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Lower Order Terms of the Discrete Minimal Riesz Energy on Smooth Closed Curves

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Abstract. We consider the problem of minimizing the energy of N points repelling each other on curves in $\mathbb{R}^d$ with the potential $|x-y|^{-s}$, $s \geq 1$, where $|\cdot|$ is the Euclidean norm. For a sufficiently smooth, simple, closed, regular curve, we find the next order term in the asymptotics of the minimal s-energy. On our way, we also prove that at least for $s \geq 2$, the minimal pairwise distance in optimal configurations asymptotically equals L/N, $N \to \infty$, where L is the length of the curve.

1 Introduction

The energy minimizing problem we consider in this paper originates from Thomson's problem of finding the positions of N classical electrons on the sphere corresponding to the ground state (the absolute minimum of the potential energy). In mathematical literature, this problem has also been considered for different classes of compact sets in d-dimensional space (curves, manifolds, rectifiable sets, and self-similar sets) and for different potentials.

In this paper we further study this problem for smooth curves. Let $\Gamma \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be a rectifiable curve (closed or non-closed), and let $\omega_N = \{x_1, \ldots, x_N\} \subset \Gamma$ be a collection of points (we will always assume that points x_i, $i = 1, \ldots, N$, are placed on Γ in such an order that the index i grows or decreases as we move along the curve). Denote by

$$E_s(\omega_N) = \sum_{1 \leq i \neq j \leq N} \frac{1}{|x_i - x_j|^s}, \quad s > 0,$$

the *Riesz s-energy* of the configuration ω_N and by

$$\mathcal{E}_s(\Gamma, N) = \min_{\substack{\omega_N \subset \Gamma \\ \#\omega_N = N}} E_s(\omega_N) \tag{1.1}$$

the *minimal N-point Riesz s-energy* of the curve Γ (here $\#X$ is the cardinality of the set X and $|\cdot|$ is the Euclidean norm in $\mathbb{R}^d$).

The exact solution to this problem is known for the circle (see Thomson [35]). Finding the exact solution to problem (1.1) in the general case is difficult, so we will study only the asymptotic behavior of optimal configurations and of quantity (1.1) as N gets large.

The minimal energy problem has different asymptotic behavior on rectifiable curves when $0 < s < 1$ and when $s \geq 1$ due to the fact that only for $0 < s < 1$

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there exists a Radon probability measure on Γ with finite continuous s-energy. Thus, methods of potential theory cannot be applied to rectifiable curves in the case $s \geq 1$, and different techniques must be used.

We are interested in the case $s \geq 1$ and build our present work upon the results of [29], where the main term in the asymptotic representation of $\mathcal{E}_s(\Gamma, N)$ as $N \to \infty$, as well as the limit distribution of optimal configurations are obtained for rectifiable arcs and their finite unions. Paper [5] extends these results to the case of an arbitrary rectifiable curve for $s > 1$. We study the behavior of the next order term of the minimal s-energy on Γ when Γ is a sufficiently smooth closed arc and obtain more information on the asymptotic behavior of the minimal energy configurations.

2 Review of Known Results

In higher dimensions, when $0 < s < \dim A$ (where $\dim A$ is the Hausdorff dimension of a compact set $A \subset \mathbb{R}^d$), the main term of $\mathcal{E}_s(A, N)$ and the limiting distribution of minimal energy configurations as $N \to \infty$ are known for any compact set $A \subset \mathbb{R}^d$ (cf. [34] for the case of logarithmic energy on the plane and cf. [28, Ch. II, § 3, no. 12] for the general case). When $s \geq \dim A$, the main term in the asymptotics of the minimal s-energy and the weak-star limit distribution of optimal configurations are known in the following cases: on the sphere S^{d-1} in $\mathbb{R}^d$, $s = d-1$, $d \in \mathbb{N}$ (cf. [16, 26]), on m-rectifiable manifolds in $\mathbb{R}^d$, $m \leq d$, $s \geq m$ (cf. [18, 19]), and on m-rectifiable sets in $\mathbb{R}^d$, $m \leq d$, $s > m$ (cf. [5]).

The problem of minimizing the logarithmic energy

$$\sum_{1 \leq i \neq j \leq N} \ln \frac{1}{|x_i - x_j|} \tag{2.1}$$

over all N-point collections $\{x_1, \dots, x_N\}$ on Γ is often referred to as the case $s = 0$ of the minimal Riesz energy problem. The next order term of the minimal energy in this case is called the self-energy of a curve.

The symmetry breaking phenomenon for the configurations minimizing energy (2.1) on certain planar curves was studied in [4]. Papers [24] and [25] estimate the difference between the potential of the equilibrium distribution on closed smooth Jordan arcs on the plane and the potential of the minimum Riesz energy configurations for $s = 0$. The complete asymptotic expansion for the minimal Riesz s-energy of N equally spaced points on the circle in terms of powers of N was found in [9] and, for the Riemannian circle, in [8].

The next order term of the minimal logarithmic energy is known on the sphere in $\mathbb{R}^d$, $d \geq 3$ (see [7] for the case $d > 3$ and for references to results, which imply the case $d = 3$). The order of the next order term of $\mathcal{E}_s(S^{d-1}, N)$ was obtained in [6] for $0 < s < d - 1$. The order of the next order term in the asymptotics of best-packing distance on the sphere S^2 in $\mathbb{R}^3$ was found in [17]. However, obtaining the exact constants in the two latter cases is still an open problem.

We remark here that exact optimal configurations on the sphere in $\mathbb{R}^d$, $d \geq 3$, are known only for certain partial cases (cf. [2, 3, 11, 13, 21, 22, 36]). The support of the limiting distribution of minimal energy configurations on sets of revolution in $\mathbb{R}^3$ was studied in [10, 20].

3 Auxiliary Notation and Definitions

We say that a curve Γ is a C^n *arc* ($n = 1, 2$ or 3) if it is a simple and regular (tangent vector is non-zero at every point) rectifiable curve of positive length that admits an n times continuously differentiable parametrization.

Assume that Γ is a closed C^3 arc or a non-closed C^2 arc. Denote by $L(x, y)$ the length of the part of Γ between points $x, y \in \Gamma$, if Γ is non-closed, and let $L(x, y)$ be the length of the shorter arc of Γ, connecting points x and y, if Γ is closed. Denote

$$g_s(x, y) := \frac{1}{|x - y|^s} - \frac{1}{L(x, y)^s},$$

and let λ_Γ be the probability measure obtained by normalizing the arc length measure supported on Γ. Define

$$\Phi_s(\Gamma) := \int_\Gamma \int_\Gamma g_s(x, y) d\lambda_\Gamma d\lambda_\Gamma.$$

It is not difficult to see that for closed C^3 arcs Γ, this integral is convergent when $s < 3$. The integral $\Phi_s(\Gamma)$ considered for closed curves in $\mathbb{R}^3$ is known as the knot energy and has been used in [30–32] to study the knots of a curve. It was also studied in [1, 15] (see also references therein). Denote by $\kappa(x)$ the curvature of Γ at a given point $x \in \Gamma$, *i.e.*, $\kappa(x)$ is the absolute value of the second derivative of the radius vector of Γ with respect to the natural parameter. Let

$$\kappa(\Gamma) := \int_\Gamma \kappa^2(x) d\lambda_\Gamma$$

be the bend energy of Γ and let

$$\gamma := \lim_{N\to\infty} \left(\sum_{k=1}^{N} \frac{1}{k} - \ln N \right).$$

be the Euler–Mascheroni constant.

4 Main Results

When Γ is a finite union of Jordan arcs whose pairwise intersections have total length zero (in particular, when Γ is a closed Jordan arc), the following equalities hold (cf. [29]):

$$\lim_{N\to\infty} \frac{\mathcal{E}_s(\Gamma, N)}{N^{s+1}} = \frac{2\zeta(s)}{|\Gamma|^s}, \quad s > 1, \tag{4.1}$$

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and

$$\lim_{N\to\infty} \frac{\mathcal{E}_1(\Gamma, N)}{N^2 \ln N} = \frac{2}{|\Gamma|}, \tag{4.2}$$

where $|\Gamma|$ stands for the length of the curve Γ and

$$\zeta(s) := \sum_{k=1}^{\infty} \frac{1}{k^s}, \quad s > 1,$$

is the Riemann zeta-function. We obtain the following result.

Theorem 4.1 *Let $\Gamma \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be a closed C^3 arc. Then if $s > 3$, we have*

$$\lim_{N\to\infty} \frac{\mathcal{E}_s(\Gamma, N) - 2\zeta(s)|\Gamma|^{-s}N^{s+1}}{N^{s-1}} = \frac{s\zeta(s-2)}{12|\Gamma|^{s-2}}\kappa(\Gamma), \tag{4.3}$$

and for $s = 3$, there holds

$$\lim_{N\to\infty} \frac{\mathcal{E}_3(\Gamma, N) - 2\zeta(3)|\Gamma|^{-3}N^{4}}{N^{2}\ln N} = \frac{\kappa(\Gamma)}{4|\Gamma|}. \tag{4.4}$$

If $1 < s < 3$, then

$$\lim_{N\to\infty} \frac{\mathcal{E}_s(\Gamma, N) - 2\zeta(s)|\Gamma|^{-s}N^{s+1}}{N^{2}} = \Phi_s(\Gamma) - \frac{2^s}{(s-1)|\Gamma|^s}, \tag{4.5}$$

and when $s = 1$, it is true that

$$\lim_{N\to\infty} \frac{\mathcal{E}_1(\Gamma, N) - 2|\Gamma|^{-1}N^{2}\ln N}{N^{2}} = \Phi_1(\Gamma) + \frac{2}{|\Gamma|}(\gamma - \ln 2). \tag{4.6}$$

Remark 4.2 For $1 \le s < 3$, the next order term of the minimal Riesz energy of a closed C^3 arc Γ in $\mathbb{R}^3$ is related to the behavior of its knots. The results of [15, Theorem 3.3 and Corollary 3.5] imply that for $s = 2$, the quantity $\Phi_s(\Gamma)|\Gamma|^2$ gives an upper bound for the topological crossing number of the knot type of Γ, hence bounding from above the number of isomorphism classes of knots that can be represented by Γ. In particular, if the limit (4.5) is less than a certain critical value (which depends on the length), then Γ is unknotted.

Remark 4.3 Limit (4.5) for $2 \le s < 3$ and limit (4.6) are positive for any closed C^3 arc Γ, except for $s = 2$ and Γ being a circle. Indeed, it is known that among all closed C^3 arcs of a given length the integral $\Phi_s(\Gamma)$, $0 < s < 3$ (and hence the limits (4.5) and (4.6)), is uniquely minimized by the circle (cf. [1, 15]). When Γ is a circle, the limit (4.5) vanishes for $s = 2$, the limit (4.6) is positive, and since the value $\left(\frac{|\Gamma|}{2}\right)^s \Phi_s(\Gamma) - \frac{1}{s-1}$ is strictly monotone for $s \in (1, 3)$, the limit (4.5) is positive for $2 < s < 3$ for the circle and hence for any C^3-arc Γ.

Moreover, since $\left(\frac{|\Gamma|}{2}\right)^s \Phi_s(\Gamma) - \frac{1}{s-1}$ is a monotone and continuous function for $s \in (1, 3)$ and tends to $-\infty$ as $s \to 1^+$, there is a unique $s_0 \in (1, 2]$, for which the right-hand side of (4.5) is zero (we have $s_0 = 2$ only when Γ is a circle). Thus, the next order term is negative for $1 < s < s_0$ and positive for $s_0 < s < 3$. For $s = s_0$, Theorem 4.1 only implies that the next order term has order less than N^2.

Separation estimates In order to show relation (4.3) in Theorem 4.1, we will study the asymptotic behavior of the distances between neighboring points in optimal configurations on a closed C^3 arc Γ. For a collection $\omega_N = \{x_1, \dots, x_N\} \subset \Gamma$, denote $x_{N+1} := x_1$, and let $l_i = l_i(\omega_N)$, $i = 1, \dots, N$, be the length of the arc of Γ, which connects points x_i and x_{i+1} and contains no other points from ω_N. Define

$$\delta(\omega_N) := \min_{1 \le i \ne j \le N} |x_i - x_j|$$

and let

$$\Delta(\omega_N) := \max_{i=1,\dots,N} l_i(\omega_N).$$

It is known (cf. [5,29]) that for any rectifiable curve $\Gamma \subset \mathbb{R}^d$ of positive length (closed or non-closed) and $s > 1$, there is a positive constant $C = C(s, \Gamma)$ such that

$$\delta(\omega_N^*) \ge \frac{C}{N} \tag{4.7}$$

for every N sufficiently large and s-energy minimizing N-point configuration ω_N^* on Γ. For estimates analogous to (4.7) on the sphere and m-dimensional rectifiable sets in $\mathbb{R}^d$, see [5, 12, 14, 19, 26, 27] and references therein.

It is also known that for any sequence $\omega_N^* = \{x_{1,N}, \dots, x_{N,N}\}$, $N \in \mathbb{N}$, of asymptotically s-energy minimizing configurations on a rectifiable Jordan arc Γ, there holds (cf. [29])

$$\lim_{N\to\infty} \sum_{i=1}^{N} \left| l_i(\omega_N^*) - \frac{|\Gamma|}{N} \right| = 0, \quad s > 1. \tag{4.8}$$

In the case $s = 0$ sharp estimates of the discrepancy between the equilibrium measure and the normalized counting measure supported on minimal energy configurations were obtained in [23, 33] on simple closed curves of smoothness $C^{3,\epsilon}$ on the plane. We obtain the following result.

Proposition 4.4 *Let $s \ge 2$ and $\Gamma \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be a closed C^3 arc. If $\{\omega_N^*\}_{N=2}^{\infty}$ is a sequence of s-energy minimizing collections on Γ such that $\#\omega_N^* = N$, $N \ge 2$, then*

$$\lim_{N\to\infty} \delta(\omega_N^*) \cdot N = \lim_{N\to\infty} \Delta(\omega_N^*) \cdot N = |\Gamma|. \tag{4.9}$$

For closed C^3 arcs, relation (4.9) implies that the distance between any two neighboring points in optimal configurations is asymptotically of size L/N, which does not follow from a more general relation (4.8).

5 Comparison of the Minimal Energy Behavior on Closed and Non-Closed Smooth Arcs

Denote

$$\rho_s(N) := \begin{cases} N^s, & s > 2, \\ N^2 \ln N, & s = 2, \\ N^2, & 1 \le s < 2, \end{cases}$$

and let $L := |\Gamma|$.

Remark 5.1 Let $\Gamma \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be a non-closed C^2 arc. If $s \geq 2$, there exist two negative constants C_1, C_2 such that for every N sufficiently large

$$C_1 \rho_s(N) < \mathcal{E}_s(\Gamma, N) - 2\zeta(s) L^{-s} N^{s+1} < C_2 \rho_s(N). \tag{5.1}$$

When $1 < s < 2$, we have

$$\mathcal{E}_s(\Gamma, N) - 2\zeta(s) L^{-s} N^{s+1} = O(N^2) \tag{5.2}$$

and

$$\mathcal{E}_1(\Gamma, N) - 2L^{-1} N^2 \ln N = O(N^2). \tag{5.3}$$

By Theorem 4.1, on closed C^3 arcs, the next order term is non-negative at least for $s \geq 2$, while for non-closed C^2 arcs it is negative. Hence, turning a smooth non-closed C^3-arc into a closed C^3-arc of the same length increases its s-energy at least for $s \geq 2$ and N sufficiently large. Moreover, for closed C^3 arcs, the absolute value of the next order term turns out to have a lower order of growth than for non-closed ones.

Remark 5.2 As we see from (4.1) and (4.2), and from relation

$$\mathcal{E}_s(\Gamma, N) \asymp N^2, \quad N \to \infty, \quad 0 < s < 1$$

(cf. [28, Section II.3.12]), the order of the main term of $\mathcal{E}_s(\Gamma, N)$ on rectifiable arcs changes as s passes through value 1. Similar transition for the next order term takes place when $s = 3$ for closed C^3 arcs and when $s = 2$ for non-closed C^2 arcs.

Remark 5.3 For any non-closed C^2 arc in $\mathbb{R}^d$ of length L, we have

$$\lim_{N \to \infty} \frac{\mathcal{E}_s(\Gamma, N) - \mathcal{E}_s([0, L], N)}{\rho_s(N)} = 0, \quad s \geq 2, \tag{5.4}$$

and

$$\lim_{N \to \infty} \frac{\mathcal{E}_s(\Gamma, N) - \mathcal{E}_s([0, L], N)}{N^2} = \Phi_s(\Gamma), \quad 1 \leq s < 2. \tag{5.5}$$

If one bends a sufficiently smooth non-closed arc Γ preserving its length, in view of relations (5.1) and (5.4), the second term in the asymptotics of $\mathcal{E}_s(\Gamma, N)$ for $s \geq 2$ will not be affected.

Proposition 5.4 *Let Γ be a non-closed C^2 arc in $\mathbb{R}^d$ and let $\{\overline{\omega}_N\}_{N \in \mathbb{N}}$ be a sequence of configurations of N equally spaced points on Γ with respect to the arc length (we assume that each configuration contains both endpoints of Γ). Then for $s > 2$, we have*

$$\limsup_{N \to \infty} \frac{\mathcal{E}_s(\Gamma, N) - E_s(\overline{\omega}_N)}{N^s} < 0.$$

Our proofs imply that the energy of equally spaced configurations on a closed C^3 arc Γ have the same next order term as $\mathcal{E}_s(\Gamma, N)$, $s \geq 1$. According to the above statement, this is not the case when Γ is non-closed and $s > 2$.

Proofs of Remarks 5.1 and 5.3 and of Proposition 5.4 are given in Appendix A.

6 Auxiliary Statements

For an N-point collection $\omega_N = \{x_1, \dots, x_N\} \subset \Gamma$, we denote

$$F_s(\omega_N) := \sum_{1 \le i \ne j \le N} \frac{1}{L(x_i, x_j)^s}.$$

Then we can write

$$G_s(\omega_N) := \sum_{1 \le i \ne j \le N} g_s(x_i, x_j) = E_s(\omega_N) - F_s(\omega_N).$$

Throughout the remainder of the paper $\overline{\omega}_N = \{\overline{x}_1, \dots, \overline{x}_N\}$ will denote a collection of equally spaced points on a simple rectifiable curve $\Gamma \subset \mathbb{R}^d$, that is, an N-point collection such that $l_i(\overline{\omega}_N) = L/N$, $i = 1, \dots, N$, when Γ is closed, and the collection such that $l_i(\overline{\omega}_N) = L/(N-1)$, $i = 1, \dots, N-1$, when Γ is non-closed. Configuration $\overline{\omega}_N$ will be optimal on closed arcs in the following sense.

Lemma 6.1 *Let $\Gamma \subset \mathbb{R}^d$ be a simple closed rectifiable curve and $s > 0$. Then*

$$F_s(\omega_N) \ge F_s(\overline{\omega}_N)$$

for every N-point configuration $\omega_N = \{x_1, \dots, x_N\} \subset \Gamma$.

For a function $\alpha \colon \Gamma \times \Gamma \to \mathbb{R}$, we write $\alpha(x, y) \rightrightarrows 0$ as $L(x, y) \to 0$, if for every $\epsilon > 0$, there is $\delta > 0$ such that $|\alpha(x, y)| < \epsilon$ whenever $0 < L(x, y) < \delta$.

Lemma 6.2 *Let $s > 0$. If $\Gamma \subset \mathbb{R}^d$ be a closed C^3 arc, then*

$$g_s(x, y) = \frac{s \cdot \kappa^2(y)}{24} L(x, y)^{2-s} + \alpha(x, y) L(x, y)^{2-s}, \quad x, y \in \Gamma, \quad x \ne y, \tag{6.1}$$

where $\alpha(x, y) \rightrightarrows 0$ as $L(x, y) \to 0$. If Γ is a non-closed C^2 arc, then

$$g_s(x, y) = \gamma(x, y) L(x, y)^{1-s}, \quad x, y \in \Gamma, \quad x \ne y, \tag{6.2}$$

where $\gamma(x, y) \rightrightarrows 0$ as $L(x, y) \to 0$.

Let δ_x be the atomic probability measure in $\mathbb{R}^d$ centered at point x and $\omega_N := \{x_{1,N}, \dots, x_{N,N}\} \subset \Gamma$, $N \in \mathbb{N}$, be a sequence of N-point sets. Denote by

$$\nu(\omega_N) := \frac{1}{N} \sum_{k=1}^{N} \delta_{x_{k,N}}$$

the normalized counting measure supported at points of ω_N. We write

$$\nu(\omega_N) \xrightarrow{*} \lambda_\Gamma, \quad N \to \infty,$$

if for every continuous function $f \colon \Gamma \to \mathbb{R}$

$$\frac{1}{N} \sum_{k=1}^{N} f(x_{k,N}) \to \int_\Gamma f(x) d\lambda_\Gamma, \quad N \to \infty.$$

Paper [29] establishes the following statement.

Lemma 6.3 *Let $s \geq 1$ and $\Gamma = \bigcup_{j=1}^{m} \Gamma_j$, where each Γ_j is a rectifiable Jordan arc, and $|\Gamma| = \sum_{j=1}^{m} |\Gamma_j|$. If $\{\omega_N^*\}_{N=2}^{\infty}$ is a sequence of s-energy minimizing configurations on Γ such that $\#\omega_N^* = N$, $N \geq 2$, then $\nu(\omega_N^*) \xrightarrow{*} \lambda_\Gamma$, $N \to \infty$.*

We remark that for $s > 1$, this result as well as relations (4.1) and (4.7) were later extended in [5] to finite unions of arbitrary rectifiable curves.

Lemma 6.4 *Let $\Gamma \subset \mathbb{R}^d$ be a non-closed C^2 arc and $1 \leq s < 2$ or a closed C^3 arc and $1 \leq s < 3$. Assume that $\{\omega_N\}_{N=2}^{\infty}$ is a sequence of N-point configurations on Γ such that $\nu(\omega_N) \xrightarrow{*} \lambda_\Gamma$, $N \to \infty$, and when $s > 1$, there is a constant $C > 0$ such that for N sufficiently large, $\delta(\omega_N) \geq CN^{-1}$. Then*

$$\lim_{N\to\infty} \frac{G_s(\omega_N)}{N^2} = \Phi_s(\Gamma). \tag{6.3}$$

7 Proofs of Auxiliary Statements

Proof of Lemma 6.1 For every $i = -N+1, \dots, 0$, denote $x_i := x_{i+N}$ and let $x_i := x_{i-N}$ for every $i = N+1, \dots, 2N$. The notation $l_i(\omega_N)$ is extended correspondingly. Let $[t]$ be the floor function of a number t. Then

$$
\begin{aligned}
F_s(\omega_N) &= \sum_{1\leq i\neq j\leq N} L(x_i, x_j)^{-s} = \sum_{\substack{j=-[(N-1)/2]\\ j\neq 0}}^{[N/2]} \sum_{k=1}^{N} L(x_k, x_{k+j})^{-s} \\
&\geq N \sum_{\substack{j=-[(N-1)/2]\\ j\neq 0}}^{[N/2]} \left(\frac{1}{N} \sum_{k=1}^{N} L(x_k, x_{k+j}) \right)^{-s} \\
&\geq N \sum_{j=1}^{[N/2]} \left(\frac{1}{N} \sum_{k=1}^{N} \sum_{i=1}^{j} l_{k+i-1}(\omega_N) \right)^{-s} \\
&\quad + N \sum_{j=1}^{[(N-1)/2]} \left(\frac{1}{N} \sum_{k=1}^{N} \sum_{i=1}^{j} l_{k-i}(\omega_N) \right)^{-s} \\
&= N \sum_{\substack{j=-[(N-1)/2]\\ j\neq 0}}^{[N/2]} \left(\frac{|j|L}{N} \right)^{-s} = F_s(\overline{\omega}_N).
\end{aligned}
$$

Lemma 6.1 is proved.

Proof of Lemma 6.2 Let an L-periodic function $\varphi\colon \mathbb{R} \to \mathbb{R}^d$ be the arc length

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parametrization for Γ. Then

$$\varphi(t_1)-\varphi(t_2)=(t_1-t_2)\varphi'(t_2)+\frac{(t_1-t_2)^2}{2}\varphi''(t_2)$$
$$+\frac{(t_1-t_2)^3}{6}\varphi'''(t_2)+(t_1-t_2)^3\theta(t_1,t_2),$$

where $\theta(t_1,t_2)\rightrightarrows 0$ as $|t_1-t_2|\to 0$ in view of uniform continuity of φ'''. Since $|\varphi'|\equiv 1$, we get $\frac{\mathrm{d}}{\mathrm{d}t}|\varphi'|^2=2\langle\varphi',\varphi''\rangle=0$ and $|\varphi''|^2=-\langle\varphi',\varphi'''\rangle$. Hence,

$$|\varphi(t_1)-\varphi(t_2)|^2=(t_1-t_2)^2-\frac{(t_1-t_2)^4}{12}|\varphi''(t_2)|^2+(t_1-t_2)^4\beta(t_1,t_2),$$

where $\beta(t_1,t_2)\rightrightarrows 0$ as $|t_1-t_2|\to 0$. Then

$$|\varphi(t_1)-\varphi(t_2)|^{-s}=|t_1-t_2|^{-s}\Big[1+\frac{s(t_1-t_2)^2}{24}|\varphi''(t_2)|^2+(t_1-t_2)^2\gamma(t_1,t_2)\Big],$$

where $\gamma(t_1,t_2)\rightrightarrows 0$ as $|t_1-t_2|\to 0$, and (6.1) follows. Relation (6.2) is proved analogously.

Proof of Lemma 6.4 Assume that Γ is a closed C^3 arc and choose arbitrary $\epsilon\in(0,L/2)$. Let

$$U_\epsilon=\{(x,y)\in\Gamma\times\Gamma : L(x,y)\geq\epsilon\}$$

and

$$V_\epsilon=\{(x,y)\in\Gamma\times\Gamma : L(x,y)\leq\epsilon\}.$$

It is not difficult to see that the boundary of U_ϵ relative to $\Gamma\times\Gamma$ has $\lambda_\Gamma\times\lambda_\Gamma$-measure zero, function g_s is continuous in a neighborhood of U_ϵ relative to $\Gamma\times\Gamma$, and $\nu(\omega_N)\times\nu(\omega_N)\xrightarrow{*}\lambda_\Gamma\times\lambda_\Gamma$, $N\to\infty$. Then since $g_s(x,y)\geq 0$, we will obtain

$$\liminf_{N\to\infty}\frac{G_s(\omega_N)}{N^2}=\liminf_{N\to\infty}\frac{1}{N^2}\sum_{1\leq i\neq j\leq N}g_s(x_i,x_j)\geq\lim_{N\to\infty}\frac{1}{N^2}\sum_{L(x_i,x_j)\geq\epsilon}g_s(x_i,x_j)$$
$$=\int_{U_\epsilon}g_s(x,y)d\lambda_\Gamma\times\lambda_\Gamma.$$

Hence, in view of arbitrariness of ϵ, we get

$$\liminf_{N\to\infty}\frac{G_s(\omega_N)}{N^2}\geq\Phi_s(\Gamma). \tag{7.1}$$

Choose again $\epsilon\in(0,L/2)$ and let $\delta\in(0,\epsilon)$ be as in the definition of $\alpha(x,y)\rightrightarrows 0$, $L(x,y)\to 0$ in (6.1). We have

$$G_s(\omega_N)=\sum_{0<L(x_i,x_j)<\delta}g_s(x_i,x_j)+\sum_{L(x_i,x_j)\geq\delta}g_s(x_i,x_j). \tag{7.2}$$

It is not difficult to see that

$$\lim_{N\to\infty}\frac{1}{N^2}\sum_{L(x_i,x_j)\geq\delta} g_s(x_i,x_j)=\int_{U_\delta} g_s(x,y)\,d\lambda_\Gamma\times\lambda_\Gamma\leq\Phi_s(\Gamma). \tag{7.3}$$

By (6.1), for every $s>0$ there are constants $M_s>0$ and $\delta_s\in(0,L/2)$ such that

$$g_s(x,y)\leq M_s\cdot L(x,y)^{2-s},\ \text{ whenever } x,y\in\Gamma,\ \text{ and } 0<L(x,y)<\delta_s. \tag{7.4}$$

We can assume that $\delta\in(0,\delta_s)$. Using (7.4) and the fact that $\delta<\epsilon$, we have

$$\sum_{0<L(x_i,x_j)<\delta} g_s(x_i,x_j)\leq M_s\sum_{0<L(x_i,x_j)\leq\epsilon} L(x_i,x_j)^{2-s}. \tag{7.5}$$

For $1\leq s<2$, we obtain

$$\begin{aligned}\limsup_{N\to\infty}\frac{1}{N^2}\sum_{0<L(x_i,x_j)<\delta} g_s(x_i,x_j)&\leq M_s\limsup_{N\to\infty}\frac{1}{N^2}\sum_{0<L(x_i,x_j)\leq\epsilon} L(x_i,x_j)^{2-s}\\&\leq M_s\epsilon^{2-s}\limsup_{N\to\infty}\frac{\#\big((\omega_N\times\omega_N)\cap V_\epsilon\big)}{N^2}\\&\leq M_s\epsilon^{2-s}=:\mu_s(\epsilon).\end{aligned} \tag{7.6}$$

It is not difficult to verify that

$$\sum_{k=1}^{N} k^a=\frac{N^{a+1}(1+o(1))}{a+1},\quad a>-1 \tag{7.7}$$

(in what follows, notations $o(\,\cdot\,)$ and $O(\,\cdot\,)$ will be used only for $N\to\infty$).

Let $2\leq s<3$. By assumption, for N sufficiently large, whenever $0<L(x_i,x_{i+k})\leq\epsilon$ and $k>0$, we have

$$L(x_i,x_{i+k})=\sum_{j=1}^{k} L(x_{i+j-1},x_{i+j})\geq k\delta(\omega_N)\geq kCN^{-1}.$$

Then from (7.5) and (7.7) we obtain

$$\begin{aligned}\sum_{0<L(x_i,x_j)<\delta} g_s(x_i,x_j)&\leq 2M_s\sum_{i=1}^{N}\sum_{\substack{j=1\\0<L(x_i,x_{i+j})\leq\epsilon}}^{[N/2]} L(x_i,x_{i+j})^{2-s}\\&\leq\frac{2M_sN^{s-1}}{C^{s-2}}\sum_{k=1}^{[\epsilon N/C]} k^{2-s}=\frac{2M_s\epsilon^{3-s}N^2(1+o(1))}{C(3-s)}\\&=:\mu_s(\epsilon)N^2(1+o(1)).\end{aligned} \tag{7.8}$$

Thus for any $1 \le s < 3$, from (7.6) and (7.8) we obtain

$$\limsup_{N\to\infty} \frac{1}{N^2} \sum_{0<L(x_i,x_j)<\delta} g_s(x_i,x_j) \le \mu_s(\epsilon), \tag{7.9}$$

where $\mu_s(\epsilon) \to 0$, $\epsilon \to 0$. Combining (7.2), (7.3), and (7.9) we will have

$$\limsup_{N\to\infty} \frac{G_s(\omega_N)}{N^2} \le \mu_s(\epsilon) + \Phi_s(\Gamma).$$

Letting $\epsilon \to 0$ and taking into account (7.1), we get (6.3).

In order to obtain (6.3) for a non-closed C^2 arc Γ, we use (6.2). If $s = 1$, we repeat the argument for closed C^3 arcs and $1 \le s < 2$. If $1 < s < 2$, we repeat the argument for closed C^3 arcs and $2 \le s < 3$.

8 Proofs of the Results on Closed Smooth Arcs

Let $\Gamma \subset \mathbb{R}^d$ be a closed C^3 arc. Recall that $\overline{\omega}_N = \{\overline{x}_1, \dots, \overline{x}_N\}$ denotes a collection of N equally spaced points on Γ and ω_N^* is the s-energy minimizing N-point collection on Γ. To prove Theorem 4.1 we compare the minimal energy of Γ to the minimal energy with respect to the arc length distance of the circle of length $|\Gamma|$. For $s \ge 1$, we have

$$\mathcal{E}_s(\Gamma, N) - F_s(\overline{\omega}_N) \le E_s(\overline{\omega}_N) - F_s(\overline{\omega}_N) = G_s(\overline{\omega}_N),$$

and by Lemma 6.1,

$$\mathcal{E}_s(\Gamma, N) - F_s(\overline{\omega}_N) = E_s(\omega_N^*) - F_s(\overline{\omega}_N) \ge E_s(\omega_N^*) - F_s(\omega_N^*) = G_s(\omega_N^*).$$

Thus,

$$G_s(\omega_N^*) \le \mathcal{E}_s(\Gamma, N) - F_s(\overline{\omega}_N) \le G_s(\overline{\omega}_N). \tag{8.1}$$

For convenience, let

$$\sigma(s) = \begin{cases} \zeta(s), & s > 1, \\ 1, & s = 1. \end{cases}$$

Lemma 8.1 *For any closed C^3 arc $\Gamma \subset \mathbb{R}^d$, we have*

$$\lim_{N\to\infty} \frac{\mathcal{E}_s(\Gamma, N) - F_s(\overline{\omega}_N)}{N^2} = \Phi_s(\Gamma), \quad 1 \le s < 3, \tag{8.2}$$

and

$$\lim_{N\to\infty} \frac{\mathcal{E}_s(\Gamma, N) - F_s(\overline{\omega}_N)}{\rho_{s-1}(N)} = \frac{s\sigma(s-2)\kappa(\Gamma)}{12L^{s-2}}, \quad s \ge 3. \tag{8.3}$$

Proof Let $1 \le s < 3$. By Lemma 6.3, $\nu(\omega_N^*) \xrightarrow{*} \lambda_\Gamma$, $N \to \infty$, and, clearly, $\nu(\overline{\omega}_N) \xrightarrow{*} \lambda_\Gamma$, $N \to \infty$. Sequence $\{\overline{\omega}_N\}_{N=2}^\infty$ and, in view of relation (4.7), sequence $\{\omega_N^*\}_{N=2}^\infty$ both satisfy the assumptions of Lemma 6.4. Hence,

$$\lim_{N\to\infty} \frac{G_s(\omega_N^*)}{N^2} = \lim_{N\to\infty} \frac{G_s(\overline{\omega}_N)}{N^2} = \Phi_s(\Gamma). \tag{8.4}$$

Then using (8.1) we get relation (8.2).

Upper estimate in (8.3) Choose again any $\epsilon > 0$ and let $\delta \in (0, \epsilon)$ be chosen from the definiton of $\alpha(x, y) \rightrightarrows 0$, $L(x, y) \to 0$ in Lemma 6.2. Then

$$G_s(\overline{\omega}_N) = \sum_{0<L(\overline{x}_i,\overline{x}_j)<\delta} g_s(\overline{x}_i, \overline{x}_j) + \sum_{L(\overline{x}_i,\overline{x}_j)\ge\delta} g_s(\overline{x}_i, \overline{x}_j).$$

Function $g_s(x, y)$ is bounded as a continuous function on a compact set U_δ. Then

$$\sum_{L(\overline{x}_i,\overline{x}_j)\ge\delta} g_s(\overline{x}_i, \overline{x}_j) = O(N^2). \tag{8.5}$$

Taking into account Lemma 6.2 and representation

$$\sum_{k=1}^{[\delta N/L]} k^{-s} = \begin{cases} \zeta(s)(1 + o(1)), & s > 1, \\ \ln N(1 + o(1)), & s = 1, \end{cases}$$

we have

$$\begin{aligned} \sum_{0<L(\overline{x}_i,\overline{x}_j)<\delta} g_s(\overline{x}_i, \overline{x}_j) &\le \sum_{0<L(\overline{x}_i,\overline{x}_j)<\delta} \Big(\epsilon + \frac{s}{24}\kappa^2(\overline{x}_j)\Big) L(\overline{x}_i, \overline{x}_j)^{2-s} \\ &\le 2\sum_{j=1}^{N} \Big(\epsilon + \frac{s}{24}\kappa^2(\overline{x}_j)\Big) \sum_{i=1}^{[\delta N/L]} \Big(\frac{iL}{N}\Big)^{2-s} \\ &= \frac{2N^{s-1}}{L^{s-2}} \int_\Gamma \Big(\epsilon + \frac{s}{24}\kappa^2(x)\Big)\, d\lambda_\Gamma \cdot (1 + o(1)) \sum_{i=1}^{[\delta N/L]} i^{2-s} \\ &= \frac{2\sigma(s-2)}{L^{s-2}} \int_\Gamma \Big(\epsilon + \frac{s\kappa^2(x)}{24}\Big)\, d\lambda_\Gamma \cdot \rho_{s-1}(N)\big(1 + o(1)\big). \end{aligned} \tag{8.6}$$

Letting $N \to \infty$, from (8.5) and (8.6) we have

$$\limsup_{N\to\infty} \frac{G_s(\overline{\omega}_N)}{\rho_{s-1}(N)} \le \frac{2\sigma(s-2)}{L^{s-2}} \int_\Gamma \Big(\epsilon + \frac{s\kappa^2(x)}{24}\Big)\, d\lambda_\Gamma.$$

Letting $\epsilon \to 0$, we finally have

$$\limsup_{N\to\infty} \frac{G_s(\overline{\omega}_N)}{\rho_{s-1}(N)} \le \frac{s\sigma(s-2)\kappa(\Gamma)}{12L^{s-2}}. \tag{8.7}$$

We next use this inequality to obtain Proposition 4.4, which in turn is used to prove the lower estimate in (8.3).

Proof of Proposition 4.4 Let $s \geq 2$ and $\{\omega_N^*\}_{N=2}^{\infty}$ be a sequence of s-energy minimizing configurations on Γ, where $\omega_N^* = \{x_{1,N}, \dots, x_{N,N}\}$, $N \geq 2$. Recall that in our notation $x_{i-N,N} = x_{i,N}$, $x_{i+N,N} = x_{i,N}$, $i = 1, \dots, N$, and $l_i(\omega_N^*)$ is the length of the arc of Γ connecting points $x_{i,N}$ and $x_{i+1,N}$, which contains no other points from ω_N^*. Let $\{(x_{j_N,N}, x_{j_N+1,N})\}_{N=2}^{\infty}$ be any sequence of pairs of neighboring points from ω_N^*. Denote $C_N := l_{j_N}(\omega_N^*) \cdot N$. We want to show that $\lim_{N\to\infty} C_N = L$. Let $\mathcal{N} \subset \mathbb{N} \setminus \{1\}$ be any infinite set such that the limit $a := \lim_{\mathcal{N}\ni N\to\infty} C_N$ exists as a finite number or equals infinity. For every $N \in \mathcal{N}$, we have

$$
\begin{aligned}
\mathcal{E}_s(\Gamma, N) = E_s(\omega_N^*) &= \sum_{\substack{k=-[(N-1)/2]\\ k\neq 0}}^{[N/2]} \sum_{i=1}^{N} L(x_i, x_{i+k})^{-s} + G_s(\omega_N^*) \\
&\geq \sum_{k=1}^{[(N-1)/2]} \sum_{i=1}^{N} \Big(\sum_{j=0}^{k-1} l_{i-k+j}(\omega_N^*) \Big)^{-s} + \sum_{k=2}^{[N/2]} \sum_{i=1}^{N} \Big(\sum_{j=0}^{k-1} l_{i+j}(\omega_N^*) \Big)^{-s} \\
&\quad + \big(l_{j_N}(\omega_N^*) \big)^{-s} + \sum_{i=1,\ i\neq j_N}^{N} \big(l_i(\omega_N^*) \big)^{-s} + G_s(\omega_N^*).
\end{aligned}
$$

Then using convexity of the function $y(t) = t^{-s}$, $s > 0$, we have

$$
\begin{aligned}
&\mathcal{E}_s(\Gamma, N) - G_s(\omega_N^*) \\
&\quad \geq \sum_{k=1}^{[(N-1)/2]} N^{s+1} \Big(\sum_{i=1}^{N} \sum_{j=0}^{k-1} l_{i-k+j}(\omega_N^*) \Big)^{-s} + \sum_{k=2}^{[N/2]} N^{s+1} \Big(\sum_{i=1}^{N} \sum_{j=0}^{k-1} l_{i+j}(\omega_N^*) \Big)^{-s} \\
&\quad\quad + N^s C_N^{-s} + (N-1)^{s+1} \Big(\sum_{\substack{i=1\\ i\neq j_N}}^{N} l_i(\omega_N^*) \Big)^{-s} \\
&\quad \geq N^{s+1} \sum_{\substack{k=-[(N-1)/2]\\ k\neq 0,1}}^{[N/2]} (|k|L)^{-s} + N^s C_N^{-s} + (N-1)^{s+1} \Big(L - \frac{C_N}{N} \Big)^{-s} \\
&\quad = F_s(\overline{\omega}_N) - L^{-s} N^{s+1} + N^s C_N^{-s} + L^{-s} N^{s+1} \Big(1 - \frac{1}{N} \Big)^{s+1} \Big(1 - \frac{C_N}{LN} \Big)^{-s}.
\end{aligned}
$$

It is not difficult to see that for $b \geq 1$ and $x \geq -1/b$ or for $b < 0$ and $-1 < x \leq -1/b$, we have

$$
(1+x)^b \geq 1 + bx \geq 0.
$$

Applying this inequality with $b = s+1$ and $b = -s$ and noting that $l_{j_N}(\omega_N^*) = C_N/N \to 0$, $N \to \infty$ (*e.g.,* in view of Lemma 6.3), we can write for N sufficiently large

$$\begin{aligned}
(8.8)\qquad & \mathcal{E}_s(\Gamma, N) - F_s(\overline{\omega}_N) - G_s(\omega_N^*) \\
&= -L^{-s}N^{s+1} + N^s C_N^{-s} + \frac{N^{s+1}}{L^s}\Big(1 - \frac{s+1}{N}\Big)\Big(1 + \frac{sC_N}{LN}\Big) \\
&\geq N^s C_N^{-s} + \frac{sC_N N^s}{L^{s+1}} - \frac{(s+1)N^s}{L^s} + o(N^s).
\end{aligned}$$

On one hand, by (8.1) we have

$$\begin{aligned}
\tau_s(\Gamma) := \limsup_{\mathcal{N}\ni N\to\infty} \frac{\mathcal{E}_s(\Gamma, N) - F_s(\overline{\omega}_N) - G_s(\omega_N^*)}{N^s} \\
\leq \limsup_{\mathcal{N}\ni N\to\infty} \frac{G_s(\overline{\omega}_N) - G_s(\omega_N^*)}{N^s}.
\end{aligned}$$

If $s \geq 3$, in view of non-negativity of $G_s(\omega_N^*)$ and relation (8.7), we have

$$(8.9)\qquad \tau_s(\Gamma) \leq \limsup_{N\to\infty} \frac{G_s(\overline{\omega}_N)}{N^s} = \lim_{N\to\infty} \frac{G_s(\overline{\omega}_N)}{\rho_{s-1}(N)} \cdot \frac{\rho_{s-1}(N)}{N^s} = 0.$$

If $2 \leq s < 3$, then from (8.4) we have

$$(8.10)\qquad \tau_s(\Gamma) \leq \lim_{N\to\infty} \frac{G_s(\overline{\omega}_N) - G_s(\omega_N^*)}{N^2} \cdot \frac{N^2}{N^s} = 0.$$

On the other hand, from (8.8) we have

$$0 \geq \tau_s(\Gamma) \geq a^{-s} + \frac{sa}{L^{s+1}} - \frac{s+1}{L^s} =: f(a).$$

Function $f(a)$ has a unique global minimum $f(L) = 0$ on $[0, \infty]$. Then, in view of (8.9) and (8.10), we can only have $a = L$. In view of the arbitrariness of the subsequence $\{C_N\}_{N\in\mathcal{N}}$ we have

$$(8.11)\qquad \lim_{N\to\infty} C_N = \lim_{N\to\infty} l_{j_N}(\omega_N^*) \cdot N = L.$$

Recall that sequence of indexes $\{j_N\}$, $1 \leq j_N \leq N$, was chosen arbitrarily. Taking now $\{j_N\}$ so that $l_{j_N}(\omega_N^*) = \Delta(\omega_N^*)$, $N \geq 2$, we get the second equality in (4.9).

It is known that uniformly over $x, y \in \Gamma$

$$(8.12)\qquad \lim_{|x-y|\to 0} \frac{|x-y|}{L(x,y)} = 1.$$

Let now $\{j_N\}$ and another sequence of indexes $\{p_N\}$ be such that $j_N < p_N$,

$$\delta(\omega_N^*) = |x_{j_N,N} - x_{p_N,N}| \quad \text{and} \quad L(x_{j_N,N}, x_{p_N,N}) = \sum_{k=j_N}^{p_N-1} l_k(\omega_N^*), \qquad N \geq 2$$

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(p_N can be greater than N). Then

$$\frac{\delta(\omega_N^*)\cdot N}{L} \le \frac{N}{L}\min_{i=1,\dots,N} l_i(\omega_N^*) \le 1.$$

Since $\delta(\omega_N^*) \le \Delta(\omega_N^*)$ and, by the second equality in (4.9), $\Delta(\omega_N^*) \to 0$, $N \to \infty$, we also have $|x_{j_N,N} - x_{p_N,N}| \to 0$, $N \to \infty$. In view of (8.11) and (8.12), we get $L(x_{j_N,N}, x_{p_N,N}) \ge l_{j_N}(\omega_N^*) = LN^{-1}(1+o(1))$, and

$$\frac{\delta(\omega_N^*)N}{L} \ge \frac{\delta(\omega_N^*)(1+o(1))}{L(x_{j_N,N}, x_{p_N,N})} = \frac{|x_{j_N,N} - x_{p_N,N}|}{L(x_{j_N,N}, x_{p_N,N})}\big(1+o(1)\big) = 1+o(1).$$

Thus,

$$\delta(\omega_N^*) = \frac{L}{N}\big(1+o(1)\big),$$

and we get the first equality in (4.9). Proposition 4.4 is proved.

Lower estimate in (8.3) Let $\{\omega_N^*\}_{N=2}^{\infty}$, $\omega_N^* = \{x_{1,N},\dots,x_{N,N}\}$, $N \ge 2$, be a sequence of s-energy minimizing configurations on Γ.

Choose any $\epsilon > 0$ and take $0 < h < \min\{\epsilon, L/4\}$ so that $|\alpha(x,y)| \le \frac{s\epsilon}{24}$ whenever $0 < L(x,y) < h$ in (6.1). Proposition 4.4 implies that $\Delta(\omega_N^*) < 2L/N$ for every N sufficiently large. Then, for every N large and $1 \le k \le m_N := [hN/(2L)]$, we have

$$L(x_{i,N}, x_{i+k,N}) \le \sum_{j=0}^{k-1} l_{i+j}(\omega_N^*) \le k\Delta(\omega_N^*) < \frac{2kL}{N} \le h. \tag{8.13}$$

Hence, by Lemma 6.2

$$\begin{aligned} G_s(\omega_N^*) &\ge 2\sum_{k=1}^{m_N}\sum_{i=1}^{N} g_s(x_{i,N}, x_{i+k,N}) \\ &\ge \frac{s}{12}\sum_{k=1}^{m_N}\sum_{i=1}^{N}\big(\kappa^2(x_{i,N}) - \epsilon\big)\, L(x_{i,N}, x_{i+k,N})^{2-s}. \end{aligned}$$

For every N large and $1 \le k \le m_N$, since $\sum_{j=0}^{k-1} l_{i+j}(\omega_N^*) < L/2$, we have

$$L(x_{i,N}, x_{i+k,N}) = \sum_{j=0}^{k-1} l_{i+j}(\omega_N^*) \ge k\delta(\omega_N^*),$$

and in view of (8.13), Lemma 6.3, and Proposition 4.4 we get

$$\begin{aligned} G_s(\omega_N^*) &\ge \frac{s}{12}\sum_{k=1}^{m_N}\sum_{i=1}^{N}\kappa^2(x_{i,N})\cdot\big(k\Delta(\omega_N^*)\big)^{2-s} - \frac{\epsilon s}{12}\sum_{k=1}^{m_N}\sum_{i=1}^{N}\big(k\delta(\omega_N^*)\big)^{2-s} \\ &= \frac{sN}{12}\kappa(\Gamma)\big(1+o(1)\big)\,\Delta(\omega_N^*)^{2-s}\sum_{k=1}^{m_N} k^{2-s} - \frac{\epsilon sN}{12}\delta(\omega_N^*)^{2-s}\sum_{k=1}^{m_N} k^{2-s} \\ &= \frac{s\sigma(s-2)(\kappa(\Gamma)-\epsilon)}{12L^{s-2}}\rho_{s-1}(N)\big(1+o(1)\big). \end{aligned}$$

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Then

$$\liminf_{N\to\infty} \frac{G_s(\omega_N^*)}{\rho_{s-1}(N)} \geq \frac{s\sigma(s-2)(\kappa(\Gamma)-\epsilon)}{12L^{s-2}}. \tag{8.14}$$

Letting $\epsilon \to 0$ in (8.14) and combining it with (8.7) and (8.1), we obtain (8.3). Lemma 8.1 is proved.

Proof of Theorem 4.1 Using known representation

$$\sum_{k=1}^{N} \frac{1}{k^s} = \zeta(s) - \frac{1}{s-1} \cdot \frac{1}{N^{s-1}} + o\Big(\frac{1}{N^{s-1}}\Big), \quad s > 1, \tag{8.15}$$

one can show that

$$\begin{aligned} F_s(\overline{\omega}_N) &= 2L^{-s}N^{s+1} \sum_{k=1}^{[N/2]} k^{-s} + O(N) \\ &= 2\zeta(s)L^{-s}N^{s+1} - \frac{2^s N^2}{(s-1)L^s} + o(N^2), \quad s > 1. \end{aligned} \tag{8.16}$$

Taking into account equality

$$\sum_{k=1}^{N} \frac{1}{k} = \ln N + \gamma + o(1), \tag{8.17}$$

where γ is the Euler–Mascheroni constant, one can also derive that

$$F_1(\overline{\omega}_N) = 2L^{-1}N^2 \ln N + 2L^{-1}(\gamma - \ln 2)N^2 + o(N^2). \tag{8.18}$$

If $1 < s < 3$, applying relation (8.2) and representation (8.16), we get (4.5). When $s = 1$, taking into account relation (8.2) with $s = 1$ and equality (8.18) we have (4.6). If $s \geq 3$, taking into account representation (8.16), relation (8.3), and the definition of the function $\sigma(s)$, we get (4.3) and (4.4). Theorem 4.1 is proved.

A Appendix

Using (8.15) and (8.17), it is not difficult to verify the following statement.

Lemma A.1 *Let* $\overline{\omega}_N = \{0, \frac{L}{N-1}, \dots, \frac{(N-2)L}{N-1}, L\} \subset [0, L]$. *Then*

$$E_s(\overline{\omega}_N) = 2\zeta(s)L^{-s}N^{s+1} - 2(\zeta(s-1) + s\zeta(s))L^{-s}N^s + o(N^s), \quad s > 2,$$

$$E_2(\overline{\omega}_N) = 2\zeta(2)L^{-2}N^3 - 2L^{-2}N^2 \ln N + O(N^2),$$

$$E_s(\overline{\omega}_N) = 2\zeta(s)L^{-s}N^{s+1} - \frac{2L^{-s}N^2}{(s-1)(2-s)} + o(N^2), \quad 1 < s < 2,$$

and

$$E_1(\overline{\omega}_N) = 2L^{-1}N^2 \ln N + 2(\gamma - 1)L^{-1}N^2 + o(N^2).$$

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Proof of Remark 5.3 For $s \geq 1$, denote by $\omega_N^* := \{x_1, \dots, x_N\}$ an s-energy minimizing configuration on Γ and let $\omega_N^{**} := \{t_1, \dots, t_N\}$ be an s-energy minimizing collection on $[0, L]$, where $L = |\Gamma|$. Denote also by $\omega_N' := \{y_1, \dots, y_N\}$ such a configuration on Γ that $L(y_i, y_j) = |t_i - t_j|$, $1 \leq i \neq j \leq N$, and let $\omega_N'' := \{u_1, \dots, u_N\}$ be a configuration on $[0, L]$ such that $|u_i - u_j| = L(x_i, x_j)$, $1 \leq i \neq j \leq N$. Then

$$\begin{aligned}(A.1)\qquad \mathcal{E}_s(\Gamma, N) - \mathcal{E}_s([0, L], N) &\leq E_s(\omega_N') - E_s(\omega_N^{**})\\ &= \sum_{1 \leq i \neq j \leq N} \Big(\frac{1}{|y_i - y_j|^s} - \frac{1}{|t_i - t_j|^s}\Big)\\ &= \sum_{1 \leq i \neq j \leq N} g_s(y_i, y_j)\end{aligned}$$

and

$$(A.2)\qquad \mathcal{E}_s(\Gamma, N) - \mathcal{E}_s([0, L], N) \geq E_s(\omega_N^*) - E_s(\omega_N'') = \sum_{1 \leq i \neq j \leq N} g_s(x_i, x_j) \geq 0.$$

Applying Lemma 6.3 to the curve Γ and interval $[0, L]$, we get that for $s \geq 1$, $\nu(\omega_N^*) \xrightarrow{*} \lambda_\Gamma$ and $\nu(\omega_N^{**}) \xrightarrow{*} dt/L$ as $N \to \infty$. Then $\nu(\omega_N') \xrightarrow{*} \lambda_\Gamma$, $N \to \infty$. By (4.7), for $s > 1$ and N sufficiently large, we have $\delta(\omega_N^*) \geq c_1/N$ and $\delta(\omega_N^{**}) \geq c_2/N$, where positive constants c_1 and c_2 are independent of N. Taking into account (8.12), we then have $\delta(\omega_N') \geq c_2/(2N)$ for every N large enough. If $1 \leq s < 2$, by Lemma 6.4 we get that

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{1 \leq i \neq j \leq N} g_s(x_i, x_j) = \lim_{N\to\infty} \frac{1}{N^2} \sum_{1 \leq i \neq j \leq N} g_s(y_i, y_j) = \Phi_s(\Gamma).$$

Hence, relation (5.5) holds.

Let $s \geq 2$, and for any $\epsilon > 0$ let δ be from the definition of the fact that $\gamma(x, y) \rightrightarrows 0$, $L(x, y) \to 0$ in Lemma 6.2. Then, using Lemma A.1 and the fact that $|t_i - t_j| \geq c_2|i - j|/N$, we have

$$\begin{aligned}\sum_{1 \leq i \neq j \leq N} g_s(y_i, y_j) &\leq \epsilon \sum_{0 < |t_i - t_j| < \delta} \frac{1}{|t_i - t_j|^{s-1}} + \sum_{L(y_i, y_j) \geq \delta} g_s(y_i, y_j)\\ &\leq \epsilon \sum_{0 < |t_i - t_j| < \delta} \frac{N^{s-1}}{c_2^{s-1}|i - j|^{s-1}} + O(N^2) \leq \epsilon E_{s-1}(\overline{\overline{\omega}}_N) + O(N^2)\\ &\leq \epsilon M \rho_s(N) + O(N^2),\end{aligned}$$

where $\overline{\overline{\omega}}_N = \{c_2/N, \dots, (N-1)c_2/N, c_2\}$ and $M > 0$ is a constant independent of N. Then in view of (A.1),

$$\limsup_{N\to\infty} \frac{\mathcal{E}_s(\Gamma, N) - \mathcal{E}_s([0, L], N)}{\rho_s(N)} \leq \epsilon M.$$

Letting $\epsilon \to 0$, and taking into account inequality $\mathcal{E}_s(\Gamma, N) - \mathcal{E}_s([0, L], N) \geq 0$ (which follows from (A.2)), we will have (5.4).

Proof of Remark 5.1 First, show these relations for the case $\Gamma = [0, L]$. Show the lower estimates. Let ω_N^{**}, $N \geq 2$, be an s-energy minimizing configuration on $[0, L]$ as above. Denote its points by $t_{1,N}, \dots, t_{N,N}$. Then using convexity of the function $y(t) = t^{-s}$ we get

$$\begin{aligned}\mathcal{E}_s([0,L],N) &= 2\sum_{k=1}^{N-1}\sum_{i=1}^{N-k}\frac{1}{|t_{i,N}-t_{i+k,N}|^s}\\ &\geq 2\sum_{k=1}^{N-1}(N-k)^{s+1}\Big(\sum_{i=1}^{N-k}|t_{i,N}-t_{i+k,N}|\Big)^{-s} \geq 2\sum_{k=1}^{N-1}\frac{(N-k)^{s+1}}{(kL)^s}.\end{aligned}$$

Using the inequality $(1-t)^{s+1} \geq 1-(s+1)t$, $0 < t < 1$, we have

$$\mathcal{E}_s([0,L],N) \geq \frac{2N^{s+1}}{L^s}\sum_{k=1}^{N-1}\Big(1-\frac{(s+1)k}{N}\Big)\cdot\frac{1}{k^s}. \tag{A.3}$$

If $s \geq 2$, using (8.15) or (8.17), we will get

$$\mathcal{E}_s\big([0,L],N\big) - 2L^{-s}N^{s+1}\zeta(s) \geq -2(s+1)\sigma(s-1)L^{-s}\rho_s(N) + o\big(\rho_s(N)\big). \tag{A.4}$$

If $1 < s < 2$, from (A.3) and (8.15) we have

$$\mathcal{E}_s\big([0,L],N\big) - 2L^{-s}N^{s+1}\zeta(s) \geq O(N^2), \tag{A.5}$$

and for $s = 1$, using (8.16), we will get

$$\mathcal{E}_1\big([0,L],N\big) - 2L^{-1}N^2\ln N \geq O(N^2). \tag{A.6}$$

Upper estimates for the quantity $\mathcal{E}_s([0,L],N) - 2\sigma(s)L^{-s}\rho_{s+1}(N)$, $s \geq 1$ follow from Lemma A.1. Combining them with lower estimates (A.4)–(A.6) we get relations (5.1), (5.2), and (5.3) for the segment $[0, L]$. Taking into account relations (5.5) and (5.4), we obtain (5.1), (5.2), and (5.3) for any non-closed C^2 arc.

Proof of Proposition 5.4 First, prove this statement for the segment [0,1]. For any $t \in (0,2)$ consider the following configuration $\omega_N^t := \{0, \frac{t}{N-1}, \frac{2}{N-1}, \dots, \frac{N-2}{N-1}, 1\}$. When $t = 1$ we get the configuration $\overline{\omega}_N$ of N equally spaced points on $[0, 1]$. Consider the difference

$$E_s(\omega_N^t) - E_s(\omega_N^1) = -2(N-1)^s\Big(1+\sum_{k=1}^{N-2}\frac{1}{k^s}\Big) + 2(N-1)^s\Big(\frac{1}{t^s}+\sum_{k=2}^{N-1}\frac{1}{(k-t)^s}\Big).$$

Let

$$d(t) := \lim_{N\to\infty}\frac{E_s(\omega_N^t)-E_s(\omega_N^1)}{N^s} = 2\Big(\frac{1}{t^s}+\sum_{k=2}^{\infty}\frac{1}{(k-t)^s} - 1 - \zeta(s)\Big).$$

Then it is not difficult to see that $d'(1) = 2s(\zeta(s+1) - 1) > 0$. Hence, there is $t_0 \in (0, 1)$ such that $d(t_0) < d(1) = 0$, and we have

$$\text{(A.7)} \qquad \limsup_{N\to\infty} \frac{\mathcal{E}_s([0,1],N) - E_s(\omega_N^1)}{N^s} \leq \lim_{N\to\infty} \frac{E_s(\omega_N^{t_0}) - E_s(\omega_N^1)}{N^s} = d(t_0) < 0.$$

Now let Γ be an arbitrary non-closed C^2 arc of length L. For the collection $\overline{\omega}_N$ of N equally spaced points on Γ, which contains the endpoints of Γ and the collection $\Omega_N = \{0, L/(N-1), \dots, (N-2)L/(N-1), L\}$, there holds

$$\lim_{N\to\infty} \frac{E_s(\overline{\omega}_N) - E_s(\Omega_N)}{N^s} = 0$$

(this can be verified using for example (6.2)). Relation analogous to (A.7) can be written for a segment of any length L:

$$\limsup_{N\to\infty} \frac{\mathcal{E}_s([0,L],N) - E_s(\Omega_N)}{N^s} < 0.$$

Then taking into account (5.4), we obtain Proposition 5.4.

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References

[1] A. Abrams, J. Cantarella, J. H. G. Fu, M. Ghomi, and R. Howard, *Circles minimize most knot energies.* Topology **42**(2003), 381–394. http://dx.doi.org/10.1016/S0040-9383(02)00016-2

[2] N. N. Andreev, *An extremal property of the icosahedron.* East J. Approx. **2**(1996), no. 4, 459–462.

[3] ———, *Location of points on a sphere with minimal energy. (Russian)* Tr. Mat. Inst. Steklova **219**(1997), Teor. Priblizh. Garmon. Anal., 27–31.

[4] M. K. Berkenbusch, I. Claus, C. Dunn, L. P. Kadanoff, M. Nicewicz, and S. C. Venkataramani, *Discrete charges on a two dimensional conductor.* J. Statist. Phys. **116**(2004), no. 5–6, 1301–1358. http://dx.doi.org/10.1023/B:JOSS.0000041741.27244.ac

[5] S. V. Borodachov, D. P. Hardin, and E. B. Saff, *Asymptotics for discrete weighted minimal Riesz energy problems on rectifiable sets.* Trans. Amer. Math. Soc. **360**(2008), no. 3, 1559–1580. http://dx.doi.org/10.1090/S0002-9947-07-04416-9

[6] J. S. Brauchart, *About the second term of the asymptotics for optimal Riesz energy on the sphere in the potential-theoretical case.* Integral Transforms Spec. Funct. **17**(2006), no. 5, 321–328. http://dx.doi.org/10.1080/10652460500431859

[7] ———, *Optimal logarithmic energy points on the unit sphere.* Math. Comp. **77**(2008), no. 263, 1599–1613. http://dx.doi.org/10.1090/S0025-5718-08-02085-1

[8] J. S. Brauchart, D. P. Hardin, and E. B. Saff, *Discrete energy asymptotics on a Riemannian circle.* Unif. Distrib. Theory **6**(2011), to appear.

[9] ———, *The Riesz energy of the N-th roots of unity: an asymptotic expansion for large N.* Bull. Lond. Math. Soc. **41**(2009), no. 4, 621–633. http://dx.doi.org/10.1112/blms/bdp034

[10] ———, *The support of the limit distribution of optimal Riesz energy points on sets of revolution in* $\mathbb{R}^3$. J. Math. Phys. **48**(2007), no. 12, 122901, 24 pp. http://dx.doi.org/10.1063/1.2817823

[11] H. Cohn and A. Kumar, *Universally optimal distribution of points on spheres.* J. Amer. Math. Soc. **20**(2007), no. 1, 99–148. http://dx.doi.org/10.1090/S0894-0347-06-00546-7

[12] B. E. J. Dahlberg, *On the distribution of Fekete points.* Duke Math. J. **45**(1978), 537–542. http://dx.doi.org/10.1215/S0012-7094-78-04524-6

[13] P. D. Dragnev, D. A. Legg, and D. W. Townsend, *Discrete logarithmic energy on the sphere.* Pacific J. Math. **207**(2002), no. 2, 345–358. http://dx.doi.org/10.2140/pjm.2002.207.345

[14] P. D. Dragnev and E. B. Saff, *Riesz spherical potentials with external fields and minimal energy points separation.* Potential Anal. **26**(2007), no. 2, 139–162. http://dx.doi.org/10.1007/s11118-006-9032-2

[15] M. H. Freedman, Z.-X. He, and Z. Wang, *Möbius energy of knots and unknots.* Ann. of Math. **139**(1994), no. 1, 1–50. http://dx.doi.org/10.2307/2946626

[16] M. Götz and E. B. Saff, *Note on d-extremal configurations for the sphere in R^{d+1}.* In: Recent progress in multivariate approximation (Witten-Bommerholz, 2000), Internat. Ser. Numer. Math., 137, Birkhäuser, Basel, 2001, pp. 159–162.

[17] W. Habicht and B. L. van der Waerden, *Lagerung von Punkten auf dem Kugel.* Math. Ann. **123**(1951), 223–234. http://dx.doi.org/10.1007/BF02054950

[18] D. P. Hardin and E. B. Saff, *Discretizing manifolds via minimum energy points.* Notices Amer. Math. Soc. **51**(2004), no. 10, 1186–1194.

[19] ———, *Minimal Riesz energy point configurations for rectifiable d-dimensional manifolds.* Adv. Math. **193**(2005), no. 1, 173–204. http://dx.doi.org/10.1016/j.aim.2004.05.006

[20] D. P. Hardin, E. B. Saff, and H. Stahl, *Support of the logarithmic equilibrium measure on sets of revolution in $\mathbb{R}^3$.* J. Math. Phys. **48**(2007), no. 2, 022901, 14 pp. http://dx.doi.org/10.1063/1.2435084

[21] A. V. Kolushov and V. A. Yudin, *Extremal dispositions of points on the sphere.* Anal. Math. **23**(1997), no. 1, 25–34. http://dx.doi.org/10.1007/BF02789828

[22] ———, *On the Korkin-Zolotarev construction (Russian).* Diskret. Mat. **6**(1994), no. 1, 155–157.

[23] J. Korevaar and R. A. Kortram, *Equilibrium distributions of electrons on smooth plane conductors.* Nederl. Akad. Wetensch. Indag. Math. **45**(1983), no. 2, 203–219.

[24] J. Korevaar and M. A. Monterie, *Approximation of the equilibrium distribution by distributions of equal point charges with minimal energy.* Trans. Amer. Math. Soc. **350**(1998), no. 6, 2329–2348. http://dx.doi.org/10.1090/S0002-9947-98-02187-4

[25] ———, *Fekete potentials and polynomials for continua.* J. Approx. Theory **109**(2001), no. 1, 110–125. http://dx.doi.org/10.1006/jath.2000.3535

[26] A. B. J. Kuijlaars and E. B. Saff, *Asymptotics for minimal discrete energy on the sphere.* Trans. Amer. Math. Soc. **350**(1998), no. 2, 523–538. http://dx.doi.org/10.1090/S0002-9947-98-02119-9

[27] A. B. J. Kuijlaars, E. B. Saff, and X. Sun, *On separation of minimal Riesz energy points on spheres in Euclidean spaces.* J. Comput. Appl. Math. **199**(2007), no. 1, 172–180. http://dx.doi.org/10.1016/j.cam.2005.04.074

[28] N. S. Landkof, *Foundations of modern potential theory.* Die Grundlehren der mathematischen Wissenschaften, 180, Springer-Verlag, New York-Heidelber, 1972.

[29] A. Martinez-Finkelstein, V. Maymeskul, E. A. Rakhmanov, and E. B. Saff, *Asymptotics for minimal discrete Riesz energy on curves in $\mathbb{R}^d$.* Canad. J. Math. **56**(2004), no. 3, 529–552. http://dx.doi.org/10.4153/CJM-2004-024-1

[30] J. O'Hara, *Energy of a knot.* Topology **30**(1991), no. 2, 241–247. http://dx.doi.org/10.1016/0040-9383(91)90010-2

[31] ———, *Family of energy functionals of knots.* Topology Appl. **48**(1992), no. 2, 147–161. http://dx.doi.org/10.1016/0166-8641(92)90023-S

[32] ———, *Energy functionals of knots. II.* Topology Appl. **56**(1994), no. 1, 45–61. http://dx.doi.org/10.1016/0166-8641(94)90108-2

[33] Ch. Pommerenke, *Über die Verteilung der Fekete-Punkte. II. (German)* Math. Ann. **179**(1969), 212–218. http://dx.doi.org/10.1007/BF01358488

[34] G. Szegö, *Bemerkungen zu einer Arbeit von Herrn M. Fekete: Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten.* Mathematische Zeitschrift **21**(1924), no. 1, 203–208. http://dx.doi.org/10.1007/BF01187465

[35] J. J. Thomson, *On the structure of the atom: an investigation of the stability and periods of oscillation of a number of corpuscles arranged at equal intervals around the circumference of a circle; with application of the results to the theory of atomic structure.* Philosophical Magazine (6) **7**(1904), no. 39, 237–265.

[36] V. A. Yudin, *The minimum of potential energy of a system of point charges.* Discrete Math. Appl. **3**(1993), no. 1, 75–81.

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