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TRIVIAL SET-STABILIZERS IN FINITE PERMUTATION GROUPS

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For which permutation groups does there exist a subset of the permuted set whose stabilizer in the group is trivial?

The permuted set has so many subsets that one might expect that subsets with trivial stabilizer usually exist. The symmetric and alternating groups are obvious exceptions to this expectation. Another, more interesting, infinite family of exceptions are the 2-Sylow subgroups of the symmetric groups on 2^n symbols, in their natural representations on 2^n points.

One of our main results, Corollary 1, sheds some light on this last family of groups. We show that when the permutation group has odd order, there is indeed a subset of the permuted set whose stabilizer in the group is trivial. Corollary 1 follows easily from Theorem 1, which completely classifies the primitive solvable permutation groups in which every subset of the permuted set has non-trivial stabilizer. All such groups have degree at most nine.

The proof of Theorem 1 uses Wolf's upper bound on the order of primitive solvable groups, as well as Suprunenko's work on maximal irreducible solvable linear groups of low degree.

In this paper we consider only solvable groups, but the question in the first paragraph seems at least as interesting, though more difficult, for nonsolvable groups. Our original interest in this question was to prove that for solvable groups G, monomial G-modules usually contain vectors whose centralizer in G is abelian. Corollary 2 gives a result of this type.

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1. Primitive solvable groups. Throughout this section G will be a primitive solvable permutation group on a finite set Ω. We denote by M a point stabilizer, so that M is a maximal subgroup of G. Then G has a regular normal elementary abelian subgroup K, which is self-centralizing in G. We have $G = MK$, $M \cap K = 1$, and M acts faithfully and irreducibly on K by conjugation. The conjugation action of M on K is permutation isomorphic to the action of $M \subseteq G$ on Ω. We may consider M as a subgroup of $GL(n, p)$ where $|K| = p^n$ for a prime number p.

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To avoid repeating the same phrase over and over, we shall say that (G, Ω) satisfies $(*)$ if some subset of Ω has trivial stabilizer in G. Equivalently, (G, Ω) satisfies $(*)$ if and only if G has a regular orbit on $P(\Omega)$, the power set of Ω.

We shall say that (G, Ω) satisfies $(**)$ if there is a subset $S \subseteq \Omega$ with $|S| \neq |\Omega|/2$ whose stabilizer in G is trivial. Of course $(*)$ and $(**)$ are equivalent when $|\Omega|$ is odd. Since $g \in G$ stabilizes S if and only if g stabilizes $\Omega - S$, the condition $|S| \neq |\Omega|/2$ in the definition of $(**)$ could be replaced by $|S| < |\Omega|/2$.

LEMMA 1. *Let $g \in G^{\#}$. Then g fixes at most $|\Omega|/2$ points of Ω and is therefore a product of at most $3|\Omega|/4$ disjoint cycles on Ω.*

Proof. We may assume that g fixes at least one point, and then that $g \in M$. Since $|C_K(g)| \leqq |K|/2$, and the conjugation action of M on K is permutation isomorphic to the action of M on Ω, it follows that g fixes at most $|\Omega|/2$ points of Ω. The statement about cycles is clear.

LEMMA 2. *If $|\Omega| > 80$, then (G, Ω) satisfies $(**)$.*

Proof. We count the number of ordered pairs (g, S) where $g \in G^{\#}$, $S \subseteq \Omega$, and g stabilizes S. By [**2**, Corollary 3.3], we have

$$|G| < 24^{-1/3}|\Omega|^{\alpha+1},$$

where $2.20 < \alpha < 2.25$. If g is the product of n disjoint cycles, then g will stabilize precisely 2^n subsets of Ω. Thus Lemma 1 implies that the number of ordered pairs (g, S) as above is at most

$$24^{-1/3}\, 2^{3|\Omega|/4}\, |\Omega|^{\alpha+1}.$$

Since Ω has $2^{|\Omega|}$ subsets we can certainly find a subset $S \subseteq \Omega$ with trivial stabilizer in G provided that

$$24^{-1/3}\, 2^{3|\Omega|/4}\, |\Omega|^{\alpha+1} < 2^{|\Omega|}$$

or equivalently

$$(\alpha + 1) \log_2 |\Omega| - (1/3) \log_2 (24) < |\Omega|/4.$$

One checks easily that this holds for $|\Omega| > 80$ (in fact, even for $|\Omega| \geqq 75$).

To prove the slightly stronger assertion that (G, Ω) satisfies $(**)$ we may assume that $|\Omega| \geqq 128$. The number of ordered pairs (g, S) as above is still at most

$$(24)^{-1/3}\, 2^{3|\Omega|/4}\, |\Omega|^{\alpha+1}.$$

The number of subsets of Ω of cardinality different from $|\Omega|/2$ is greater than $2^{|\Omega|-1}$. One checks easily that

$$(24)^{-1/3} 2^{3|\Omega|/4} |\Omega|^{\alpha+1} < 2^{|\Omega|-1}$$

for $|\Omega| \geqq 128$.

LEMMA 3. *If* $|\Omega| = 64$, *then* (G, Ω) *satisfies* $(**)$.

Proof. Consider the action of $GL(6, 2)$ on its natural 6-dimensional module over $GF(2)$. The centralizer in $GL(6,2)$ of a hyperplane is elementary abelian of order 32. There are 63 hyperplanes, so the number of hyperplane-centralizing elements of $GL(6, 2)^{\#}$ is precisely 63.31.

It follows from the second sentence of the preceding paragraph that any element of $G^{\#}$ has either 48 or at most 40 cycles on Ω. Let

$$G_0 = \{g \in G |\ g \text{ has 48 cycles on } \Omega\},$$

so that G_0 is a set of involutions of G, and $|G_0| \leqq 63.31$.

Thus the number of subsets of Ω stabilized by elements of G_0 is at most $(63.31)\ (2^{48})$. Let $G_1 = G^{\#} - G_0$. The number of subsets of Ω stabilized by elements of G_1 is at most

$$2^{40}|G_1| < 2^{40}(24)^{-1/3}(64)^{3.25},$$

as in the proof of Lemma 2. Thus the number of subsets of Ω stabilized by elements of $G^{\#}$ is at most

$$64 \cdot 63 \cdot 2^{48} + 24^{-1/3} \cdot 64^{3.25} \cdot 2^{40} < 2^{60} + 2^{58} < 2^{64}.$$

Therefore (G, Ω) satisfies $(*)$.

Less than half the subsets of Ω have cardinality 32. Since $2^{60} + 2^{58} < 2^{63}$, the preceding argument shows that (G, Ω) satisfies $(**)$ also.

LEMMA 4. *If* $|\Omega| = 25$ *or* 49, *then* (G, Ω) *satisfies* $(*)$.

Proof. First suppose $|\Omega| = 49$. By [**1**, Section 21, Theorem 6], we have $|M| \leqq 144$. Elements of $G^{\#}$ have at most 28 cycles on Ω, so the number of ordered pairs (g, S), where $g \in G^{\#}$, $S \subseteq \Omega$, and g stabilizes S is at most $2^{28}|G^{\#}| < 144 \cdot 49 \cdot 2^{28} < 2^{49}$.

Next suppose $|\Omega| = 25$. By [**1**, Section 21, Theorem 6], M is isomorphic to a subgroup of $GL(2, 5)$ of order 32, 48, or 96. Let G_2 be the subset of all elements of order 2 in G and define G_3 and G_5 similarly. Any subset of Ω which is stabilized by an element of $G^{\#}$ is stabilized by an element of $G_2 \cup G_3 \cup G_5$. Define G_2^+ to be the set of all elements of G_2 which fix exactly one point of Ω and define G_2^- to be the set of all elements of G_2 which fix exactly five points of Ω. We have $G_2 = G_2^+ \cup G_2^-$. Clearly $|M \cap G_2^+| \leqq 1$ so $|G_2^+| \leqq 25$ and each element of G_2^+ has exactly 13 cycles on Ω. Each element of G_2^- has 15 cycles on Ω and lies in exactly 5 conjugates of M. Each element of G_3 has 9 cycles on Ω. Since $M \cap G_5$ is empty, $|G_5| = 24$ and each element of G_5 has 5 cycles on Ω. Thus the number of ordered pairs (g, S), where $g \in G_2 \cup G_3 \cup G_5$, $S \subseteq \Omega$, and

g stabilizes S is at most

$$\begin{aligned} &2^{13}|G_2^+| + 2^{15}|G_2^-| + 2^9|G_3| + 2^5|G_5| \\ &\quad < 25 \cdot 2^{13} + (1/5) \cdot 96 \cdot 25 \cdot 2^{15} + 96 \cdot 25 \cdot 2^9 + 24 \cdot 2^5 \\ &\quad = 2^{13}(25 + 1920 + 150) + 768 < 2^{25}. \end{aligned}$$

LEMMA 5. *If* $|\Omega| = 16, 27,$ *or* 32, *then* (G, Ω) *satisfies* (**).

Proof. First we introduce some notation. For a prime power q we denote by $S(q)$ the group of all semilinear transformations on the field $GF(q)$. Thus $S(q)$ consists of all transformations $x \to ax^\sigma$, where $a \in GF(q)^\times$, the multiplicative group of $GF(q)$, and $\sigma \in \mathrm{Aut}\ (GF(q))$. We denote by $AS(q)$ the affine semilinear group consisting of all transformations on $GF(q)$ of the form $x \to ax^\sigma + b$, where a and σ are as above, and $b \in GF(q)$. Note that $S(q)$ is an irreducible solvable subgroup of $GL(n, p)$, where $q = p^n$ and p is prime.

By [**1**, Section 21, Theorems 2 and 6], every maximal irreducible solvable subgroup of $GL(5, 2)$ is conjugate in $GL(5, 2)$ to $S(32)$, every maximal irreducible solvable subgroup of $GL(4, 2)$ is conjugate in $GL(4, 2)$ either to $S(16)$ or to the wreath product $GL(2, 2) \wr \mathbf{Z}_2$, and every maximal irreducible solvable subgroup of $GL(3, 3)$ is conjugate in $GL(3, 3)$ either to $S(27)$ or to the monomial group $GF(3)^\times \wr S_3$.

Now M, considered as a subgroup of the appropriate general linear group, is conjugate to a subgroup of one of the five groups listed in the previous paragraph, so we may assume that M is one of those five groups. First assume that M is one of the three semilinear groups. Then the action of G on Ω is permutation isomorphic to the action of $AS(q)$ on $GF(q)$.

Suppose that $G = AS(16)$ and $\Omega = GF(16)$. Let Tr be the trace map from $GF(16)$ to $GF(4)$. Then Tr is a $GF(4)$-linear map which is surjective and has kernel $GF(4)$. Choose elements x and y in $GF(16)$-$GF(4)$ such that $\mathrm{Tr}(x) \neq \mathrm{Tr}(y)$. Define $S \subseteq GF(16)$ to be $GF(4) \cup \{x, y\}$. We will show that S has trivial stabilizer in $AS(16)$.

Let g be an arbitrary element of $AS(16)$. Let $zg = az^\sigma + b$ for $z \in GF(16)$. To show that g does not stabilize S unless $g = 1$, we may and will assume that σ has order $\leqq 2$. Otherwise g has order divisible by 4 and so g^2, which sends $z \in GF(16)$ to $a_1 z^{\sigma^2} + b_1$, for some a_1 and b_1, would also be a non-identity element of the stabilizer in G of S. Since g sends at least 2 elements of $GF(4)$ into $GF(4)$, subtracting equations shows that a and b are in $GF(4)$. Thus g stabilizes $GF(4)$ and $\{x, y\}$.

We now suppose, to get a contradiction, that the automorphism σ has order 2. If $ax^\sigma + b = y$ and $ay^\sigma + b = x$ we add these two equations and apply Tr, obtaining $a\mathrm{Tr}(x + y) = \mathrm{Tr}(x + y)$. Since we are assuming $\mathrm{Tr}(x + y) \neq 0$, we have $a = 1$. We apply Tr to the equation $x^\sigma + b = y$ and obtain $\mathrm{Tr}(x) = \mathrm{Tr}(y)$, a contradiction. If $ax^\sigma + b = x$ and $ay^\sigma + b$

$= y$ we add these equations and as before obtain $a = 1$. Then $x^\sigma + x = b = y^\sigma + y$ so $\mathrm{Tr}(x) = \mathrm{Tr}(y)$, again a contradiction.

Therefore $\sigma = 1$. As in the preceding paragraph, we easily obtain $a = 1$. If $x + b = y$, then $\mathrm{Tr}(x) = \mathrm{Tr}(y)$, which is not the case. Thus $x + b = x$, so $b = 0$ and $g = 1$.

Now suppose $G = AS(27)$ and $\Omega = GF(27)$. Let Tr be the trace map from $GF(27)$ to $GF(3)$. Let x be an element of $GF(27)$ such that $\mathrm{Tr}(x) \neq 0$ and let $S = GF(3) \cup \{x\}$. We will show that S has trivial stabilizer in G. Suppose $g \in G$ stabilizes S. Write $zg = az^\sigma + b$ for $z \in GF(27)$. As before we have $a, b \in GF(3)$. Thus g stabilizes $GF(3)$ and fixes x. We apply Tr to the equation $ax^\sigma + b = x$ and obtain $a\mathrm{Tr}(x) = \mathrm{Tr}(x)$, so $a = 1$. If $\sigma \neq 1$ and $b = \pm 1$, then $x^\sigma = x \pm 1$ and

$$\mathrm{Tr}(x) = x + x^\sigma + x^{\sigma^2} = x + (x + 1) + (x + 2) \quad \text{or} \quad x + (x - 1) + (x - 2),$$

so $\mathrm{Tr}(x) = 0$, a contradiction. If $\sigma \neq 1$ and $b = 0$, then $x^\sigma = x$, which is not the case. Thus $\sigma = 1$ and $x + b = x$, so $b = 0$ and $g = 1$.

Next suppose $G = AS(32)$ and $\Omega = GF(32)$. Let S be any 3-element subset of Ω. G is transitive on Ω, so after replacing S by another 3-element set in the same G orbit, we may assume that $0 \in S$. Since every element of the stabilizer of S must fix every element of S, the stabilizer of S lies in $S(32)$. But $S(32)$ is Frobenius on $GF(32)^\times$, so the stabilizer of S in G is trivial. This shows, by the way, that G acts regularly on the 3-element subsets of Ω.

The next case to consider is $|\Omega| = 16$ and $M = GL(2, 2) \wr \mathbf{Z}_2$. Let $F = \{1, 2, 3, 4\}$. Then the action of G on Ω is permutation isomorphic to the product action of $S_4 \wr \mathbf{Z}_2$ on the Cartesian product $F \times F$. By the definition of product action, the base group $S_4 \times S_4$ acts component-wise on $F \times F$, while $\mathbf{Z}_2$ permutes the coordinates. Let S be the subset $\{(1, 1), (1, 2), (1, 3), (2, 2), (2, 4), (3, 1)\}$ of $F \times F$. Since three different numbers appear as first coordinates of elements of S and four different numbers appear as second coordinates of elements of S, any $g \in G$ which stabilizes S must lie in the base group $S_4 \times S_4$. We write $g = (g_1, g_2) \in S_4 \times S_4$. No two elements of F appear the same number of times as first coordinates of elements of S, so $g_1 = 1$. Similarly g_2 must stabilize $\{1, 2\}$ so there are four possibilities for g_2. One checks that none of the four resulting possibilities for $(1, g_2)$ stabilizes S. Thus S has trivial stabilizer in G.

The last case to consider is $|\Omega| = 27$ and $M = GF(3)^\times \wr S_3$. Let $T = \{1, 2, 3\}$. Then the action of G on Ω is permutation isomorphic to the product action of $S_3 \wr S_3$ on $T \times T \times T$. Let

$$S = \{(1, 1, 3), (1, 2, 1), (1, 2, 3), (2, 2, 2), (2, 2, 3)\} \subseteq T \times T \times T.$$

As above, one shows easily that an element of G which stabilizes S must lie in the base group, and then that each of its components must be 1. Thus S has trivial stabilizer in G.

LEMMA 6. *Suppose* $|\Omega|$ *is a prime number* p. *Then* (G, Ω) *fails to satisfy* $(*)$ *if and only if* $|\Omega| = 3, 5,$ *or* 7 *and* G *is a Frobenius group of order* 6, 10, 20, *or* 42.

Proof. Clearly G is Frobenius with complement M and kernel K. Every subset of Ω whose stabilizer in G is nontrivial is stabilized by a subgroup whose order is a prime divisor of $p - 1$. Each such subgroup lies in exactly one of the p conjugates of M, so we have at most $p\sum_q 2^{((p-1)/(q))+1}$ subsets of Ω with nontrivial stabilizer, where the index of summation q ranges over the prime divisors of $p - 1$.

We first show that this number is less than 2^p for $p \geqq 11$. If $p = 11$, the sum is $11(8 + 64) = 792 < 2^{11}$. For $p > 11$, we observe that the number of prime divisors of $p - 1$ is less than $\log_2 p$, so the sum is less than $p(\log_2 p)2^{(p+1)/2}$, which is less than 2^p for $p \geqq 13$.

On the other hand, if $p = 7$ and $|G| = 42$, then we have $\binom{7}{i} \leqq \binom{7}{3} = 35$ for all i between 0 and 7. Thus G has no regular orbit on $P(\Omega)$, so (G, Ω) does not satisfy $(*)$. If $p = 7$ and $|G| = 21$, then any 2-element subset of Ω has trivial stabilizer, so (G, Ω) satisfies $(*)$. If $p = 7$ and $|G| = 14$, each of the 7 involutions in G stabilizes exactly 3 subsets of order 3 in Ω. Thus 21 of the 35 subsets of order 3 in Ω have nontrivial stabilizers, and the remaining 14 form a regular G-orbit. Thus (G, Ω) satisfies $(*)$. If $p = 5$ and $|G| = 10$, then each of the 5 involutions in G stabilizes two 2-element subsets of Ω. It follows that every 2-element subset of Ω has nontrivial stabilizer in G, so (G, Ω) does not satisfy $(*)$.

The rest of the lemma is clear.

We now discuss the remaining cases, where $|\Omega| = 4, 8,$ or 9.

If $|\Omega| = 4$, then G is isomorphic to A_4 or S_4, and in either case (G, Ω) does not satisfy $(*)$.

If $|\Omega| = 8$, then by [**1**, Section 21, Theorem 6], M is conjugate in $GL(3, 2)$ to a subgroup of $S(8)$. If $M \cong S(8)$, then (G, Ω) does not satisfy $(*)$ because $|G| = 168 > \binom{8}{i}$ for all i between 0 and 8, so G has no regular orbit on $P(\Omega)$. If $|M| < 21$, then $|M| = 7$, G is a Frobenius group, and (G, Ω) satisfies $(**)$.

If $|\Omega| = 9$, then M is isomorphic to an irreducible subgroup of $GL(2, 3)$. Note that $GL(2, 3)$ is solvable of order 48. If $|M| > 14$, then $(*)$ fails for (G, Ω) because $|G| > \binom{9}{i}$ for all i between 0 and 9. We claim that $|M|$ cannot be 12. If $|M|$ were 12 and $O_3(M) > 1$, then M could not act faithfully and irreducibly on $\mathbf{Z}_3 \times \mathbf{Z}_3$. If $|M|$ were 12 and $O_3(M) = 1$, then M would be isomorphic to A_4, and so

$$O_2(M) \subseteq M' \subseteq GL(2, 3)' = SL(2, 3),$$

so $M \subseteq SL(2, 3)$. This is impossible because $SL(2, 3)$ has no subgroup of order 12. Thus $|M| \neq 12$. Also $|M| \neq 6$ and $|M| \neq 3$.

Thus we may assume M is a 2-subgroup of $GL(2, 3)$ of order $\leqq 8$. If $M \cong D_8$ and T is a 2-Sylow of $GL(2, 3)$ containing M, then T is semidihedral and so every involution of T lies in M, so every involution of the semidirect product $T(\mathbf{Z}_3 \times \mathbf{Z}_3)$ lies in $M(\mathbf{Z}_3 \times \mathbf{Z}_3) = G$, so every element of prime order in $T(\mathbf{Z}_3 \times \mathbf{Z}_3)$ lies in G. Thus (*) fails for (G, Ω) because $(*)$ fails for $(T(\mathbf{Z}_3 \times \mathbf{Z}_3), \Omega)$.

In the remaining cases we will show that (G, Ω) satisfies $(*)$. Since M acts irreducibly, $|M| \geqq 4$ and M contains an element of order 4. The square of any element of order 4 in $GL(2, 3)$ is the central involution z of $GL(2, 3)$. From the structure of the semidihedral group of order 16, we conclude that z is the unique involution in M. It suffices to show that the number of subsets of Ω fixed by an element of order 2 or 3 in G is less than $2^9 = 512$. Since z is the unique involution of M, G is a Frobenius group with complement M. Thus G has 9 involutions and 8 elements of order 3. The involutions have 5 cycles on Ω and the elements of order 3 have 3 cycles on Ω. Thus the elements of prime order in G stabilize at most $9 \cdot 2^5 + 8 \cdot 2^3 = 352$ sets.

THEOREM 1. *Let G, M and Ω be as in the first paragraph of this section. Then (G, Ω) satisfies $(*)$ except in the following cases. If (G, Ω) satisfies $(*)$ and $|\Omega| \neq 2$, then (G, Ω) satisfies $(**)$.*

(1) $|\Omega| = 3$, $|M| = 2$
(2) $|\Omega| = 4$, $|M| = 3$ *or* 6
(3) $|\Omega| = 5$, $|M| = 2$ *or* 4
(4) $|\Omega| = 7$, $|M| = 6$
(5) $|\Omega| = 8$, $|M| = 21$
(6) $|\Omega| = 9$, $|M| = 16, 24$, *or* 48, *or* $M \cong D_8$.

Proof. This is clear from Lemmas 1–6 and the discussion preceding the statement of this theorem.

2. Imprimitive solvable groups. The phrase "(G, Ω) satisfies (*)" will have the same meaning as in the previous section, even though we are no longer assuming that G is primitive.

THEOREM 2. *Let G be a transitive solvable permutation group on a finite set Ω. Let $H \subseteq G$ be a point stabilizer. Suppose there exists a chain $H = H_0 < H_1 < \ldots < H_n = G$ of subgroups, each maximal in the next, such that*

(1) $|H_i : H_{i-1}| \neq 2$ *for* $i \leqq i \leqq n$.

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(2) *The primitive permutation group induced by the action of* H_i *on the cosets of* H_{i-1} *in* H_i *satisfies* $(*)$ *for* $1 \leqq i \leqq n$.

Then (G, Ω) *satisfies* $(*)$.

Proof. If $n = 1$ there is nothing to prove. Otherwise identify Ω with the set of right cosets Hx of H in G. The distinct right cosets of H_1 constitute a partition of Ω into blocks of imprimitivity $B_1, \ldots, B_t$. Let G_j be the stabilizer in G of B_j, for $1 \leqq j \leqq t$, so that each G_j is conjugate in G to H_1, each $|B_j| = |H_1 : H| \neq 2$, and G_j acts on B_j like H_1 acts on the cosets of H in H_1.

By condition (2) above each B_j contains a subset S_j whose stabilizer in G_j fixes every point in B_j. Let $\bar{G}_j$ denote the primitive permutation group induced by the action of G_j on B_j. Since $(\bar{G}_j, B_j)$ satisfies $(**)$ by Theorem 1, we may assume that $|S_j| < |B_j|/2$. By induction on n there is a subset $\mathcal{S}$ of $\{B_1, \ldots, B_t\}$ such that any element of G which stabilizes $\mathcal{S}$ fixes every block. Now let

$$S_1 = \bigcup_{B_j \in \mathcal{S}} S_j,$$

let

$$S_2 = \bigcup_{B_k \notin \mathcal{S}} (B_k - S_k),$$

and let $S = S_1 \cup S_2$. Since no $|S_j|$ equals any $|B_k - S_k|$, any element $g \in G$ which stabilizes S must stabilize $\mathcal{S}$, so g must stabilize every block. Therefore g stabilizes S_j for $1 \leqq j \leqq t$, so g fixes every point in every block, so $g = 1$.

COROLLARY 1. *If G is a permutation group on Ω and $|G|$ is odd, then* (G, Ω) *satisfies* $(*)$.

Proof. We may assume G is transitive. The result then follows from the solvability of groups of odd order and Theorems 1 and 2.

Our next proposition shows that condition (1) in the statement of Theorem 2 is not superfluous.

PROPOSITION 1. *Let* $n \geqq 2$. *Let* $m = 2^n$. *Let* $\Omega = \{1, 2, \ldots, m\}$ *and let G be a 2-Sylow subgroup of* S_m. *Then* (G, Ω) *does not satisfy* $(*)$.

Proof. G is an iterated wreath product of order 2^{m-1}, so $|G| > \binom{m}{k}$ for all k between 0 and m, so G has no regular orbit on $P(\Omega)$.

Our final result concerns centralizers of vectors in monomial G-modules.

COROLLARY 2. *Let F be a field. Let G be a finite group. Let H be a subgroup of G and let* $L = \operatorname{core}_G(H)$. *Suppose* $|G : L|$ *is odd. Let V be a* *faithful right $F[G]$-module. Suppose $V = W^G$ for some one-dimensional $F[H]$-module W. Then there is a vector $v \in V$ such that $C_G(v)$ is abelian.*

Proof. Let $\{g_1, \ldots, g_n\}$ be a set of right coset representatives for H in G. Then G transitively permutes the one-dimensional subspaces $W \otimes g_i$ of v. The kernel of this action is L. By Corollary 1 there is a subset S of $\{1, 2, \ldots, n\}$ such that the stabilizer in G of $\{W \otimes g_i | i \in S\}$ is L. Let w be a nonzero vector in W. Let $v = \sum_{i \in S} w \otimes g_i$. Then $C_G(v) \subseteq L$. Since V is a direct sum of one-dimensional subspaces invariant under L, the faithfulness of V implies that L is abelian.

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