ON THE FUNDAMENTAL THEOREM OF AFFINE GEOMETRY

P. Scherk

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The fundamental theorem of affine geometry is an easy corollary of the corresponding projective theorem 2.26 in Artin's Geometric Algebra. However, a simple direct proof based on Lipman's paper [this Bulletin, 4, 265-278] and his axioms 1 and 2 may be of some interest.

Lipman's [desarguian] affine geometry G determined a left linear vector space $L = \{a, b, \ldots\}$ over a skew field F. We wish to construct 1-1 transformations γ of G onto itself such that γ and γ^{-1} map straight lines onto straight lines preserving parallelism. Designate any point 0 as the origin of G. Multiplying γ with a suitable translation, we may assume $\gamma 0 = 0$. Thus γ will then be equivalent to a 1-1 transformation Γ of L onto itself which preserves linear dependence. Since Γ^{-1} will have the same properties, Γ must also preserve linear independence. By our assumptions

$$\Gamma(0) = 0 ,$$

$$\Gamma(\lambda a) = \varphi(\lambda, a) \cdot \Gamma(a)$$

for all $\lambda \in F$, $a \in L$.

Let a and b be linearly independent. The straight lines through a parallel to b and through b parallel to a intersect at a+b. Hence the lines through $\Gamma(a)$ parallel to $\Gamma(b)$ and through $\Gamma(b)$ parallel to $\Gamma(a)$ intersect in $\Gamma(a+b)$. This yields

$$(1) \qquad \Gamma(a+b) = \Gamma(a) + \Gamma(b)$$

if a and b are linearly independent. Replacing a by a+b, we obtain

$$(2) \qquad \Gamma(a-b) = \Gamma(a) - \Gamma(b) .$$

Suppose now that a and b are linearly dependent but that a, b, and a+b do not vanish. By axiom 2, there exists a c such that a and c are linearly independent. Since a+c and b as well as a+b and c are linearly independent, we have

$$\Gamma(a+b) + \Gamma(c) = \Gamma(a+b+c) = \Gamma((a+c)+b) = \Gamma(a+c) + \Gamma(b) = \Gamma(a) + \Gamma(b) + \Gamma(c).$$

This yields (1). Furthermore, (1) is trivial if $a = 0$ or $b = 0$. It remains to prove (1) if $a+b = 0$, i. e.

$$(3) \qquad \Gamma(a) + \Gamma(-a) = 0.$$

Choose b such that a and b are linearly independent. Then (1) and (2) imply

$$\Gamma(b) - \Gamma(a) = \Gamma(b-a) = \Gamma(b+(-a)) = \Gamma(b) + \Gamma(-a).$$

This verifies (3). Thus (1) - (3) are always valid.

Let a and b be linearly independent. Then

$$\begin{aligned}(\lambda, a)\,\Gamma(a) + \varphi(\lambda, b)\,\Gamma(b) &= \Gamma(\lambda a) + \Gamma(\lambda b) \\ &= \Gamma(\lambda a + \lambda b) = \Gamma(\lambda(a+b)) \\ &= \varphi(\lambda, a+b)\,\Gamma(a+b) \\ &= \varphi(\lambda, a+b)\,(\Gamma(a) + \Gamma(b)) .\end{aligned}$$

Since $\Gamma(a)$ and $\Gamma(b)$ are also linearly independent, this implies

$$\varphi(\lambda, a) = \varphi(\lambda, b) .$$

If a and b are linearly dependent, let a and c be independent. Then $\varphi(\lambda, a) = \varphi(\lambda, c) = \varphi(\lambda, b)$. Thus φ is independent of the vector, and we may write

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(4) $$\Gamma(\lambda a) = \lambda^{\varphi} \cdot \Gamma(a)$$

for all $a \in L$, $\lambda \in F$.

We have

$$(\lambda+\mu)^{\varphi} \Gamma(a) = \Gamma((\lambda+\mu)a) = \Gamma(\lambda a+\mu a)$$

$$= \Gamma(\lambda a) + \Gamma(\mu a)$$

$$= \lambda^{\varphi} \Gamma(a) + \mu^{\varphi} \Gamma(a) = (\lambda^{\varphi} + \mu^{\varphi}) \Gamma(a)$$

and

$$(\lambda\mu)^{\varphi} \Gamma(a) = \Gamma(\lambda\mu a) = \Gamma(\lambda(\mu a)) = \lambda^{\varphi} \Gamma(\mu a) = \lambda^{\varphi} \mu^{\varphi} \Gamma(a) .$$

Hence

(5) $$(\lambda+\mu)^{\varphi} = \lambda^{\varphi} + \mu^{\varphi} , \quad (\lambda\mu)^{\varphi} = \lambda^{\varphi} \mu^{\varphi}.$$

Thus φ is an automorphism of F.

Let $\{a_\alpha\}$ be a base of L. Thus every vector of L can be written in one and only one way as a left linear combination of finitely many a_αs. The $\Gamma(a_\alpha)$ will also form a base of L and

$$a = \Sigma \lambda_i a_{\alpha_i} \quad \text{implies} \quad \Gamma(a) = \Sigma \lambda_i^{\varphi} \Gamma(a_{\alpha_i}) .$$

This leads to the following construction of the transformations Γ: Let $\{a_\alpha\}$ be a fixed base of L. Let $\{b_\alpha\}$ be any base of L and let φ be an automorphism of L. Then

$$\Gamma : \Sigma \lambda_i a_{\alpha_i} \rightarrow \Sigma \lambda_i^{\varphi} b_{\alpha_i}$$

will be the most general 1-1 transformation of L onto itself such that the corresponding 1-1 transformation γ of G onto itself with $\gamma 0 = 0$ maps straight lines onto straight lines and preserves parallelism.

Orillia, Ont.