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Maximal Sublattices of Finite Distributive Lattices. III: A Conjecture from the 1984 Banff Conference on Graphs and Order

Jonathan David Farley

Abstract. Let L be a finite distributive lattice. Let $\mathrm{Sub}_0(L)$ be the lattice

$$\{S \mid S \text{ is a sublattice of } L\} \cup \{\varnothing\}$$

and let $\ell_*[\mathrm{Sub}_0(L)]$ be the length of the shortest maximal chain in $\mathrm{Sub}_0(L)$. It is proved that if K and L are non-trivial finite distributive lattices, then

$$\ell_*[\mathrm{Sub}_0(K \times L)] = \ell_*[\mathrm{Sub}_0(K)] + \ell_*[\mathrm{Sub}_0(L)].$$

A conjecture from the 1984 Banff Conference on Graphs and Order is thus proved.

1 Motivation

Let L be a finite lattice. Let $\mathrm{Sub}_0(L)$ denote the lattice

$$\{S \mid S \text{ is a sublattice of } L\} \cup \{\varnothing\}$$

ordered by inclusion. (Recall that a lattice or sublattice is by definition non-empty; if $|L| = 1$, we say L is *trivial*.) Let $\ell_*[\mathrm{Sub}_0(L)]$ be the length of the shortest maximal chain in this lattice. Figures 1 through 4 illustrate maximal chains in $\mathrm{Sub}_0(L)$ where L equals $\mathbf{3}$, $\mathbf{2} \times \mathbf{2}$, and $\mathbf{3} \times \mathbf{3}$. (For $n \geq 0$, $\mathbf{n}$ is the n-element chain.) We exhibit two maximal chains of $\mathrm{Sub}_0(\mathbf{3}^2)$ of different lengths, one of length 9, one of length 6. How do we know there are not maximal chains that are shorter still?

In [3, Theorem 2(i)], Chen, Koh, and Lee proved the following.

Theorem 1.1 *Let $m \geq 1$; let $n_1, \dots, n_m \geq 2$. Then*

$$\ell_*[\mathrm{Sub}_0(\mathbf{n_1} \times \cdots \times \mathbf{n_m})] = \sum_{i=1}^{m} n_i.$$

(Hence the maximal chain of Figure 4 is the shortest possible.)

The papers [1, 6, 7] deal with maximal sublattices of finite distributive lattices.

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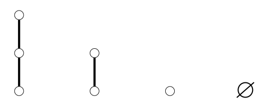

Figure 1: A shortest maximal chain in $\mathrm{Sub}_0(\mathbf{3})$; it has length 3.

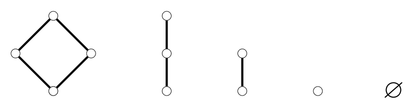

Figure 2: A shortest maximal chain in $\mathrm{Sub}_0(\mathbf{2}^2)$; it has length 4.

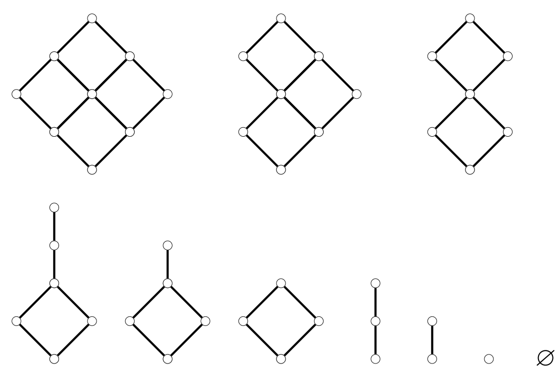

Figure 3: A maximal chain in $\mathrm{Sub}_0(\mathbf{3}^2)$; it has length 9.

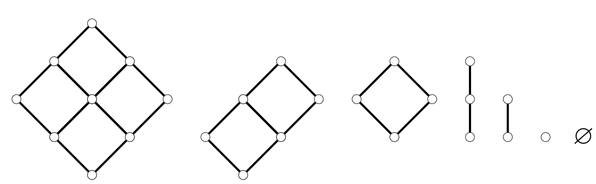

Figure 4: Is this a shortest maximal chain in $\mathrm{Sub}_0(\mathbf{3}^2)$?

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The following was posed in [3, Problem 1].

Problem 1.2 *Let K and L be (non-trivial) finite distributive lattices. Is it always true that* $\ell_*[\mathrm{Sub}_0(K \times L)] = \ell_*[\mathrm{Sub}_0(K)] + \ell_*[\mathrm{Sub}_0(L)]$*?*

Chen, Koh, and Lee [3] add, "The equality holds if both L and K are products of chains by Theorem 2(i), and up till now we are still unable to find a counterexample."

At the 1984 Banff Conference on Graphs and Order, Koh stated the above as a conjecture [8, p. 554], adding, "It would be nice to prove it if either L or K is a chain." (Note that in neither [3] nor [8] was the word "non-trivial" inserted, though it is clearly needed, as $\ell_*[\mathrm{Sub}_0(\mathbf{1})] = 1$ but $K \times \mathbf{1} \cong K$. Note also that even Figure 2 already shows that Problem 1.2 cannot be solved by naively "splicing" together a maximal chain in $\mathrm{Sub}_0(K)$ with a maximal chain in $\mathrm{Sub}_0(L)$.)

We solve Problem 1.2 below (Theorem 3.3).

2 Notation and Basic Results

For notation and terminology not explained here, see [2, 4].

Let P be a poset. For $p, q \in P$ such that $p \leq q$, define

$$\downarrow p := \{r \in P \mid r \leq p\}, \quad \overset{\circ}{\downarrow} p := (\downarrow p) \setminus \{p\},$$

$$\uparrow p := \{r \in P \mid r \geq p\}, \quad \overset{\circ}{\uparrow} p := (\uparrow p) \setminus \{p\}.$$

We say p is a *lower cover* of q (and q is an *upper cover* of p), denoted $p \lessdot q$, if $p < q$ and $\uparrow p \cap \downarrow q = \{p, q\}$. For $k \geq 0$, let

$$\mathcal{J}_k(P) := \{r \in P \mid r \text{ has exactly } k \text{ lower covers}\},$$

$$\mathcal{M}_k(P) := \{r \in P \mid r \text{ has exactly } k \text{ upper covers}\}.$$

A subset Q of P is a *down-set* of P if $\downarrow r \subseteq Q$ for all $r \in Q$. Let $\mathcal{O}(P)$ denote the bounded distributive lattice of all down-sets of P.

Note that sometimes we will deal with two partial orderings at once, for instance, P and $L = \mathcal{O}(P)$. Occasionally, when Q is a subset of a poset P, we will give Q the partial ordering inherited from P and call Q a *subposet* of P; but sometimes Q will have a different partial ordering. Poset notation relevant to one partial order in cases where there may be confusion will be designated with a subscript, *e.g.*, $a \leq_Q b$ or $\downarrow_L x$. We view the partial order relation $\leq$ as a set of ordered pairs.

Let P and Q be finite posets whose underlying sets are disjoint. Let $P + Q$ be the poset whose underlying set is the disjoint union $P \uplus Q$ and such that for all r and s in $P \uplus Q$ $r \leq_{P+Q} s$ if and only if either $r, s \in P$ and $r \leq_P s$, or else $r, s \in Q$ and $r \leq_Q s$. That is, for all $p \in P$ and $q \in Q$, p and q are incomparable (denoted $p \parallel q$). Note that $\mathcal{O}(P + Q) \cong \mathcal{O}(P) \times \mathcal{O}(Q)$.

Now we come to the first new definition. Let P be a finite poset. A *maximal sublattice sequence for P of size k* (where $k \geq 1$) is a sequence of subsets of P (not necessarily subposets)

$$(P_k, P_{k-1}, \ldots, P_2, P_1)$$

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such that $P_k = P$, $P_1 = \varnothing$, and, for $1 \leq i < k$, at least one of the following holds (where, for $1 \leq i \leq k$, we let $\leq_i$ denote the partial ordering of P_i).

(I) P_{i+1} has a least element 0_{i+1} and P_i is the subposet $P_{i+1} \setminus \{0_{i+1}\}$. Let $c_i := 1$.

(II) P_{i+1} has a greatest element 1_{i+1} and P_i is the subposet $P_{i+1} \setminus \{1_{i+1}\}$. Let $c_i := 2$.

(III) There exist $x, y \in P_{i+1}$ such that $x \parallel_{i+1} y$, $\overset{\circ}{\downarrow}_{i+1} y \subseteq \overset{\circ}{\downarrow}_{i+1} x$, and $\overset{\circ}{\uparrow}_{i+1} x \subseteq \overset{\circ}{\uparrow}_{i+1} y$; P_i has underlying set P_{i+1} and $\leq_i = \leq_{i+1} \cup \{\,(y,x)\,\}$. Let $c_i := 3$.

(IV) There exist $x \in \mathcal{M}_1(P_{i+1})$ and $y \in \mathcal{J}_1(P_{i+1})$ such that $x \lessdot_{i+1} y$ and P_i is the subposet $P_{i+1} \setminus \{x\}$ or $P_{i+1} \setminus \{y\}$. Call x and y the *key* elements and let $c_i = 4$.

We call $(c_{k-1}, \dots, c_1)$ the *maximal sublattice coding of size* $k-1$ *associated with* the maximal sublattice sequence.

The point of the above definition is as follows: Birkhoff's theorem says every finite distributive lattice L is isomorphic to $\mathcal{O}(P)$ for some finite poset P, which must necessarily be isomorphic to $\mathcal{J}_1(L)$. *Priestley duality* is the dual equivalence between the categories of bounded distributive lattices with $\{0,1\}$-preserving homomorphisms and Priestley spaces with continuous order-preserving maps. Hence we can describe a maximal $\{0,1\}$-sublattice (a maximal sublattice containing 0 and 1) M of a finite distributive lattice L by describing the relationship between $P \cong \mathcal{J}_1(M)$ and $Q \cong \mathcal{J}_1(L)$. That relationship must take the form of (III) or (IV). (If M does not contain 0_L, we get (I); if M does not contain 1_L, we get (II).)

Remark. The description of the "duals" of maximal $\{0,1\}$-sublattices of finite distributive lattices ((III) and (IV) above) can be gleaned from [1, §3]. The authors do not provide proofs, but state that "Hashimoto [5] was the first to observe that there is a bijective correspondence between the critical pairs of P on one side ... [and] with the proper maximal sublattices of $\mathcal{O}(P)$". (The ordered pairs (y,x) in (III) or (IV) satisfy the definition of *criticality* in [1].) We do not find this in [5], although Hashimoto does prove the related theorem [5, Theorem 9.2]. Nevertheless, once one knows what result to aim for, it is routine to prove that the above characterization of maximal $\{0,1\}$-sublattices is correct. One notes that, except for the beginning, the proof of [7, Theorem 2] applies to any maximal $\{0,1\}$-sublattice. (This proof itself depends on [6, Theorem 2, Theorem 3], and a converse, which comes from [7, Theorem 1] and the comments at the beginning of [7, §3].) One observes that the element c in the statement of [7, Theorem 2], as the cover of a join-irreducible element of a finite distributive lattice, belongs to $\mathcal{J}_1(L)$ or $\mathcal{J}_2(L)$. In the former case, M is type (IV); in the latter, type (III).

Hence we get the following.

Lemma 2.1 *Let L be a finite distributive lattice. Let $P := \mathcal{J}_1(L)$. Then $\mathrm{Sub}_0(L)$ has a maximal chain of length k if and only if P has a maximal sublattice sequence of size k if and only if P has a maximal sublattice coding of size $k-1$.*

If L is non-trivial and $(P_k, \dots, P_1)$ is a maximal sublattice sequence, then $k \geq 2$ and $|P_2| = 1$.

Proof If $L = L_k \supsetneqq L_{k-1} \supsetneqq \cdots \supsetneqq L_1 \supsetneqq L_0 = \varnothing$ is a maximal chain in $\mathrm{Sub}_0(L)$, then L_1 is trivial. If L is non-trivial, L_2 must be **2**. ■

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3 Proof of a Conjecture from the 1984 Banff Conference on Graphs and Order

Proposition 3.1 *Let P and Q be disjoint, non-empty, finite posets. Let $K := \mathcal{O}(P)$ and let $L := \mathcal{O}(Q)$. Let $k := \ell_*[\mathrm{Sub}_0(K)]$ and let $l := \ell_*[\mathrm{Sub}_0(L)]$; let $j := \ell_*[\mathrm{Sub}_0(K \times L)]$. Then $j \geq k + l$.*

Proof Suppose for a contradiction that $j < k+l$. Let $(R_j, R_{j-1}, \dots, R_1)$ be a maximal sublattice sequence for $P + Q$; let $(e_{j-1}, \dots, e_1)$ be the associated maximal sublattice coding and let $\leq_i$ be the partial order of R_i $(1 \leq i \leq j)$. Let

$$\begin{aligned} k' = 1 + \Big| &\{1 \leq i \leq j-1 \mid e_i = 1 \text{ or } 2, \text{ and } R_{i+1} \setminus R_i \subseteq P\} \\ &\cup \{1 \leq i \leq j-1 \mid e_i = 3, \text{ and } \leq_i \setminus \leq_{i+1} \subseteq P \times P\} \\ &\cup \{1 \leq i \leq j-1 \mid e_i = 4, \text{ and both key elements are in } P\} \Big| . \end{aligned}$$

Let l' be the corresponding number for Q. Then $k'-1+l'-1 \leq j-1 \leq k-1+l-1$. If $k' \geq k$ and $l' \geq l$, then $k'-1+l'-1 = j-1$. So there would be no $i \in \{1, \dots, j-1\}$ such that $e_i = 3$ and $\leq_{i+1} \setminus \leq_i \subseteq (P \times Q) \cup (Q \times P)$. But this is impossible since P and Q are non-empty, while $R_1 = \varnothing$ and for all $p \in P$ and $q \in Q$, $p \parallel q$ in $R_j = P + Q$. Thus, without loss of generality, $k' < k$.

For $1 \leq i \leq j$, let P_i be the subposet $R_i \cap P$ of $(R_i, \leq_i)$. Except for $k'-1$ values of $i \in \{1, \dots, j-1\}$, we have $(P_{i+1}, \leq_{i+1}) = (P_i, \leq_i)$ (without loss of generality in case $e_i = 4$). Let the posets corresponding to the exceptions be, in order,

$$\big((\overline{P}_{k'}, \sqsubseteq_{k'}), (\overline{P}_{k'-1}, \sqsubseteq_{k'-1}), \dots, (\overline{P}_1, \sqsubseteq_1) \big) .$$

This is a maximal sublattice sequence for P of size $k' < k$, so, by Lemma 2.1, $\ell_*[\mathrm{Sub}_0(K)] < k$, which is a contradiction. ■

Lemma 3.2 *Let P be a non-empty finite poset. If, for some $k \geq 1$, P has a maximal sublattice coding of size $k-1$, then P has a maximal sublattice coding $(c_{k-1}, \dots, c_1)$ where, for some $a \in \{1, \dots, k-1\}$,*

$$c_{k-1}, \dots, c_{a+1} \in \{3, 4\} \text{ and } c_a, \dots, c_1 \in \{1, 2\}.$$

Moreover, if the latter's associated maximal sublattice sequence is $(P_k, \dots, P_1)$, then $P_{a+1}, P_a, \dots, P_1$ are chains of size $a, a-1, \dots, 0$, respectively.

Proof If $(d_{k-1}, \dots, d_1)$ is a maximal sublattice coding and, for some

$$i \in \{1, \dots, k-2\}, \quad d_{i+1} \in \{1, 2\}, \quad d_i \in \{3, 4\},$$

then $(d_{k-1}, \dots, d_{i+2}, d_i, d_{i+1}, d_{i-1}, \dots, d_1)$ is also a maximal sublattice coding. By Lemma 2.1, we have $k \geq 2$ and $c_1 \in \{1, 2\}$. ■

Theorem 3.3 *Let K and L be non-trivial finite distributive lattices. Then*

$$\ell_*[\mathrm{Sub}_0(K \times L)] = \ell_*[\mathrm{Sub}_0(K)] + \ell_*[\mathrm{Sub}_0(L)].$$

Proof Let $k := \ell_*[\mathrm{Sub}_0(K)]$ and $l := \ell_*[\mathrm{Sub}_0(L)]$. Let $P := \mathcal{J}_1(K)$ and let $Q := \mathcal{J}_1(L)$ (which we can assume to be disjoint). By Lemma 2.1 and Proposition 3.1, we need only show that there is a maximal sublattice sequence for $P + Q$ of size $k + l$.

Applying Lemma 3.2, let $1 \leq a \leq k - 1$ be such that P has a maximal sublattice coding $(c_{k-1}, \dots, c_1)$ where $c_{k-1}, \dots, c_{a+1} \in \{3, 4\}$ and $c_a, \dots, c_1 \in \{1, 2\}$. Let $1 \leq b \leq l - 1$ be such that Q has a maximal sublattice coding $(d_{l-1}, \dots, d_1)$ where $d_{l-1}, \dots, d_{b+1} \in \{3, 4\}$ and $d_b, \dots, d_1 \in \{1, 2\}$.

Now

$$(c_{k-1}, \dots, c_{a+1}, d_{l-1}, \dots, d_{b+1}, 4, \dots, 4, 3, 1, 1),$$

where the 4's displayed appear $a - 1 + b - 1$ times, is a maximal sublattice coding for $P + Q$ of size

$$\begin{aligned}[(k-1) - (a+1) + 1] + [(l-1) - (b+1) + 1] + (a-1) + (b-1) + 3 \\ = k + l - a - b + a + b - 1 - 1 - 1 - 1 + 3 \\ = k + l - 1.\end{aligned}$$

By Lemma 2.1, we are done. (The associated maximal sublattice sequence $(R_{k+l}, \dots, R_1)$ is such that, by the time the 4's start, we have a disjoint sum of two chains by Lemma 3.2; the 4's reduce the poset to a two-element antichain; the 3 makes it a two-element chain; and the final 1's remove the elements of this chain.) ■

Thus we have proven the conjecture from the 1984 Banff Conference on Graphs and Order.

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Institut für Algebra, Johannes Kepler Universität Linz, A-4040 Linz, Österreich

and

Center for International Security and Cooperation, Stanford University, Stanford, California 94305, USA

and

Department of Mathematics and Computer Science, The University of the West Indies, Mona, Kingston 7, Jamaica

and

Department of Mathematics, California Institute of Technology, Pasadena, California 91125, USA
e-mail: lattice@caltech.edu