THE ANGULAR MOMENTUM OF AN ELECTROMAGNETIC FIELD

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1. It is known that the behaviour of an electromagnetic field is consistent with it possessing a linear momentum density (**1**, p. 103)

$$\boldsymbol{g} = \boldsymbol{D} \times \boldsymbol{B} = c^{-2} \boldsymbol{E} \times \boldsymbol{H} \qquad \text{(1.1)}$$

(Girogi Units are used and the system of notation is the usual one).

This leads to the suggestion that the electromagnetic field has a density of angular momentum,

$$\boldsymbol{m} = \boldsymbol{r} \times \boldsymbol{g}. \qquad \text{(1.2)}$$

This is consistent with a variational principle in field theory (**2**). Corresponding to this is a flux of angular momentum across any surface, and it may be shown (**3**) that the rate of flux of angular momentum across unit surface in the direction of normal $\boldsymbol{n}$ is

$$\boldsymbol{c}_n = -\boldsymbol{r} \times \boldsymbol{E}(\boldsymbol{D}\,.\,\boldsymbol{n}) - \boldsymbol{r} \times \boldsymbol{H}(\boldsymbol{B}\,.\,\boldsymbol{n}) + U\boldsymbol{r} \times \boldsymbol{n} \qquad \text{(1.3)}$$

where $2U = \boldsymbol{E}\,.\,\boldsymbol{D} + \boldsymbol{B}\,.\,\boldsymbol{H}$. Unfortunately there arises from general field theory a further definition of angular momentum density (**3**)

$$\boldsymbol{m}' = \boldsymbol{D} \times \boldsymbol{A} + \boldsymbol{r} \times \{\boldsymbol{D}\,.(\boldsymbol{A}\nabla)\} \qquad \text{(1.4)}$$

the corresponding flux being

$$\boldsymbol{c}_n' = (\boldsymbol{H} \times \boldsymbol{n}) \times \boldsymbol{A} + \boldsymbol{r} \times [(\boldsymbol{H} \times \boldsymbol{n})\,.(\boldsymbol{A}\nabla) + (\boldsymbol{n}\,.\,\boldsymbol{D})\nabla\phi + L\boldsymbol{n}] \qquad \text{(1.5)}$$

where $\boldsymbol{A}$ is the vector potential, ϕ the scalar potential (**1**, p. 23) and L is the so-called Lagrangian density

$$2L = \boldsymbol{E}\,.\,\boldsymbol{D} - \boldsymbol{B}\,.\,\boldsymbol{H}.$$

The question immediately arises, under what conditions are the forms for $\boldsymbol{m}$ and $\boldsymbol{m}'$, and the forms for $\boldsymbol{c}_n$ and $\boldsymbol{c}_n'$ equivalent. Clearly the conditions necessary are that $\boldsymbol{m}' - \boldsymbol{m}$ vanishes when integrated over the whole of space and that $\boldsymbol{c}_n' - \boldsymbol{c}_n$ vanishes when integrated over a closed surface. This is in accordance with the principle that the energy of electromagnetic fields cannot actually be localised. All that is possible is to find the value over the whole of space (**1**, p. 110). It has already been shown (**3**) that the forms for $\boldsymbol{m}'$ and $\boldsymbol{m}$ are in general equivalent and it has been shown (**4**) that the forms for $\boldsymbol{c}_n'$ and $\boldsymbol{c}_n$ are in fact equivalent for a TE_{11} wave in a circular waveguide if the flux is taken across a normal cross section the centre of which is the origin. It is proposed

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to show in this paper that the equivalence of $\boldsymbol{c}_n'$ and $\boldsymbol{c}_n$ is much more general and to give general conditions for equivalence.

For completeness, the relation for the equivalence of $\boldsymbol{m}'$ and $\boldsymbol{m}$ is given here. The proof is given in (3).

$$\int_V (\boldsymbol{m}'-\boldsymbol{m})d\tau = \int_S (\boldsymbol{r}\times \boldsymbol{A})\boldsymbol{D}\,.\,d\boldsymbol{S} - \int_V (\boldsymbol{r}\times \boldsymbol{A})\nabla\,.\,\boldsymbol{D}d\tau \quad \text{.........(1.6)}$$

S being the surface bounding V. If the region is charge-free the second integral on the right-hand side of (1.6) vanishes, and for a number of fields such as a plane wave, or a field vanishing in a suitable way at infinity, the first integral vanishes if S is a sphere of infinite radius.

2. Using standard methods of vector analysis it is easily shown that

$$\boldsymbol{c}_n'-\boldsymbol{c}_n = \nabla\,.\,[(\boldsymbol{n}\times\boldsymbol{H})(\boldsymbol{A}\times\boldsymbol{r})] - \boldsymbol{r}\times\frac{\partial}{\partial t}(\boldsymbol{A}[\boldsymbol{D}\,.\,\boldsymbol{n}]).$$

It follows that

$$\int_S (\boldsymbol{c}_n'-\boldsymbol{c}_n)dS = \int_S \nabla\,.\,[(\boldsymbol{n}\times\boldsymbol{H})(\boldsymbol{A}\times\boldsymbol{r})]dS - \frac{\partial}{\partial t}\int_S (\boldsymbol{r}\times\boldsymbol{A})\boldsymbol{D}\,.\,d\boldsymbol{S} \quad \text{...(2.1)}$$

provided that the surface S is fixed.

Suppose now that S is a plane surface. Then, using Stokes' theorem, we have

$$\begin{aligned}\int_S \nabla\,.\,&[(\boldsymbol{n}\times\boldsymbol{H})(\boldsymbol{A}\times\boldsymbol{r})]dS \\ &= \int_S d\boldsymbol{S}\,.\,(\boldsymbol{n}\,.\,\nabla)[(\boldsymbol{n}\times\boldsymbol{H})(\boldsymbol{A}\times\boldsymbol{r})] \\ &\quad - \int_C (\boldsymbol{n}\times d\boldsymbol{r})\,.\,[(\boldsymbol{n}\times\boldsymbol{H})(\boldsymbol{A}\times\boldsymbol{r})] \\ &= -\int_C (\boldsymbol{A}\times\boldsymbol{r})(\boldsymbol{H}\,.\,d\boldsymbol{r}).\end{aligned}$$

The result is independent of $\boldsymbol{n}$ and so by addition of elementary plane surfaces, it holds for any arbitrary surface and its associated rim. Thus

$$\int (\boldsymbol{c}_n'-\boldsymbol{c}_n)dS = \int_C (\boldsymbol{r}\times\boldsymbol{A})(\boldsymbol{H}\,.\,d\boldsymbol{r}) - \frac{\partial}{\partial t}\int_S (\boldsymbol{r}\times\boldsymbol{A})(\boldsymbol{D}\,.\,d\boldsymbol{S}). \quad \text{.........(2.2)}$$

The equivalence of $\boldsymbol{c}_n'$ and $\boldsymbol{c}_n$ follows if the sum of the two integrals on the right-hand side of (2.2) vanishes. Sufficient conditions for these are as follows.

(*a*) If the field is periodic, the mean value (over a period) of the second term vanishes and if the field is finite at $|t| = \infty$, the mean value of the second term over all time vanishes. This condition is generally satisfied in practice.

(*b*) If the surface is closed, the first integral vanishes.

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(*c*) Either integral may vanish under suitable radiation conditions at infinity.

(*d*) The origin may be chosen so that the first term always vanishes. If having chosen the origin accordingly, it is found that $\int A(\boldsymbol{H}\,.\,d\boldsymbol{r})$ also vanishes, then the first term will vanish. In a large number of cases of interest this is in fact found to be true but it does not necessarily follow. It may be noted that ϕ may be chosen to be zero if desired.

3. If we use the potentials $\boldsymbol{A}^*$, ϕ^* defined by

$$\boldsymbol{H} = -\nabla\phi^* - \frac{\partial \boldsymbol{A}^*}{\partial t}$$

$$\boldsymbol{D} = -\nabla\times\boldsymbol{A}^*$$

and the dual property of Maxwell's Equations (**1**, p. 25), we may derive alternative expressions for the angular momentum density and flux.

$$\boldsymbol{m}'' = \boldsymbol{B}\times\boldsymbol{A}^* + \boldsymbol{r}\times\{\boldsymbol{B}\,.\,(\boldsymbol{A}^*\nabla)\} \quad \text{..........................(3.1)}$$

$$\boldsymbol{c}_n'' = -(\boldsymbol{E}\times\boldsymbol{n})\times\boldsymbol{A}^* + \boldsymbol{r}\times\{-(\boldsymbol{E}\times\boldsymbol{n})\,.\,(\boldsymbol{A}^*\nabla) + (\boldsymbol{n}\,.\,\boldsymbol{B})\nabla\phi^* + L\boldsymbol{n}\}. \quad \text{.........(3.2)}$$

It follows that

$$\int_V (\boldsymbol{m}'' - \boldsymbol{m})d\tau = -\int_S (\boldsymbol{r}\times\boldsymbol{A}^*)(\boldsymbol{B}\,.\,d\boldsymbol{S}) \text{.....................(3.3)}$$

$$\int_S (\boldsymbol{c}_n'' - \boldsymbol{c}_n)dS = -\int_C (\boldsymbol{r}\times\boldsymbol{A}^*)(\boldsymbol{E}\,.\,d\boldsymbol{r}) - \frac{\partial}{\partial t}\int_S (\boldsymbol{r}\times\boldsymbol{A}^*)(\boldsymbol{B}\,.\,d\boldsymbol{S}). \text{......(3.4)}$$

The conditions for the vanishing of the integrals on the right-hand sides of (3.3) and (3.4) are similar to those mentioned above. There is, however, one further condition which may prove useful. If the boundary curve C lies on a perfect conductor $\boldsymbol{E}\,.\,d\boldsymbol{r} = 0$. Similarly if a portion of S is a perfect conductor, then $\boldsymbol{B}\,.\,d\boldsymbol{S} = 0$.

Clearly in this case also, it will be convenient to take ϕ^* zero.

4. It has been shown (**2**, p. 23) that the expressions (1.3) and (1.5) for angular momentum flux are equivalent for a TE_{11} wave in a circular waveguide if the flux is taken across a normal cross section the centre of which is the origin. It may be shown that this is true much more generally, and the proof given below—which uses (2.2) and avoids a heavy manipulation of Bessel Functions—is applicable for a coaxial guide also. It may be remarked that the formulations (1.3) and (3.2) are equivalent because the first integral on the right-hand side of (3.4) vanishes and the mean value of the second term vanishes. It may be noted that this would be true for a waveguide of any cross section because $\boldsymbol{E}\,.\,d\boldsymbol{r}$ vanishes everywhere on the wall. Thus $\boldsymbol{c}_n''$ and $\boldsymbol{c}_n$ are equivalent here. The equivalence of $\boldsymbol{c}_n'$ and $\boldsymbol{c}_n$ for the circular guide may be proved as follows. The mean value of the second integral of the right-hand side of equation (2.2) vanishes and it is only necessary to consider the first integral. If the origin is at the centre of the guide, then, with an obvious notation

$$\int_C (\boldsymbol{r}\times \boldsymbol{A})(\boldsymbol{H}.d\boldsymbol{r}) = \int_{-\pi}^{\pi} r^2[A_\theta \boldsymbol{i}_z - A_z \boldsymbol{i}_\theta]H_\theta d\theta \quad \text{.................. (4.1)}$$

where the integrand is evaluated at $r = a$, and if the guide is coaxial at $r = b$ also. $\boldsymbol{i}_p$ denotes unit vector in direction p. It can be shown that A_θ and A_z vanish on the walls and so the integral on the right-hand side of (4.1) vanishes. For the *TEM* mode in a coaxial guide

$$\boldsymbol{E} = E\boldsymbol{i}_r$$

and so $\boldsymbol{r}\times \boldsymbol{A}$ vanishes identically.

Thus $\boldsymbol{c}_n'$ and $\boldsymbol{c}_n$ are equivalent for a circular waveguide or a coaxial waveguide, provided that the centre of the cross section be taken as origin. If the origin is not at the centre of the cross section, but at a point whose position vector is $\boldsymbol{r}_0$ referred to the centre of the cross section, an extra term

$$\boldsymbol{r}_0 \times \left(\int A_r \boldsymbol{i}_r H_\theta a d\theta\right)$$

is added to the right side of (4.1), the other components of $\boldsymbol{A}$ vanishing on the wall. The quantity in brackets can be rewritten as

$$\int_{-\pi}^{\pi} A_r H_\theta(\boldsymbol{i}_x \cos\theta + \boldsymbol{i}_y \sin\theta) a d\theta.$$

This quantity will vanish for any mode by the usual circular symmetry properties.

The actual value of the angular momentum flux along the guide may be obtained most easily from $\boldsymbol{c}_n$. Taking cylindrical polar coordinates, it becomes

$$\boldsymbol{C} = \int r[(E_z \boldsymbol{i}_\phi - E_\phi \boldsymbol{i}_z)D_z + (H_z \boldsymbol{i}_\phi - H_\phi \boldsymbol{i}_z)B_z - \boldsymbol{i}_\phi U]dS.$$

It may be shown without much difficulty that this reduces, for any mode, to

$$\boldsymbol{C} = -\boldsymbol{i}_z \int r(E_\phi D_z + H_\phi B_z)dS.$$

This is zero for the *TEM* mode in a coaxial guide as might be expected.

If the field is periodic with a time factor $\exp\{i\omega t\}$, the mean value of C over a period is given by

$$\bar{\boldsymbol{C}} = -i_z \tfrac{1}{2}\mathscr{R} \int r(E_\phi D_z^* + H_\phi^* B_z)dS$$

where $\mathscr{R}$ denotes the real part of and the asterisk denotes the complex conjugate value. Using the z component in Maxwell's equations we find that

$$\bar{c} = \mathbf{i}_z \tfrac{1}{2}\mathscr{R}\,\frac{1}{i\omega}\int -\left\{\frac{\partial E_r}{\partial \phi} H_\phi^* + \frac{\partial H_r^*}{\partial \phi} E_\phi\right\} dS$$

$$+\mathbf{i}_z \tfrac{1}{2}\mathscr{R}\,\frac{1}{i\omega}\int_{-\pi}^{\pi} d\phi \int_b^a rH_\phi^* \frac{\partial}{\partial r}(rE_\phi) + rE_\phi \frac{\partial}{\partial r}(rH_\phi^*)dr.$$

The second integral on the right-hand side can easily be shown to be zero.

If the ϕ behaviour is $\exp\{-in\,\phi\}$

$$\bar{C} = \mathbf{i}_z \tfrac{1}{2}\mathscr{R}\,\frac{n}{\omega}\int (E_r H_\phi^* - H_r^* E_\phi) dS$$

$$= \mathbf{i}_z \tfrac{1}{2}\mathscr{R}\,\frac{n}{\omega}\int (\mathbf{E}\times\bar{\mathbf{H}}).\,d\mathbf{S}.$$

In particular, if $n = 1$ as in the TE_{11} mode which has been previously discussed (4), this becomes

$$\bar{C} = \mathbf{i}_z \tfrac{1}{2}\mathscr{R}\,\frac{1}{\omega}\int (\mathbf{E}\times\bar{\mathbf{H}}).\,d\mathbf{S}$$

in agreement with the previous result.

5. If A_j is a component of the four potential, the Lagrangian density is given, using the gauge condition that $\dfrac{\partial A_j}{\partial x_j}$ vanishes, by (3) writing $\dfrac{\partial A_j}{\partial x_k} = A_{jk}$

$$L = -\frac{1}{2\mu_0}(A_{jk} - A_{kj})(A_{jk} - A_{kj}) \qquad \text{.......................(5.1)}$$

where the summation convention is used throughout. (There is an error of 2 in the original.) The formulæ for angular momentum density, indicated in § **1**, have associated with them tensor densities Φ'_{jkl} and Φ_{jkl} defined as follows (3)

$$\Phi_{jkl} = x_j T_{kl} - x_k T_{jl}$$

where $$\mu_0 T_{jk} = (A_{lj} - A_{jl})(A_{lk} - A_{kl}) + L\delta_{jk}.$$

$$\Phi'_{jkl} = M_{lkj} + x_j \theta_{kl} - x_k \theta_{jl}$$

where $$M_{kjl} = \frac{\partial L}{\partial A_{jk}} A_l - \frac{\partial L}{\partial A_{lk}} A_j, \qquad \text{..................................(5.2)}$$

$$\theta_{kj} = T_{kj} + \tfrac{1}{2}\frac{\partial}{\partial x_l}\{M_{kjl} - M_{jlk} - M_{lkj}\}.$$

Utilising equations (5.1) and (5.2) and the fact that $\frac{\partial L}{\partial A_{pq}}+\frac{\partial L}{\partial A_{qp}}=0$ it follows that

$$\Phi'_{jkl}-\Phi_{jkl}=\frac{\partial}{\partial x_\alpha}(x_jA_k-x_kA_j)\frac{\partial L}{\partial A_{l\alpha}}.$$

In general this expression is non zero, and while equivalence relations can be derived from this, it is easier to consider the equivalence of the vector quantities $\boldsymbol{c}_n$, and $\boldsymbol{c}'_n$ $\boldsymbol{m}$ and $\boldsymbol{m}'$.

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