-unibranched case in [**1**]. Acknowledgement for helpful comments in the formulation of this paper are also due him.

2. Some remarks and lemmas.

If $c \in A$, denote by $v(x)$ the value (or order) of the element x and by $e(A)$ the multiplicity of the maximal ideal M of A. An element x belonging to M is said to be *transversal to* M if $v(x) = e(A)$.

Due to the standard assumption that $\bar{A} = k[[t]]$ one has:

LEMMA 1. *If $x \in M \subset A$, then $I(\bar{A}/k[[x]])$ is a free $\bar{A}$ module of rank $v(x) - 1$.*

Proof. The element x is by assumption of positive degree. It follows from power series arguments [**3**, p. 211, Theorem 28] that $\bar{A}$ is finitely generated over

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$k[[x]]$. Since $k[[x]]$ is itself a local P.I.D. and since $\dim_k \bar{A}/(x)\bar{A} = v(x)$, it follows that $\bar{A}$ is a free $k[[x]]$ module of rank $v(x)$.

Since the sequence of $\bar{A}$ modules

$$0 \to I(\bar{A}/k[[x]]) \to \bar{A} \otimes_{k[[x]]} \bar{A} \xrightarrow{\pi} \bar{A} \to 0$$

is split exact and since the middle term is free of rank $v(x)$, the left term must be a free $\bar{A}$ module of rank $v(x) - 1$.

Given any three commutative rings $A \subset B \subset C$, the kernel of the canonical map $\chi(\sum_{i=1}^{s} (x_i \otimes_A y_i)) = \sum_{i=1}^{s} (x_i \otimes_B y_i)$,

$$\chi\colon C \otimes_A C \to C \otimes_B C$$

is generated as an ideal by the elements $\{1 \otimes_A x - x \otimes_A 1 : x \in B\}$.

Hence, if A' is the strict closure of A in $\bar{A}$ where A' is defined by $A' = \{x \in \bar{A}\colon 1 \otimes_A x - x \otimes_A 1 = 0\}$, then $A \subset A'$ and it is easily seen that $\bar{A} \otimes_A \bar{A} \cong \bar{A} \otimes_{A'} \bar{A}$ where the isomorphism χ is given as above. In fact, restricting χ gives $I(\bar{A}/A) \cong I(\bar{A}/A')$ as $\bar{A}$ modules. Since $\bar{A} = k[[t]]$, it is clear that A', which contains A, is local and that $\bar{A}' = k[[t]]$. Hence, any consideration of the decomposition of $I(\bar{A}/A)$ as an $\bar{A}$ module may be done under the assumption that A is strictly closed; that is, $A' = A$.

Denote by A^M the blow-up of A along M. It is important to note that if A is strictly closed, then

$$A^M = \{z/x\colon z \in M\}$$

where x is a fixed transversal element to M.

Now if x is an arbitrary element in M, let $R = k[[x]]$ and consider the diagram:

$$(*) \qquad \begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& N(A) \arrow[dr] & & I(\bar{A}/A^M) \arrow[ur] & \\
& & I(\bar{A}/R) \arrow[ur, "\varphi_2"] \arrow[dr, "\varphi_1"] & & \\
& N(A^M) \arrow[ur] & & I(\bar{A}/A) \arrow[dr] \arrow[uu, "\theta"'] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd}$$

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In this, φ_1, and φ_2 and θ are the obvious canonical maps and the free $\bar{A}$ modules $N(A)$ and $N(A^M)$ are the kernels of φ_1 and φ_2, respectively. Notice $R \subset A \subset A^M$.

The sequences

$$\begin{array}{ccccccccc} 0 \to & I(\bar{A}/A^M) & \to & \bar{A} \otimes_{A^M} \bar{A} & \to & A & \to 0 \\ & \uparrow \theta & & \uparrow \chi & & \uparrow \mathrm{Id} & \\ 0 \to & I(\bar{A}/A) & \to & \bar{A} \otimes_A \bar{A} & \to & \bar{A} & \to 0 \end{array}$$

easily show θ is onto. Likewise, φ_1 and φ_2 are also shown to be onto. Referring to this diagram one shows:

LEMMA 2. *If x is transversal to M in A, then $x\, N(A^M) = N(A)$.*

Proof. First consider the case when A is strictly closed, in which case

$$A^M = \{z/x\colon z \in M, x \text{ transversal to } M\}.$$

The kernel of φ_2 is generated as an ideal in $\bar{A} \otimes_R \bar{A}$ by $(1 \otimes z/x - z/x \otimes 1)$ where $R = k[[x]]$.

Hence, if $\eta \in N(A^M)$, then

$$\eta = \sum_{i=1}^{s} (c_i \otimes_R d_i)\left(1 \otimes_R \frac{z_i}{x} - \frac{z_i}{x} \otimes_R 1\right), \quad \text{and}$$

$$x\eta = \sum_{i=1}^{s} (c_i \otimes_R d_i)(1 \otimes_R z_i - z_i \otimes_R 1),$$

so $\varphi_1(x\eta) = 0$.

Now suppose $\xi \in N(A)$. Then

$$\xi = \sum_{i=1}^{s} (a_i \otimes_R b_i)(1 \otimes_R z_i - z_i \otimes_R 1), \quad z_i \in M,$$

and hence

$$\xi = x\left[\sum_{i=1}^{s} (a_i \otimes_R b_i)\left(1 \otimes_R \frac{z_i}{x} - \frac{z_i}{x} \otimes_R 1\right)\right].$$

But the right hand factor of ξ is an element in $N(A^M)$, so $\xi \in N(A^M)$. Hence, $x\, N(A^M) = N(A)$.

In the general case, if x is transversal to M in A, it is also transversal to the maximal ideal M' of A' since $e(A) = e(A')$. Hence, the aforementioned isomorphism between $I(\bar{A}/A)$ and $I(A/A')$ easily shows $N(A) = N(A')$, and likewise $N(A^M) = N((A^M)')$. But since $(A^M)' = (A')^{M'}$, it follows that $x\, N(A^M) = N(A)$.

3. The theorem. Let $A = A_0$ and denote A^M by A_1. The ring A_1 is local once again, and one may form its blow up along its maximal ideal M_1. Denote

$A_1^{M_1}$ by A_2; this process stops at $\bar{A}$ after a finite number of steps, and it shall be assumed that A_n is singular and that $A_n^{M_n} = k[[t]]$.

The sequence $A = A_0 \subset A_1 \subset \ldots \subset A_n \subset \bar{A}$ is called the *blow up sequence of* A and the sequence $e(A_0) = e_0, e(A_1) = e_1, \ldots, e(A_n) = e_n$ is called the *multiplicity sequence of* A. By assumption none of these integers in this decreasing sequence is equal to one.

Given any finitely generated module T over the P.I.D. $\bar{A}$, let F be a free module of rank s mapping onto T and consider the $(s \times s)$ matrix determined by the relations for T. If σ_i is the greatest common divisor of all $i \times i$ subdeterminants then $\sigma_1 = E_1, \sigma_2/\sigma_1 = E_2, \ldots, \sigma_s/\sigma_{s-1} = E_s$ are the invariant factors of T which are unique up to units from $k[[t]]$. These completely determine the structure of T over $\bar{A}$.

With these preliminaries the theorem stated at the outset may be proved.

THEOREM. *Let A be the complete local ring of an algebraic curve at a one-branch singularity defined over an algebraically closed field k of arbitrary characteristic. Then the module $D^n(\bar{A}/A)$ of* nth *order differentials for $n \gg 1$ is uniquely determined by the multiplicity sequence of A.*

Proof. Since $I(\bar{A}/A)$ is nilpotent [**2**, Theorem 1.2] one need only consider the decomposition of $I(\bar{A}/A)$. It will be shown recursively how the invariant factors of $I(\bar{A}/A)$ are found.

Using the previous notation, let A_n be the last singular ring in the blow up sequence of A, and let x be transversal to M_n in A_n. Hence, $A_n^{M_n} = \bar{A}$. Setting $R = k[[x]]$ gives that $I(\bar{A}/R)$ is free of rank $e_n - 1$. Referring to diagram (*), let (α) be the matrix determined by the relations for $I(\bar{A}/\bar{A})$ and σ_i the greatest common divisor of the $i \times i$ subdeterminants of (α). Since $I(\bar{A}/\bar{A}) = 0$, the invariant factors $\sigma_1, \sigma_2/\sigma_1, \ldots, \sigma_{e_n-1}/\sigma_{e_n-2}$ of $I(\bar{A}/\bar{A})$ must all be units. Lemma 2 shows that $xN(\bar{A}) = N(A_n)$. Hence, the matrix determined by the relations for $I(A/A_n)$ is just $x(\alpha)$ and it follows that the greatest common divisor of the $i \times i$ subdeterminants of this matrix is $x^i\sigma_i$. This shows that each of the $e_n - 1$ invariant factors of $I(\bar{A}/A_n)$ is just x multiplied by a unit.

For the general step denote by B with maximal ideal P the ring under consideration in the blow up sequence of A. Let $E_1, \ldots, E_s$ be the invariant factors of $I(\bar{A}/B^P)$ and choose x to be transversal to P in B. Then $v(x) = e$, the multiplicity of B. Let $R = k[[x]]$ and note that $I(\bar{A}/R)$ is free of rank $e - 1$. Once again referring to diagram (*), let (α) be the $(e - 1) \times (e - 1)$ sized matrix determined by the relations for $I(\bar{A}/B^P)$. If σ_i is the greatest common divisor of the $i \times i$ subdeterminants of (α), one may assume that the ratios $\sigma_1, \sigma_2/\sigma_1, \ldots, \sigma_{e-1}/\sigma_{e-2}$ are just

$$u_1, u_2, \ldots, u_{(e-1)-s}, E_1u_{(e-1)-(s-1)}, \ldots, E_su_{e-1}$$

respectively where the u_i's are units.

But since $xN(B^P) = N(B)$, the matrix determined by the relations for

$I(\bar{A}/B)$ is $x(\alpha)$ and it follows that the invariant factors of $I(\bar{A}/B)$ are, up to units,

$$\underbrace{x, \ldots, x}_{(e-1)-s}, E_1 x, \ldots, E_s x$$

This recursive process gives the invariant factors of $I(\bar{A}/A)$. In fact, let $E_i = t^{e_i}$ where $e_i = e(A_i)$, $i = 0, \ldots, n$. Then, up to units, the factors of $I(\bar{A}/A)$ are given by:

$$(**) \quad \underbrace{E_0, \ldots, E_0}_{e_0 - e_1}, \underbrace{E_1E_0, \ldots, E_1E_0}_{e_1 - e_2}, \ldots, \underbrace{E_nE_{n-1} \ldots, E_0, \ldots, E_nE_{n-1} \ldots, E_0}_{e_n - 1}$$

This formula shows that the multiplicity sequence determines the decomposition of $I(\bar{A}/A)$.

Conversely, given the ring A with a known decomposition of $I(\bar{A}/A)$ one may read off the multiplicity sequence of A in the following way.

If all the invariant factors of $I(\bar{A}/A)$ are simply units, then the uniqueness of these factors and formula (**) imply that $e(A) = 1$, that is $A = k[[t]]$. These same considerations show in the general case that if G is the non-trivial factor of highest order appearing in the decomposition, then this order must equal $e_0 \ldots e_n$, where $e_0 = e(A), \ldots, e_n = e(A)$ again represents the non-trivial multiplicity sequence of A. In fact, the number of times the factor G appears (up to units) equals $e_n \dot{-} 1$. Hence, e_n is determined.

Now consider $F = G/t^{e_n}$. Once again the uniqueness of the factors and formula (**) imply that apart from G, the order of F is the largest that can possibly appear among the factors of $I(\bar{A}/A)$. If $v(F) = 0$, then since $v(F) = e_0 \ldots e_{n-1}$ and since $e_i > 1$ for all i, $e(A) = e_n$ and this must be the complete multiplicity sequence of A. ($n = 0$ in this case).

If $v(F) > 0$, then the number of times a factor of order $v(F)$ appears among the invariant factors of $I(\bar{A}/A)$ must equal $e_{n-1} - e_n$. Such a factor may not exist in which case $e_{n-1} = e_n$. In either case, e_{n-1} is determined.

One continues the process by considering $F/t^{e_{n-1}} = G/t^{e_n}t^{e_{n-1}}$ until such division produces an element of order zero in which case the complete multiplicity sequence will have been determined. This completes the proof of the theorem.

For arbitrary characteristic of k, the formula for the length of $I(\bar{A}/A)$ over $\bar{A}$, denoted by $\lambda_{\bar{A}}(I(\bar{A}/A))$, is easily found to be as before [**2**, Theorem 3.6].

Corollary. $\lambda_{\bar{A}}(I(\bar{A}/A)) = \sum_{i=1}^{n} e_i(e_i - 1)$.

Proof. If $E_0, \ldots, E_n$ are the invariant factors of $I(\bar{A}/A)$, then formula (**)

shows

$$\lambda_{\bar{A}}(I(\bar{A}/A)) = (e_0 - e_1)v(E_0) + (e_1 - e_2)[v(E_0) + v(E_1)] + \ldots + (e_n - 1)[v(E_0) + \ldots + v(E_n)] = \sum_{i=0}^{n} e_i(e_i - 1)$$

since $v(E_i) = e_i$.

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