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GLOBAL EXISTENCE AND BLOW-UP FOR NON-NEWTON POLYTROPIC FILTRATION SYSTEM COUPLED WITH LOCAL SOURCE

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Abstract. This paper deals with the global existence and blow-up properties of the following non-Newton polytropic filtration system coupled with local source: $u_t - \triangle_{m,p}u = av^\alpha,\ v_t - \triangle_{n,q}v = bu^\beta$. Under appropriate hypotheses, we prove that the solution either exists globally or blows up in finite time depending on the initial data and the relations between $\alpha\beta$ and $mn(p-1)(q-1)$.

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1. Introduction. In this paper, we consider the following doubly degenerate parabolic system:

$$\begin{array}{lll} u_t - \triangle_{m,p}u = av^\alpha, \quad v_t - \triangle_{n,q}v = bu^\beta, & (x,t) \in \Omega \times (0,T], \\ u(x,t) = 0, \quad v(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T], \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{array} \tag{1.1}$$

where $\triangle_{k,\gamma}\Theta = \nabla \cdot (|\nabla \Theta^k|^{\gamma-2} \nabla \Theta^k)$, $\nabla\Theta^k = k\Theta^{k-1}(\Theta_{x_1}, \ldots, \Theta_{x_N})\ k \geq 1,\quad \gamma > 2$, $N \geq 1$. $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$; $m, n \geq 1$, $p, q > 2,\ \alpha, \beta > 0$ are parameters; a, b are positive constants.

Throughout this paper, we denote $Q_T = \Omega \times (0,T],\ S_T = \partial\Omega \times [0,T],\ T > 0$ and make the following assumption on initial data:

(H) The non-negative initial data satisfies compatibility conditions and $u_0^m(x) \in C(\bar{\Omega}) \cap W_0^{1,p}(\Omega)$, $v_0^n(x) \in C(\bar{\Omega}) \cap W_0^{1,q}(\Omega)$ and $\nabla u_0^m \cdot \nu < 0$, $\nabla v_0^n \cdot \nu < 0$ on the boundary $\partial\Omega$, where ν is unit outer normal vector on $\partial\Omega$.

Parabolic systems like (1.1) appear in population dynamics, chemical reactions, heat transfer and so on. In particular, equations (1.1) may be used to describe the nonstationary flows in a porous medium of fluids with a power dependence of the tangential stress on the velocity of displacement under polytropic conditions. In this case, equations (1.1) are called the non-Newton polytropic filtration equations (see [**15**, **29**, **31**] and references therein). The problems with non-linear reaction term and non-linear diffusion include blow-up and global existence conditions of solutions, blow-up

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rates and blow-up sets, etc. (see the surveys [**3**, **13**, **17**, **25**]). Here, we say solution blows up in finite time if the solution becomes unbounded (in the sense of maximum norm) at that time.

System (1.1) has been studied by many authors. For $p = q = 2$, it is called porous medium equations (see [**23**, **24**, **33**] for non-linear boundary conditions; see [**7**, **10**, **11**] for local non-linear reaction terms; see [**1**, **6**, **8**, **9**, **19**] for non-local non-linear reaction terms). In [**10**, **11**], V. A. Galaktionov et al. consider the following problem:

$$\begin{array}{ll} u_t - \triangle u^m = v^p,\ v_t - \triangle v^n = u^q, & (x,t) \in \Omega \times (0,T], \\ u(x,t) = 0,\ \ v(x,0) = 0, & x \in \partial\Omega \times (0,T], \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \Omega, \end{array} \tag{1.2}$$

where $m, n > 1$ and $p, q > 0$. They get the following results: (a) If $pq < mn$, then all solutions of (1.2) with continuous, bounded initial values are global; (b) If $pq > mn$, then there exist both non-trivial global solutions and non-global solutions of (1.2); (c) When $pq = mn$, all solutions are global if the diameter of the domain is sufficiently small. In particular, when $p = n$, $q = m$, all solutions of (1.2) are global if $\lambda > 1$, while if $\lambda < 1$, there are no non-trivial global solutions of (1.2), where λ is the first eigenvalue of operator $-\triangle$ with homogeneous Dirichlet boundary condition.

When $m = n = 1$, problem (1.1) is called p-Laplace equations. In the last three decades, many authors have studied the following degenerate parabolic problem:

$$\begin{array}{ll} u_t - \bigtriangledown \cdot (|\bigtriangledown u|^{p-2}) = f(u), & (x,t) \in \Omega \times (0,T], \\ u(x,t) = 0, & x \in \partial\Omega \times (0,T], \\ u(x,0) = u_0(x), & x \in \Omega, \end{array} \tag{1.3}$$

under different conditions (see [**12**, **35**] for non-linear boundary conditions; see [**14**, **16**, **27**, **34**] for local non-linear reaction terms; see [**18**] for non-local non-linear reaction terms). In the book discussed in ref. [**4**], the existence, uniqueness and regularity of solutions were obtained. When $f(u) = -u^q$, $q > 0$ or $f(u) \equiv 0$ extinction phenomenon of the solution may appear (see [**28**, **32**]); however, if $f(u) = u^q$, $q > 1$ the solution may blow up in finite time (see [**14**, **16**, **27**, **34**]). Roughly speaking, the results read: (1) the solution u exists globally if $q < p - 1$; (2) u blows up in finite time if $q > p - 1$ and $u_0(x)$ is sufficiently large.

For general $m, n \geq 1$, $p, q > 2$, $\alpha, \beta > 0$, problem (1.1) is called non-Newton polytropic filtration system and only the case of non-linear boundary condition has been considered extensively (see [**26**, **30**, **36**]). However, it seems that the case of local (non-local) non-linear reaction terms is studied less. In [**26**], Sun & Wang consider the following doubly degenerate equation:

$$\begin{array}{ll} (u^m)_t = \bigtriangledown \cdot (|\bigtriangledown u|^{p-2}), & (x,t) \in \Omega \times (0,T], \\ |\bigtriangledown u|^{p-2} \bigtriangledown u \cdot \nu = u^\alpha, & x \in \partial\Omega \times (0,T], \\ u(x,0) = u_0(x), & x \in \Omega; \end{array} \tag{1.4}$$

by using upper and lower solution methods, they proved that all positive solutions of (1.4) exist globally if and only if $\alpha \leq m$ when $m \leq p - 1$ or $\alpha \leq \frac{(p-1)(m+1)}{p}$ when $m > p - 1$.

Motivated by the paper cited above, in this paper, we investigate the blow-up properties of solutions of the problem (1.1) and extend the results of [**10**, **11**, **14**, **16**, **27**, **34**] to more generalized cases. Our main results are stated as follows:

THEOREM 1.1. *Suppose the initial data* $(u_0(x), v_0(x))$ *satisfies the assumption* (H), *the solution of problem* (1.1) *exists globally if one of the following conditions holds:* (i) $\alpha\beta < mn(p-1)(q-1)$; ($ii$) $\alpha\beta = mn(p-1)(q-1)$ *and* $\|\Omega\|$ *is sufficiently small, where* $\|\Omega\|$ *is the measure of* Ω; (iii) $\alpha\beta > mn(p-1)(q-1)$ *and the initial data is sufficiently small.*

THEOREM 1.2. *Suppose the initial data* $(u_0(x), v_0(x))$ *satisfies the assumption* (H), *if* $\alpha\beta > mn(p-1)(q-1)$ *and the initial data is sufficiently large, then the solution of problem* (1.1) *blows up in finite time.*

This paper is organized as follows: In Section 2, we give some preliminaries, which is the basement to prove our theorems. The proof of Theorem 1.1 is the subject of Section 3. In Section 4, we consider the blow-up properties of problem (1.1) and give the proof of Theorem 1.2.

2. Preliminaries. As it is well known that degenerate equations need not have classical solutions, we give a precise definition of a weak solution for problem (1.1)

DEFINITION OF WEAK SOLUTION. A pair of functions $(u(x,t), v(x,t))$ is called an upper (lower) solution of problem (1.1) in $\bar{Q}_T \times \bar{Q}_T$ if and only if $u^m(x,t) \in C(0,T;L^\infty(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$, $v^n(x,t) \in C(0,T;L^\infty(\Omega)) \cap L^q(0,T;W_0^{1,q}(\Omega))$, $u_t \in L^2(0,T;L^2(\Omega))$, $v_t \in L^2(0,T;L^2(\Omega))$, $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$, and the following inequalities:

$$\begin{aligned}
&\int_\Omega u(x,t_2)\psi(x,t_2)\,dx - \int_\Omega u(x,t_1)\psi(x,t_1)\,dx \ge (\le) \int_{t_1}^{t_2}\int_\Omega u\psi_t\,dx\,dt \\
&\quad - \int_{t_1}^{t_2}\int_\Omega |\nabla u^m|^{p-2}\nabla u^m \cdot \nabla\psi\,dx\,dt + a\int_{t_1}^{t_2}\int_\Omega \psi(x,t)v^\alpha(x,t)\,dx\,dt, \\
&\int_\Omega v(x,t_2)\psi(x,t_2)\,dx - \int_\Omega v(x,t_1)\psi(x,t_1)\,dx \ge (\le) \int_{t_1}^{t_2}\int_\Omega v\psi_t\,dx\,dt \\
&\quad - \int_{t_1}^{t_2}\int_\Omega |\nabla v^n|^{q-2}\nabla v^n \cdot \nabla\psi\,dx\,dt + b\int_{t_1}^{t_2}\int_\Omega \psi(x,t)u^\beta(x,t)\,dx\,dt,
\end{aligned}$$

hold for all $0 < t_1 < t_2 < T$, where $\psi(x,t) \ge 0 \in C^{1,1}(\bar{Q}_T)$ such that $\psi(x,T) = 0$ and $\psi(x,t) = 0$ on S_T. In particular, $(u(x,t), v(x,t))$ is called a weak solution of (1.1) if it is both a weak upper and a weak lower solution.

The local existence of weak solutions to problem (1.1) under the assumption (H) and the following comparison principle is standard (see [**4**, **15**, **29**, **31**]).

COMPARISON PRINCIPLE. Suppose that $(\underline{u}(x,t), \underline{v}(x,t))$ and $(\bar{u}(x,t), \bar{v}(x,t))$ are upper and lower solutions of problem (1.1) on $\bar{Q}_T \times \bar{Q}_T$, respectively, then $(\underline{u}(x,t), \underline{v}(x,t)) \le (\bar{u}(x,t), \bar{v}(x,t))$ a.e. on $\bar{Q}_T \times \bar{Q}_T$.

In order to study the globally existing solutions to problem (1.1), we need to study the following elliptic system:

$$\begin{cases} -\triangle_{k,\gamma}\Theta = \lambda_{k,\gamma}\Theta^{k(\gamma-1)}, & x \in \Omega, \\ \Theta = 1, & x \in \partial\Omega, \end{cases} \tag{2.1}$$

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where $\triangle_{k,\gamma}\Theta$ is defined in (1.1), and we get the following lemma:

LEMMA 2.1. *Problem* (2.1) *has a solution* $\Theta(x)$ *with* $\lambda_{k,\gamma} = \lambda > 0$, *which satisfies the following relations:*

$$\Theta(x) \geq 1 \text{ in } \Omega, \quad \nabla\Theta \cdot \nu < 0 \text{ on } \partial\Omega, \quad \lim_{\operatorname{diam}(\Omega)\to 0} \lambda \to +\infty, \ \sup_{x\in\Omega} \Theta = M < +\infty,$$

where M *is a positive constant.*

Proof. Set $\Theta^k = \Phi$, then Φ satisfies the following equation:

$$\begin{array}{ll} -\nabla \cdot (|\nabla \Phi|^{\gamma-2} \nabla \Phi) = \lambda_{k,\gamma} \Phi^{\gamma-1}, & x \in \Omega, \\ \Phi = 1, & x \in \partial\Omega. \end{array} \tag{2.2}$$

References [**20**–**22**] mention that problem (2.2) has a solution $\Phi(x)$ with first eigenvalue $\lambda > 0$ and satisfies the following relations:

$$\Phi(x) \geq 1 \text{ in } \Omega, \quad \nabla\Phi \cdot \nu < 0 \text{ on } \partial\Omega, \quad \lim_{\operatorname{diam}(\Omega)\to 0} \lambda \to +\infty, \ \sup_{x\in\Omega} \Phi = M' < +\infty,$$

where M' is a positive constant. Since $\Theta = \Phi^{1/k}$, the conclusion of Lemma 2.1 comes directly. □

3. Global existence of solution. In this section, we investigate the global existence property of the solutions to problem (1.1) and prove Theorem 1.1. The main method is constructing a globally upper solution and using comparison principle to achieve our purpose.

Proof of Theorem 1.1. Let $\varphi(x)$ and $\psi(x)$ be the solution of the following elliptic problem with first eigenvalues $\lambda_1 > 0, \ \lambda_2 > 0$, respectively,

$$\begin{cases} -\triangle_{m,p}\, \varphi = \lambda_1 \varphi^{m(p-1)}, & x \in \Omega, \\ \varphi = 1, & x \in \partial\Omega, \end{cases} \qquad \begin{cases} -\triangle_{n,q}\, \psi = \lambda_2 \psi^{n(q-1)}, & x \in \Omega, \\ \psi = 1, & x \in \partial\Omega. \end{cases}$$

Then from Lemma 2.1, we get following relations:

$$\begin{array}{l} \varphi(x), \psi(x) \ \geq \ 1 \text{ in } \Omega, \quad \nabla\varphi \cdot \nu, \nabla\psi \cdot \nu < 0 \text{ on } \partial\Omega, \\ \displaystyle\lim_{\operatorname{diam}(\Omega)\to 0} \lambda_i \to +\infty \ (i = 1, 2), \ \sup_{x\in\Omega} \varphi = M_1 < +\infty, \ \sup_{x\in\Omega} \psi = M_2 < +\infty, \end{array}$$

where M_1, M_2 are positive constants. Let $\bar{u}(x, t) = \Lambda_1 \varphi(x), \quad \bar{v}(x, t) = \Lambda_2 \psi(x)$, where $\Lambda_1, \Lambda_2 > 0$ will be determined later. Then from direct computation we get

$$\bar{u}_t - \triangle_{m,p}\bar{u} = \lambda_1 (\Lambda_1 \varphi)^{m(p-1)} \geq \lambda_1 \Lambda_1^{m(p-1)}, \quad a\bar{v}^{\alpha} \leq aM_2 \Lambda_2^{\alpha},$$

$$\bar{v}_t - \triangle_{n,q}\bar{v} = \lambda_2 (\Lambda_2 \psi)^{n(q-1)} \geq \lambda_2 \Lambda_2^{n(q-1)}, \quad b\bar{u}^{\beta} \leq bM_1^{\beta} \Lambda_1^{\beta}.$$

So, $(\bar{u}(x, t), \bar{v}(x, t))$ is an upper solution of problem (1.1), if

$$\begin{array}{l} \lambda_1 \Lambda_1^{m(p-1)} \geq aM_2^{\alpha} \Lambda_2^{\alpha}, \quad \lambda_2 \Lambda_2^{n(q-1)} \geq bM_1^{\beta} \Lambda_1^{\beta}, \\ \bar{u}(x, t)|_{\partial\Omega} \geq 0, \ \bar{v}(x, t)|_{\partial\Omega} \geq 0, \ \bar{u}(x, 0) \geq u_0(x), \quad \bar{v}(x, 0) \geq v_0(x). \end{array} \tag{3.1}$$

Next we will prove (3.1) in three cases.

(i) When $\alpha\beta < mn(p-1)(q-1)$, then (3.1) holds if we choose Λ_1, Λ_2 large enough such that

$$\Lambda_1 > \max\left\{\max_{x\in\bar\Omega} u_0(x),\ \left((a/\lambda_1)(b/\lambda_2)^{\frac{\alpha}{n(q-1)}} M_1^{\frac{\alpha\beta}{n(q-1)}} M_2^{\alpha}\right)^{\frac{n(q-1)}{mn(p-1)(q-1)-\alpha\beta}}\right\},$$
$$\Lambda_2 > \max\left\{\max_{x\in\bar\Omega} v_0(x),\ \left((a/\lambda_1)^{\frac{\beta}{m(p-1)}} (b/\lambda_2) M_1^{\beta} M_2^{\frac{\alpha\beta}{m(p-1)}}\right)^{\frac{m(p-1)}{mn(p-1)(q-1)-\alpha\beta}}\right\},$$

(ii) When $\alpha\beta = mn(p-1)(q-1)$, we can choose Λ_1, Λ_2 large enough such that

$$\Lambda_1 > \max_{x\in\bar\Omega} u_0(x),\quad \Lambda_2 > \max_{x\in\bar\Omega} v_0(x),$$

then (3.1) holds if $\|\Omega\|$ is small enough such that λ_1, λ_2 are large enough to satisfy

$$1 > \max\left\{(a/\lambda_1)(b/\lambda_2)^{\frac{\alpha}{n(q-1)}} M_1^{\frac{\alpha\beta}{n(q-1)}} M_2^{\alpha},\ (a/\lambda_1)^{\frac{\beta}{m(p-1)}} (b/\lambda_2) M_1^{\beta} M_2^{\frac{\alpha\beta}{m(p-1)}}\right\}.$$

(iii) When $\alpha\beta > mn(p-1)(q-1)$, we can take Λ_1, Λ_2 small enough such that

$$\Lambda_1 < \left((a/\lambda_1)(b/\lambda_2)^{\frac{\alpha}{n(q-1)}} M_1^{\frac{\alpha\beta}{n(q-1)}} M_2^{\alpha}\right)^{\frac{n(1-q)}{\alpha\beta-mn(p-1)(q-1)}},$$
$$\Lambda_2 < \left((a/\lambda_1)^{\frac{\beta}{m(p-1)}} (b/\lambda_2) M_1^{\beta} M_2^{\frac{\alpha\beta}{m(p-1)}}\right)^{\frac{m(1-p)}{\alpha\beta-mn(p-1)(q-1)}}.$$

Furthermore, if the initial data is sufficiently small such that $u_0(x) \le \Lambda_1$ and $v_0(x) \le \Lambda_2$, then (3.1) holds. The proof of Theorem 1.1 is complete. $\square$

4. Blow-up of solution. In this section, we investigate the blow-up property of the solutions to problem (1.1) and prove Theorem 1.2. The main method is constructing a blowing-up lower solution and using comparison principle to achieve our purpose.

Proof of Theorem 1.2. Set

$$\underline{u}(x,t) = (T-t)^{-\gamma_1} V_1(\xi),\ \xi = |x|(T-t)^{-\ell_1},\ V_1(\xi) = \left(1 + A/2 - \xi^2/(2A)\right)_+^{1/m},$$
$$\underline{v}(x,t) = (T-t)^{-\gamma_2} V_2(\eta),\ \eta = |x|(T-t)^{-\ell_2},\ V_2(\eta) = \left(1 + A/2 - \eta^2/(2A)\right)_+^{1/n},$$

where $\gamma_i, \ell_i > 0 (i = 1, 2)$, $A > 1$ and $0 < T < 1$ are parameters to be determined. It is easy to see that $\underline{u}(x,t), \underline{v}(x,t)$ blow up at time T, so it is enough to prove $(\underline{u}(x,t), \underline{v}(x,t))$ is a lower solution of problem (1.1). If we choose T small enough such that

$$\operatorname{supp} \underline{u}(\cdot,t) = \overline{B(0, R(T-t)^{\ell_1})} \subset \overline{B(0, RT^{\ell_1})} \subset \Omega,$$
$$\operatorname{supp} \underline{v}(\cdot,t) = \overline{B(0, R(T-t)^{\ell_2})} \subset \overline{B(0, RT^{\ell_2})} \subset \Omega,$$

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where $R = (A(2+A))^{1/2}$, then $\underline{u}(x,t)|_{\partial\Omega} = 0$, $\underline{v}(x,t)|_{\partial\Omega} = 0$. Next if we choose the initial data large enough such that

$$u_0(x) \geq \frac{1}{T^{\gamma_1}} V_1\left(\frac{|x|}{T^{\ell_1}}\right), \quad v_0(x) \geq \frac{1}{T^{\gamma_2}} V_2\left(\frac{|x|}{T^{\ell_2}}\right),$$

then $(\underline{u}(x,t), \underline{v}(x,t))$ is a lower solution of problem (1.1) if

$$\underline{u}_t - \triangle_{m,p}\underline{u} \leq a\underline{v}^{\alpha}, \quad \underline{v}_t - \triangle_{n,q}\underline{v} \leq b\underline{u}^{\beta}, \quad (x,t) \in \Omega \times (0,T]. \tag{4.1}$$

By direct computation, we obtain

$$\begin{aligned}
\underline{u}_t &= \frac{\gamma_1 V_1(\xi) + \ell_1 \xi V_1'(\xi)}{(T-t)^{\gamma_1+1}}, \quad \underline{v}_t = \frac{\gamma_2 V_2(\eta) + \ell_2 \eta V_2'(\eta)}{(T-t)^{\gamma_2+1}},\\
\nabla \underline{u}^m &= \frac{x}{A(T-t)^{m\gamma_1+2\ell_1}}, \quad -\triangle\, \underline{u}^m = \frac{N}{A(T-t)^{m\gamma_1+2\ell_1}},\\
\nabla \underline{v}^n &= \frac{x}{A(T-t)^{n\gamma_2+2\ell_2}}, \quad -\triangle\, \underline{v}^n = \frac{N}{A(T-t)^{n\gamma_2+2\ell_2}},
\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
\triangle_{m,p}\underline{u} &= |\nabla \underline{u}^m|^{p-2} \triangle\, \underline{u}^m + (p-2)|\nabla \underline{u}^m|^{p-4} (\nabla \underline{u}^m)^T \cdot (H_x(\underline{u}^m)) \cdot \nabla \underline{u}^m\\
&= |\nabla \underline{u}^m|^{p-2} \triangle\, \underline{u}^m + (p-2)|\nabla \underline{u}^m|^{p-4} \sum_{j=1}^{N}\sum_{i=1}^{N} \frac{\partial \underline{u}^m}{\partial x_i} \frac{\partial^2 \underline{u}^m}{\partial x_i x_j} \frac{\partial \underline{u}^m}{\partial x_j},
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
\triangle_{n,q}\underline{v} &= |\nabla \underline{v}^n|^{q-2} \triangle\, \underline{v}^n + (q-2)|\nabla \underline{v}^n|^{q-4} (\nabla \underline{v}^n)^T \cdot (H_x(\underline{v}^n)) \cdot \nabla \underline{v}^n\\
&= |\nabla \underline{v}^n|^{q-2} \triangle\, \underline{v}^n + (q-2)|\nabla \underline{v}^n|^{q-4} \sum_{j=1}^{N}\sum_{i=1}^{N} \frac{\partial \underline{v}^n}{\partial x_i} \frac{\partial^2 \underline{v}^n}{\partial x_i x_j} \frac{\partial \underline{v}^n}{\partial x_j},
\end{aligned} \tag{4.4}$$

where $H_x(\underline{u}^m)$, $H_x(\underline{v}^n)$ denote the Hessian matrix of $\underline{u}^m(x,t)$, $\underline{v}^n(x,t)$, respectively.

Denote $d(\Omega) = \mathrm{diam}(\Omega)$, then from (4.2) and (4.3), we get

$$\begin{aligned}
|\triangle_{m,p}\, \underline{u}| &\leq \frac{N}{A(T-t)^{m\gamma_1+2\ell_1}} \left(\frac{d(\Omega)}{(T-t)^{m\gamma_1+2\ell_1}}\right)^{p-2}\\
&\quad + (p-2)\left(\frac{d(\Omega)}{(T-t)^{m\gamma_1+2\ell_1}}\right)^{p-4} \left(\frac{d(\Omega)}{(T-t)^{m\gamma_1+2\ell_1}}\right)^{2} \frac{N}{A(T-t)^{m\gamma_1+2\ell_1}}\\
&= \frac{N(p-1)(d(\Omega))^{p-2}}{A(T-t)^{(m\gamma_1+2\ell_1)(p-1)}}.
\end{aligned} \tag{4.5}$$

Similarly, from (4.2) and (4.4) we obtain

$$\begin{aligned}
|\triangle_{n,q}\, \underline{v}| &\leq \frac{N}{A(T-t)^{n\gamma_2+2\ell_2}} \left(\frac{d(\Omega)}{(T-t)^{n\gamma_2+2\ell_2}}\right)^{q-2}\\
&\quad + (q-2)\left(\frac{d(\Omega)}{(T-t)^{n\gamma_2+2\ell_2}}\right)^{q-4} \left(\frac{d(\Omega)}{(T-t)^{n\gamma_2+2\ell_2}}\right)^{2} \frac{N}{A(T-t)^{n\gamma_2+2\ell_2}}\\
&= \frac{N(q-1)(d(\Omega))^{q-2}}{A(T-t)^{(n\gamma_2+2\ell_2)(q-1)}}.
\end{aligned} \tag{4.6}$$

If $0 \le \xi, \eta \le A$, then $1 \le V_1(\xi) \le (1+\frac{A}{2})^{1/m}$, $1 \le V_2(\eta) \le (1+\frac{A}{2})^{1/n}$, and $V_1'(\xi) \le 0$, $V_2'(\eta) \le 0$. Then from (4.2)–(4.6) we get

$$\underline{u}_t - \triangle_{m,p}\underline{u} - a\underline{v}^\alpha \le \frac{\gamma_1\left(1+\frac{A}{2}\right)^{1/m}}{(T-t)^{\gamma_1+1}} + \frac{N(p-1)(d(\Omega))^{p-2}}{A(T-t)^{(m\gamma_1+2\ell_1)(p-1)}} - \frac{a}{(T-t)^{\alpha\gamma_2}}, \tag{4.7}$$

$$\underline{v}_t - \triangle_{n,q}\underline{v} - b\underline{u}^\beta \le \frac{\gamma_2\left(1+\frac{A}{2}\right)^{1/n}}{(T-t)^{\gamma_2+1}} + \frac{N(q-1)(d(\Omega))^{q-2}}{A(T-t)^{(n\gamma_2+2\ell_2)(q-1)}} - \frac{b}{(T-t)^{\beta\gamma_1}}. \tag{4.8}$$

If $\xi, \eta \ge A$, since m, $n \ge 1$, we obtain $V_1(\xi) \le 1$, $V_2(\eta) \le 1$ and $V_1'(\xi) \le -\frac{1}{m}$, $V_2'(\eta) \le -\frac{1}{n}$. Combining the above inequalities and (4.2)–(4.6), we get

$$\underline{u}_t - \triangle_{m,p}\underline{u} - a\underline{v}^\alpha \le \frac{\gamma_1 - \frac{1}{m}\ell_1 A}{(T-t)^{\gamma_1+1}} + \frac{N(p-1)(d(\Omega))^{p-2}}{A(T-t)^{(m\gamma_1+2\ell_1)(p-1)}}, \tag{4.9}$$

$$\underline{v}_t - \triangle_{n,q}\underline{v} - b\underline{u}^\beta \le \frac{\gamma_2 - \frac{1}{n}\ell_2 A}{(T-t)^{\gamma_2+1}} + \frac{N(q-1)(d(\Omega))^{q-2}}{A(T-t)^{(n\gamma_2+2\ell_2)(q-1)}}. \tag{4.10}$$

If $0 \le \xi \le A$ and $\eta \ge A$, we have (4.7) and (4.10) hold. If $\xi \ge A$ and $0 \le \eta \le A$, we have (4.8) and (4.9) hold.

So, from the above discussions, (4.1) holds if the right-hand sides of (4.7)–(4.10) are non-positive.

Since $p, q > 2$, $m, n \ge 1$ and $\alpha\beta > mn(p-1)(q-1) > mn \ge 1$, we can choose two constants $\ell_1, \ell_2 > 0$ small enough such that

$$\frac{1+\alpha}{\alpha\beta-1} < \frac{1-2\ell_1(p-1)}{m(p-1)-1}, \quad \frac{1+\beta}{\alpha\beta-1} < \frac{1-2\ell_2(q-1)}{n(q-1)-1}.$$

Then we can choose two constants γ_1, γ_2 such that

$$\frac{1+\alpha}{\alpha\beta-1} < \gamma_1 < \frac{1-2\ell_1(p-1)}{m(p-1)-1}, \quad \frac{1+\beta}{\alpha\beta-1} < \gamma_2 < \frac{1-2\ell_2(q-1)}{n(q-1)-1},$$

i.e.,

$$(m\gamma_1 + 2\ell_1)(p-1) < \gamma_1 + 1 < \alpha\gamma_2, \quad (n\gamma_2 + 2\ell_2)(q-1) < \gamma_2 + 1 < \beta\gamma_1.$$

Furthermore, if we choose $A > \max\{1, m\gamma_1/\ell_1, n\gamma_2/\ell_2\}$, then for $T > 0$ sufficiently small, the right-hand sides of (4.9)–(4.12) are non-positive, so (4.1) holds. The proof of Theorem 1.2 is complete. □

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