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FINITE GROUPS WITH SOME $\mathfrak{Z}$-PERMUTABLE SUBGROUPS*

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Abstract. Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a finite group G; that is to say for each prime p dividing the order of G, $\mathfrak{Z}$ contains one and only one Sylow p-subgroup of G. A subgroup H of G is said to be $\mathfrak{Z}$-permutable in G if H permutes with every member of $\mathfrak{Z}$. In this paper we characterise the structure of finite groups G with the assumption that (1) all the subgroups of $G_p \in \mathfrak{Z}$ are $\mathfrak{Z}$-permutable in G, for all prime $p \in \pi(G)$, or (2) all the subgroups of $G_p \cap F^*(G)$ are $\mathfrak{Z}$-permutable in G, for all $G_p \in \mathfrak{Z}$ and $p \in \pi(G)$, where $F^*(G)$ is the generalised Fitting subgroup of G.

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1. Introduction and statements of results. All groups considered in this paper are finite. We use conventional notions and notation, as in Huppert [**5**]. Throughout this paper, G stands for a finite group and $\pi(G)$ represents the set of distinct primes dividing $|G|$.

A subgroup of G is called *quasi-normal* in G if it permutes with every subgroup of G. We say, following Kegel [**8**], that a subgroup of G is *S-quasi-normal* in G if it permutes with every Sylow subgroup of G. Recently, Asaad and Heliel [**1**] introduced a new embedding property, namely the $\mathfrak{Z}$-permutability of subgroups of a group; $\mathfrak{Z}$ is called *a complete set of Sylow subgroups of* G if for each prime $p \in \pi(G)$, $\mathfrak{Z}$ contains exactly one Sylow p-subgroup of G, say G_p. A subgroup of G is said to be $\mathfrak{Z}$-*permutable* in G if it permutes with every member of $\mathfrak{Z}$. Obviously, every S-quasi-normal subgroup is $\mathfrak{Z}$-permutable. In contrast to the fact that every S-quasi-normal subgroup is subnormal (see [**8**]), it does not hold in general that every $\mathfrak{Z}$-permutable subgroup of G is subnormal in G. It suffices to consider the alternating group of degree 4.

Many authors have investigated the structure of a group G under the assumption that some subgroups of G are well situated in G. Srinivasan [**14**] proved that a group G is supersolvable if every maximal subgroup of any Sylow subgroup of G is normal. Later on, Wall [**15**] gave a complete classification of finite groups under the assumption

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of Srinivasan. In [**10**], the authors obtained the following results ([**10**], Theorems 3.1 and 3.4): Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a group G and p the smallest prime dividing $|G|$. Then G is p-nilpotent if one of the following hold: (1) the maximal subgroups of $G_p \in \mathfrak{Z}$ are $\mathfrak{Z}$-permutable subgroups of G; (2) G is A_4-free and the 2-maximal subgroups of G_p are $\mathfrak{Z}$-permutable subgroups of G. In [**11**], the authors obtained the following ([**11**], Theorems 3.1 and 3.4): Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a group G and p the smallest prime dividing $|G|$. Then G is p-nilpotent if one of following holds: (1) every cyclic subgroup of prime order or order 4 (when $p = 2$) of $G_p \in \mathfrak{Z}$ is $\mathfrak{Z}$-permutable in G; (2) G is A_4-free and every subgroup of prime square order of $G_p \in \mathfrak{Z}$ is $\mathfrak{Z}$-permutable in G. We know that if every subgroup of G is normal in G; then G is the Dedikind group [**13**]; every subgroup of G is quasi-normal in G; then G is the quasi-Hamilton group [**2**]. It is easy to see that that G is nilpotent if and only if every subgroup of G of prime power order is $\mathfrak{Z}$-permutable in G, where $\mathfrak{Z}$ is a complete set of Sylow subgroups of G. In view of the above results, it is interesting to give the structure of G under the assumption that for any $G_p \in \mathfrak{Z}$, every subgroup of G_p is $\mathfrak{Z}$-permutable in G. We get the following.

THEOREM 1.1. *Let G be a finite group and $\mathfrak{Z}$ a complete set of Sylow subgroups of G. Then every subgroup of $G_p \in \mathfrak{Z}$, for any prime $p \in \pi(G)$, is $\mathfrak{Z}$-permutable in G if and only if there exists a normal subgroup L of G satisfying the following*:

(1) L is an abelian Hall subgroup of G and G/L is nilpotent;

(2) the elements of G induce power automorphisms in L;

(3) for any two distinct primes $p, q \notin \pi(L)$, $[G_p, G_q] = 1$, where $G_p, G_q \in \mathfrak{Z}$.

It is interesting to limit the hypotheses to a smaller subgroup of G. By [**4**] and [**9**], we know the following: Let G be a finite group and $\mathfrak{Z}$ a complete set of Sylow subgroups of G and $F^*(G)$ is the generalised Fitting subgroup of G. Then G is supersolvable under one of following assumptions: (1) the maximal subgroups of $G_p \cap F^*(G)$ are $\mathfrak{Z}$-permutable subgroups of G, for all $G_p \in \mathfrak{Z}$; (2) the cyclic subgroups of $G_p \cap F^*(G)$ of prime order or order are $\mathfrak{Z}$-permutable subgroups of G, for all $G_p \in \mathfrak{Z}$. Hence, it is interesting to investigate the structure of G under the assumption that all the subgroups of $G_p \cap F^*(G)$ are $\mathfrak{Z}$-permutable subgroups of G, for all $G_p \in \mathfrak{Z}$. Here we get the following.

THEOREM 1.2. *Let G be a finite group and $\mathfrak{Z}$ a complete set of Sylow subgroups of G, and $F^*(G)$ is the generalised Fitting subgroup of G. Then every subgroup of $G_p \cap F^*(G)$, for any $G_p \in \mathfrak{Z}$ and any $p \in \pi(G)$, is $\mathfrak{Z}$-permutable in G if and only if there exists a normal subgroup L of G satisfying the following*:

(1) L is abelian and G/L is nilpotent;

(2) L is a Hall subgroup of $F^*(G)$;

(3) p'-elements of G induce power automorphisms in L_p, the Sylow p-subgroup of L.

COROLLARY 1.3. *Let G be a finite group, and $F^*(G)$ is the generalised Fitting subgroup of G. Then every subgroup of $F^*(G)$ is S-quasi-normal in G if and only if there exists a normal subgroup L of G satisfying the following*:

(1) L is abelian and G/L is nilpotent;

(2) L is a Hall subgroup of $F^*(G)$;

(3) p'-elements of G induce power automorphisms in L_p, the Sylow p-subgroup of L.

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Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a group G. If $N \lhd G$, we denote

$$\mathfrak{Z}N = \{G_pN : G_p \in \mathfrak{Z}\},$$
$$\mathfrak{Z}N/N = \{G_pN/N : G_p \in \mathfrak{Z}\},$$
$$\mathfrak{Z} \cap N = \{G_p \cap N : G_p \in \mathfrak{Z}\}.$$

The generalised Fitting subgroup $F^*(G)$ of G is the unique maximal normal quasi-nilpotent subgroup of G. Its important properties can be found in [**7**], Chapter X, Section 13.

Now, $G^{\mathcal{N}}$ denotes the nilpotent residual of G, which some authors prefer to write as $K_\infty(G)$; it is the last term in the lower central series of G.

2. Preliminaries.

The following lemmas will be used in the proofs of our results.

LEMMA 2.1. *([1], Lemma 2.1) Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of G, U a $\mathfrak{Z}$-permutable subgroup of G and N a normal subgroup of G. Then*

(1) $\mathfrak{Z} \cap N$ and $\mathfrak{Z}N/N$ are complete sets of Sylow subgroups of N and G/N, respectively;
(2) UN/N is a $\mathfrak{Z}N/N$-permutable subgroup of G/N;
(3) U is a $\mathfrak{Z} \cap N$-permutable subgroup of N if $U \leq N$.

LEMMA 2.2. *Let G be a finite group and $\mathfrak{Z}$ a complete set of Sylow subgroups of G. Suppose N is a normal p-subgroup of G; then every subgroup of N is $\mathfrak{Z}$-permutable in G if and only if every subgroup of N is S-quasi-normal in G.*

Proof. We only need to prove the necessity. Suppose any subgroup of N is $\mathfrak{Z}$-permutable in G. Let L be an arbitrary subgroup of N. Then LG_{p_i} is a subgroup of G for every $G_{p_i} \in \mathfrak{Z}$. Since $N \lhd G$, it follows that $L \leq N^x$ for all $x \in G$. Hence $L^{x^{-1}} \leq N$, and therefore $L^{x^{-1}}G_{p_i} \leq G$. But $LG_{p_i}^x = (L^{x^{-1}}G_{p_i})^x$ is a subgroup of G, then L is S-quasi-normal in G. □

LEMMA 2.3. *([7]; Chapter X, Section 13) Let G be a group and M a subgroup of G.*

(1) If M is normal in G, then $F^*(M) \leq F^*(G)$.
(2) If $F^*(G)$ is solvable, then $F^*(G) = F(G)$.

LEMMA 2.4. *Let G be a finite group and $\mathfrak{Z}$ a complete set of Sylow subgroups of G. Suppose every subgroup of $F^*(G) \cap G_p$ is $\mathfrak{Z}$-permutable in G, for any $G_p \in \mathfrak{Z}$; then G is supersolvable.*

Proof. This is a corollary of results in [**4**] or [**9**]. □

LEMMA 2.5. *Suppose G is a group and P a normal p-subgroup of G. Then $P \leq Z_\infty(G)$ if and only if $C_G(P) \geq O^p(G)$.*

Proof. If $C_G(P) \geq O^p(G)$, then $G/C_G(P)$ is a p-group; so $P \leq Z_\infty(G)$ by [**16**], p. 220, Theorem 6.3. The converse is [**12**], Lemma 2.8. □

LEMMA 2.6. *Suppose P is a normal p-subgroup of G. If every subgroup of P is S-quasi-normal in G, then every p'-element of G induces a power automorphism in P.*

Proof. Take any $a \in P$. Let x be a p'-element of G. Then $x \in G_{p'}$ for some p'-Hall subgroup of G. Then $\langle a\rangle G_{p'}$ is a group by hypotheses. Hence

$$a^{\langle x\rangle} = a^{\langle x\rangle} \cap \langle a\rangle G_{p'} = \langle a\rangle(a^{\langle x\rangle} \cap G_{p'}) \leq \langle a\rangle(P \cap G_{p'}) = \langle a\rangle,$$

i.e. $a^{\langle x\rangle} = \langle a\rangle$. Therefore x induces a power automorphism in $\langle a\rangle$. □

In the next lemma we collect some properties of power automorphism.

LEMMA 2.7. *Suppose N is a non-trivial normal p-subgroup of G. Then*
(1) the p'-power automorphism of N is trivial if N is non-abelian;
(2) there exists a positive integer n such that $a^{\alpha} = a^n$, for all $a \in N$, if N is abelian and α is a power automorphism of N;
(3) the power automorphisms of N are in the centre of Aut(N) if N is abelian;
(4) $G/C_G(N)$ is nilpotent if all p'-elements of G induce power automorphisms in N by conjugate.

Proof. (1) It is **[6]**, Hilfsatz 5.
(2) See **[13]**, Chapter 13, Theorem 4.3.
(3) It is a direct corollary of (2).
(4) If N is non-abelian, then $G/C_G(N)$ is a p-group by (1); hence $G/C_G(N)$ is nilpotent. If N is abelian, then the power automorphisms are in the centre of Aut(N) by (3). It is easy to see that $G/C_G(N)$ is nilpotent.

□

LEMMA 2.8. *Suppose $L = K_{\infty}(G)$ is the nilpotent residual of G. If L is nilpotent, then $L_p = [L_p, G]$, for any $p \in \pi(G)$.*

Proof. By definition, $L = [L, G] = [L_p \times L_{p'}, G] = [L_p, G] \times [L_{p'}, G] = L_p \times L_{p'}$, for any $p \in \pi(G)$. Hence $L_p = [L_p, G]$. □

3. Proofs.

Proof of Theorem 1.1. We prove the necessity of this theorem in several steps.

(i) G is supersolvable. Hence $F^*(G) = F(G)$.

By Lemmata 2.4 and 2.3.

(ii) If N is a normal p-subgroup of G, then p'-elements of G induce power automorphisms in N.

If N is a normal p-subgroup of G, then $N \leq G_p \in \mathfrak{Z}$. Thus every subgroup of N is $\mathfrak{Z}$-permutable in G by hypotheses; then is S-quasi-normal in G by Lemma 2.2. Now applying Lemma 2.6, we get step (ii).

(iii) Pick $L = G^{\mathcal{N}}$; then G/L is nilpotent. Furthermore, L is abelian.

By (ii) and Lemma 2.7(4), for any $p \in \pi(G)$, we know that $G/C_G(O_p(G))$ is nilpotent. So $(G/C_G(O_p(G)))^{\mathcal{N}} = 1$, and it follows that $G^{\mathcal{N}} \leq C_G(O_p(G))$. Therefore $G^{\mathcal{N}} \leq \bigcap_{p\in\pi(G)} C_G(O_p(G)) = C_G(F(G)) \leq F(G)$. Then $L \leq Z(F(G))$, Hence L is abelian.

(iv) L is a Hall subgroup of G.

Let p be the largest prime dividing $|G|$, and P is a Sylow p-subgroup of G. Since G is supersolvable by step (i), we know that $P \trianglelefteq G$. Then $P = G_p \in \mathfrak{Z}$. Now, we consider the quotient group G/P. By Lemma 2.1, all subgroups of every member in $\mathfrak{Z}P/P$ are $\mathfrak{Z}P/P$-permutable in G/P. By induction, $(G/P)^{\mathcal{N}} = G^{\mathcal{N}}P/P = LP/P$ is a Hall subgroup of G/P.

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Suppose that every p'-element of G centralises P. Let $L_p \in Syl_p(L)$. If $L_p \neq 1$, then, by Lemma 2.8, $L_p = [L_p, G] = [L_p, P] < L_p$ as $L_p \leq P$, a contradiction. Hence $L_p = 1$ and L is a p'-group. Therefore, $L \cong LP/P$ is a normal Hall subgroup of G. Now suppose that there exists a p'-element x which induces a non-trivial power automorphism on P. Hence $P = [P, G] \leq L$. Therefore L is a Hall subgroup of G.

(v) The elements of G induce power automorphisms in L.

It is easy to see from (ii)–(iv).

(vi) For any two distinct primes $p, q \notin \pi(L)$, $[G_p, G_q] = 1$, where $G_p, G_q \in \mathfrak{Z}$.

By the hypotheses, G_pG_q is a group. Since $G_pG_q \cong G_pG_qL/L \leq G/L$, G_pG_q is nilpotent by (iii). Hence $[G_p, G_q] = 1$.

Conversely, it suffices to prove that $\langle x\rangle\langle y\rangle = \langle y\rangle\langle x\rangle$, for any p-element $x \in G_p$ and q-element $y \in G_q$, where $G_p, G_q \in \mathfrak{Z}$.

If $p, q \in \pi(L)$, since L is a normal abelian Hall subgroup of G, we have that $x, y \in L$ and $\langle x\rangle\langle y\rangle = \langle y\rangle\langle x\rangle$.

If $p, q \notin \pi(L)$, by (3), $[x, y] = 1$. Hence $\langle x\rangle\langle y\rangle = \langle y\rangle\langle x\rangle$.

Suppose $p \in \pi(L)$ or $q \in \pi(L)$. Without lose generality, let $p \in \pi(L)$. Then $x \in L$ and $\langle x\rangle \trianglelefteq G$ by (2). Hence $\langle x\rangle\langle y\rangle = \langle y\rangle\langle x\rangle$. Finishing the proof. □

Proof of Theorem 1.2. We first prove the necessity of Theorem 1.2. With the same arguments as in the proof of Theorem 1.1, we get steps (i)–(iii).

(i) G is supersolvable. Hence $F^*(G) = F(G)$.

(ii) If N is a normal p-subgroup of G, then p'-elements of G induce power automorphisms in N.

(iii) Pick $L = G^{\mathcal{N}}$; then G/L is nilpotent. Furthermore, L is abelian.

(iv) Denote $F = F(G)$. Then $F = C_G(L)$.

By the proof of (iii), we only need to prove $C_G(L) \leq F$. We know that $L \leq Z(C_G(L))$, and hence $C_G(L)/Z(C_G(L)) \leq G/Z(C_G(L))$ is nilpotent. Hence $C_G(L)$ is nilpotent. Therefore $C_G(L) \leq F(G)$.

(iv) $F = Z_\infty(G) \times L$.

By [**5**], Chapter VI, Satz 7.15, G splits over L, i.e. $G = X \rtimes L$, for some subgroup X of G. So $F = F \cap (XL) = L(C_G(L) \cap X) = C_X(L)L = C_X(L) \times L$.

Now we prove that $C_X(L) = Z_\infty(G)$.

Notice that $[C_X(L), G] = [C_X(L), X] \leq C_X(L)$. Since $X \cong G/L$ is nilpotent, there exists an integer n such that $K_n(X) = K_\infty(X) = 1$. Therefore $[C_X(L), G, \ldots, G] = [C_X(L), X, \ldots, X] \leq K_\infty(X) = 1$. Therefore $C_X(L) \leq Z_{n-2}(G) \leq Z_\infty(G)$. Thus $Z_\infty(G) = Z_\infty(G) \cap F = C_X(L)(Z_\infty(G) \cap L)$,

Next we want to prove that $Z_\infty(G) \cap L = 1$. If $Z_\infty(G) \cap L \neq 1$, then there exists a prime $p \in \pi(G)$ such that $Z_\infty(G) \cap L_p \neq 1$. If every p'-element of G centralises L_p, then, by Lemma 2.8, $L_p = [L_p, G] = [L_p, G_p] < L_p$, a contradiction. Hence there exists a p'-element x which induces a non-trivial power automorphism on L_p, and so $[L_p, x] \neq 1$. On the other hand, we know that $[Z_\infty(G) \cap L_p, x] = 1$ by Lemma 2.5. Hence $[L_p, x] = 1$ by (iii) and Lemma 2.7(2), a contradiction. Hence $Z_\infty(G) \cap L = 1$. So $Z_\infty(G) = C_X(L)$. Thus $F = Z_\infty(G) \times L$.

(v) L is a Hall subgroup of $F(G)$.

If L is not a Hall subgroup of $F(G)$, then there is a prime $p \in \pi(L) \cap \pi(F(G)/L)$. Denote $C = C_X(L)$.

By (iv), $F = C_G(L) = Z_\infty(G) \times L = C_X(L) \times L = C \times L$. Since $p \in \pi(C)$, we have $C_p \neq 1$; then $C_p \cap Z(C) \neq 1$. Therefore $p \in \pi(Z(C))$. For any p'-element $x \in X$, x

induces a power automorphism on $F_p = C_p \times L_p$. By Lemma 2.5, $[C_p, x] = 1$. Hence $[Z(C_p), x] = 1$. By Lemma 2.7(2), $[Z(C_p) \times L_p, x] = 1$, and in particular, $[L_p, x] = 1$. Therefore $[L_p, X_{p'}] = 1$. Then $L_p = [L_p, G] = [L_p, X] = [L_p, X_p] < L_p$, a contradiction. Therefore L is a Hall subgroup of $F(G)$.

Conversely, it is easy to see that G is solvable; hence $F^*(G) = F(G) \neq 1$. It suffices to prove that every cyclic p-subgroup $< g >$ of $F(G)$ is S-quasi-normal in G, for any prime $p \in \pi(G)$.

Suppose $g \in O_p(G)$. If $P \in Syl_p(G)$, then $g \in O_p(G) \leq P$. Thus $< g > P = P < g >= P$. Pick an arbitrary $Q \in Syl_q(G)$, where $q \neq p$. If $p \in \pi(L)$, then $g \in L_p$ by (iii). Then Q normalises $< g >$ by (ii). Therefore $Q < g >=< g > Q$. Hence suppose that $p \notin \pi(L)$. Then $[< g > L/L, QL/L] = 1$ as G/L is nilpotent by (i). So $[< g >, Q] \leq L$. It follows that $[Q, g] \leq L \cap g^G \leq L \cap O_p(G) = 1$. Hence $Q < g >=< g > Q$. Completing the proof. □

Proof of Corollary 1.3. If every subgroup of $F^*(G)$ is S-quasi-normal in G, then G is supersolvable by the results in [**4**] or [**9**]. In particular, $F^*(G) = F(G)$. For any prime $p \in \pi(G)$ and any Sylow p-subgroup G_p of G, every subgroup of $G_p \cap F^*(G) = G_p \cap F(G) = O_p(G)$ is S-quasi-normal in G by the hypotheses. By Theorem 1.2, it is easy to see the necessity of Corollary 1.3 holds. Conversely, by the proof of Theorem 1.2, we know that every cyclic p-subgroup of $F(G) = F^*(G)$ is S-quasi-normal in G, for any prime $p \in \pi(G)$. It is easy to see that every subgroup of $F^*(G)$ is S-quasi-normal in G. Completing the proof. □

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