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A remark on R. Moeckel's paper 'Geodesics on modular surfaces and continued fractions'

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Abstract. It is shown that a result by Moeckel holds not only for admissible subgroups of SL $(2, \mathbb{Z})$, but also for arbitrary subgroups of finite index.

The modular group $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ acts discontinuously on a hyperbolic plane $\mathscr{H} = \{z = x + iy;\ y > 0\}$. Let G be a subgroup of finite index in Γ. In his paper [**1**] Moeckel obtained the following result [**1**, Proposition 2.1]:

Let C be a G-cusp. If G is admissible, then for almost every irrational number β,

$$\lim_{N\to\infty} \frac{1}{N} |\{n \le N;\ \beta_n \in C\}| = w(C)/[\Gamma : G], \tag{1}$$

where β_n is the n*th approximant $[b_0, b_1, \ldots, b_n]$ of the continued fraction espansion of $\beta = [b_0, b_1, \ldots,]$, and $w(C)$ denotes the width of C.*

The objective of the present note is to show that Moeckel's proposition holds without the assumption that G is admissible.

It is necessary to say a few words about the correct statement of our generalization of Moeckel's proposition. Let $\bar{G}$ be the inhomogenized group of G in $\bar{\Gamma} = \mathrm{PSL}(2, \mathbb{Z})$ ([**2**, p. 71]). As Γ is actually viewed as a group of linear fractional transformations, the statement of the generalized proposition is the same as Moeckel's, but (1) is replaced by

$$\lim_{N\to\infty} \frac{1}{N} |\{n \le N;\ \beta_n \in C\}| = w(C)/[\bar{\Gamma} : \bar{G}].$$

Here $2[\Gamma : G] = [\bar{\Gamma} : \bar{G}]$ if $-\mathbf{I} \notin G$ and $[\Gamma : G] = [\bar{\Gamma} : \bar{G}]$ if $-\mathbf{I} \in G$, where $\mathbf{I}$ is the unit matrix.

Let $\mathscr{Q}$ be the fundamental quadrilateral defined by

$$\mathscr{Q} = \{z = x + iy;\ 0 \le x < 1, |z| \ge 1, |z-1| > 1\} \cup \{(1+\sqrt{3}i)/2\}.$$

Let $S(z) = 1/(-z+1)$. An *elementary triangle* is the image of $\mathscr{Q} \cup S\mathscr{Q} \cup S^2\mathscr{Q}$ under an element of Γ. The group G partitions the rational numbers into equivalence classes called *G-cusps.*

Let C be a *G-cusp* and Δ be an elementary triangle with a vertex in C. The triangle Δ may be left invariant by a cyclic subgroup of order 3 in G, which does not occur for admissible groups. Hence the restriction of the canonical projection $\pi: \mathcal{H} \to \mathcal{H}/G$ to Δ may fail to be injective. To establish Moeckel's proposition we need a function on $T_1(\mathcal{H}/G)$, the unit tangent bundle to $\mathcal{H}/G$, of the same character as the function $f_{(C,\Delta')}$ in the proof of Proposition 3.2 of [**1**]. So we define for $\Delta' = \pi\Delta$:

$$f_{(C,\Delta')}(x, y, \theta) = \begin{cases} 1/\sigma & \text{if } (x, y, \theta) \text{ lies on an initial segment of arclength } \sigma \text{ in } \Delta', \text{ associated with } C, \\ 0 & \text{otherwise.} \end{cases}$$

Here we view the coordinates (x, y, θ) of $T_1\mathcal{H}$, the unit tangent bundle to $\mathcal{H}$ [**1**, p. 70] as local coordinates of $T_1(\mathcal{H}/G)$ except possibly for the fixed point of the cyclic group or order 3. We can neglect this point for our purpose. By replacing G by a conjugation of G in Γ, if necessary, we can assume that $\Delta = \mathcal{Q} \cup S\mathcal{Q} \cup S^2\mathcal{Q}$ and ∞ belongs to C. In this case the width $w(C)$ of C is the smallest positive integer k such that the translation $z \to z + k$, is an element of G, and Δ is left invariant by the cyclic group $\{\mathbf{I}, S, S^2\}$. Let (x, y, θ) be a point of $T_1\mathcal{H}$ lying on an initial segment in Δ, associated with ∞. If we express this point by the coordinates (α, β, s) introduced at p. 70 of [**1**], then $-1 \le \alpha < 0$ and $1 < \beta < \infty$. If $(x, y) \in \mathcal{Q}$, then (a, β) is in the set $\bigcup_{i=1}^4 \Omega_i$ depicted in figure 1. If $(x, y) \in S\mathcal{Q}$, then (α, β) is in $\Omega_3 \cup \Omega_4 \cup \Omega_5$ and if $(x, y) \in S^2\mathcal{Q}$, then (α, β) is in $\Omega_2 \cup \Omega_4 \cup \Omega_5$. For the case where S is an element of G, we need the following lemma.

LEMMA. *Let (x_1, y_1, θ_1) and (x_2, y_2, θ_2) be points each of which lies on an initial segment in Δ associated with ∞. If they are equivalent under the action of $\{I, S, S^2\}$, then $(x_1, y_1, \theta_1) = (x_2, y_2, \theta_2)$.*

Proof. Assume that the two points (x_1, y_1, θ_1) and (x_2, y_2, θ_2) are distinct. It suffices to consider the cases (1) $(x_1, y_1) \in \mathcal{Q}$ and $(x_2, y_2) \in S\mathcal{Q}$, (2) $(x_1, y_1) \in \mathcal{Q}$ and $(x_2, y_2) \in S^2\mathcal{Q}$, and (3) $(x_1, y_1) \in S\mathcal{Q}$ and $(x_2, y_2) \in S^2\mathcal{Q}$. Express the points (x_i, y_i, θ_i), $i = 1, 2$, as (α_i, β_i, s_i) with $-1 \le \alpha_i < 0$, $1 < \beta_i < \infty$. For the case (1), if the two points are equivalent under the action of $\{\mathbf{I}, S, S^2\}$, then $(\alpha_1, \beta_1) = (S^2\alpha_2, S^2\beta_2)$. However, as figure 1 shows, this is impossible. The figure also shows that other cases are impossible. □

It follows from the lemma that, even though $\pi|_\Delta$ is not injective, the tangent vectors of $T_1\mathcal{H}$ lying on initial segments in Δ, associated with ∞ and the tangent vectors of $T_1(\mathcal{H}/G)$ lying on initial segments in Δ', associated with C are in one-to-one correspondence. Hence for the present function $f_{(C,\Delta')}$ the following computation is also true [**1**, p. 82]:

$$\frac{1}{2}\int_{T_1(\mathcal{H}/G)} f_{(C,\Delta')}\, d\mu = \int_1^\infty d\beta \int_{-1}^0 \frac{2d\alpha}{(\alpha-\beta)^2} \int_{\gamma(\alpha,\beta)} \frac{1}{\sigma}\, ds$$
$$= 2 \ln 2.$$

The function $f_{(C,\Delta')}$ is defined so that its integral over a geodesic counts the number of initial segments along the geodesic which lie in Δ', are associated with C, like

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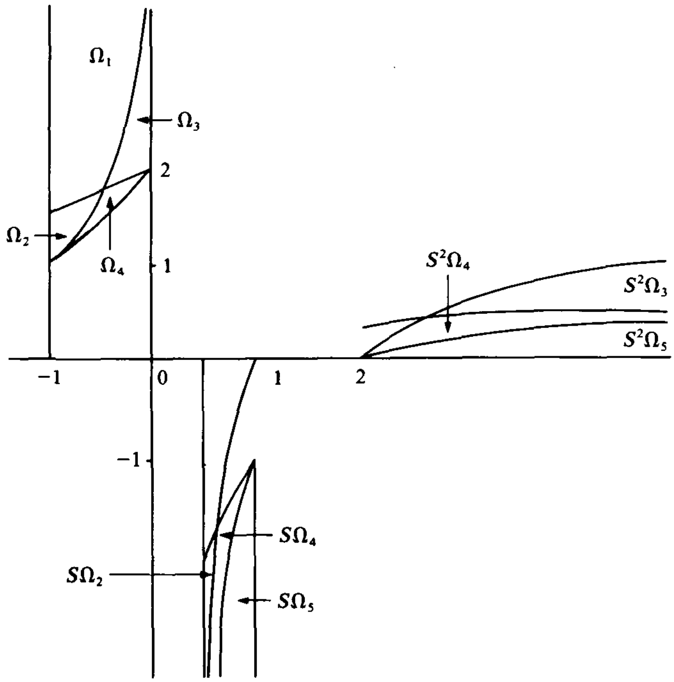

FIGURE 1

$\Omega_1=\{(\alpha,\beta);\ \beta\geq\max(-1/\alpha,(2-\alpha)/(1-\alpha)\}$,
$\Omega_2=\{(\alpha,\beta);\ \beta\geq-1/\alpha,\ \beta<(2-\alpha)/(1-\alpha)\}$,
$\Omega_3=\{(\alpha,\beta);\ \beta<-1/\alpha,\ \beta\geq(2-\alpha)/(1-\alpha)\}$,
$\Omega_4=\{(\alpha,\beta);\ \beta<\min(-1/\alpha,(2-\alpha)/(1-\alpha)),\ \beta>(2-\alpha)/(1-2\alpha)\}$,
$\Omega_5=\{(\alpha,\beta);\ \beta\leq(2-\alpha)/(1-2\alpha)\}$.

the function in Proposition 3.2 of [**1**]. Hence, by proceeding with this function, we can prove Proposition 3.2 for G which may not be admissible. Then Moeckel's Proposition 2.1 follows, because Proposition 3.2 is a rephrasing of Proposition 2.1 in terms of the symbolic description of geodesics on $\mathcal{H}/G$.

We conclude this note by offering some examples. Let

$$\Gamma_0(p)=\left\{\begin{bmatrix} a & b\\ c & d\end{bmatrix}\in\Gamma;\ c\equiv 0 \bmod p\right\},\quad\text{and}$$

$$\Gamma^0(p)=\left\{\begin{bmatrix} a & b\\ c & d\end{bmatrix}\in\Gamma;\ b\equiv 0 \bmod p\right\},$$

where p is a prime [**2**, Chap. IV, 3]. These groups are not admissible if, for example, $p=7$ and 13. But now we can apply Moeckel's Proposition to them and obtain for almost every irrational number β,

$$\lim_{N\to\infty}\frac{1}{N}|\{n\leq N;\ p|P_n\}|=\lim_{N\to\infty}\frac{1}{N}|\{n\leq N;\ p|Q_n\}|$$
$$=(1+p)^{-1},$$

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where the nth approximant of the continued fraction expansion of β is presented by a reduced ratio P_n/Q_n of integers.

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