Canad. Math. Bull. Vol. 60 (2), 2017 pp. 235–245
http://dx.doi.org/10.4153/CMB-2016-098-8


# Topology of Certain Quotient Spaces of Stiefel Manifolds

Samik Basu and B. Subhash

*Abstract.* We compute the cohomology of the right generalised projective Stiefel manifolds. Following this, we discuss some easy applications of the computations to the ranks of complementary bundles and bounds on the span and immersibility.

## 1 Introduction

The study of Stiefel manifolds and their quotients has a long history [8]. Their topology has played a fundamental role in solving many problems such as the celebrated solution of the vector field problem on spheres by Adams. The cohomology of the (real) projective Stiefel manifolds with $\mathbb{Z}/2$-coefficients was computed in [6] (in the case of the projective orthogonal and unitary groups, this was first computed in [3]). This was used to prove immersion results for real projective spaces in [5].

The cohomology of the complex projective Stiefel manifolds was the subject of the paper [9], but it turned out to contain many errors. The correct computation was achieved in [1], and a universal property was associated with these manifolds. As a consequence, the authors conclude the non-existence of certain sections to appropriate bundles over projective spaces and lens spaces. Following this, the question of parallelizability of complex projective Stiefel manifolds was settled in [2].

This paper deals with right generalised projective Stiefel manifolds, which were studied in [7]. These manifolds are interesting from a topological point of view and also since a certain amount of number theory is automatically mixed in with the topology in the very definition of these manifolds. They are obtained as quotients of Stiefel manifolds $W_{n,k}$ (the space of orthonormal $k$-frames in $\mathbb{C}^n$) by an action of the circle group $S^1$. The action is given by the formula $z \cdot (v_1, \dots, v_k) \mapsto (z^{l_1} v_1, \dots, z^{l_k} v_k)$, which can be described by right multiplication by a matrix with diagonal entries $(z^{l_1}, \dots, z^{l_k}, 1, \dots, 1)$ on $U(n)/U(n-k)$. We assume that the $k$-tuple $(l_1, \dots, l_k) \in \mathbb{Z}^k$ is primitive, which means that the gcd of $l_1, \dots, l_k$ equals 1. The corresponding quotient space is called a *right generalised projective Stiefel manifold* $P_\ell W_{n,k}$. It is a smooth real manifold of dimension $k(2n-k) - 1$ and can be realised as the homogeneous space $U(n)/S^1 \times U(n-k)$. In [7], the question of parallelizability of these manifolds is settled.

In this paper we are motivated by certain computations for complex projective Stiefel manifolds and attempt to search for similar relations for the right generalised

Received by the editors November 24, 2015; revised December 15, 2016.
Published electronically February 8, 2017.
AMS subject classification: **55R20**, 55R25, 57R20.
Keywords: projective Stiefel manifold, span, spectral sequence.

projective Stiefel manifolds. We compute the cohomology of these manifolds (*cf.* Theorem 2.3) and observe that they satisfy a universal property for complementary bundles of a certain sum of line bundles (*cf.* Theorem 2.6). Using these theorems we deduce implications for line bundles over specific manifolds. The cohomology formulas also enable us to compute the Pontrjagin classes for the manifolds $P_\ell W_{n,k}$. Working specifically with $k = 2$, we use these to bound the span and immersibility of these manifolds in Euclidean spaces. Our methods improve the results in [7].

In Section 2, we compute the cohomology of the manifolds $P_\ell W_{n,k}$ and introduce a universal property of these spaces. We follow this with some applications in Section 3.

## 2 Some Cohomology Computations

In this section we compute the cohomology of the manifolds $P_\ell W_{n,k}$. We work with $\mathbb{Z}$-coefficients up to Proposition 2.2 and $\mathbb{Z}/p$-coefficients thereafter. Our method involves an interplay between Serre spectral sequences of various fibrations and enables us to deduce a universal property for $P_\ell W_{n,k}$. Throughout we assume that the gcd of $(l_1, \dots, l_k)$ is 1.

Recall that the cohomology of the unitary group $U(n)$ is an exterior algebra with generators in degrees $1, 3, \dots, 2n-1$ ([4]). We denote this expression by $H^*(U(n)) = \Lambda(y_1, \dots, y_n)$, where the class $y_j$ lies in degree $2j-1$. The Stiefel manifold $W_{n,k}$ is homeomorphic to $U(n)/U(n-k)$, and its cohomology is given by $\Lambda(y_{n-k+1}, \dots, y_n)$. Recall that the principal $S^1$ fibration $W_{n,k} \to P_\ell W_{n,k}$ yields a fibration

$$W_{n,k} \longrightarrow P_\ell W_{n,k} \longrightarrow BS^1,$$

the latter space being $\mathbb{C}P^\infty$. Note that the Stiefel manifold $W_{n,k}$ also fibres over the flag manifold $F(1, \dots, 1, n-k)$ of sequences of flags of orthogonal subspaces of dimensions $1, \dots, 1, n-k$, that is, with $k$ subspaces of dimension 1 and one of dimension $n-k$. This fits into a principal $(S^1)^k$ fibration $W_{n,k} \to F(1, \dots, 1, n-k)$. The $S^1$ action whose orbits are the manifold $P_\ell W_{n,k}$ comes from the inclusion of $S^1$ in $(S^1)^k$ given by $\Phi_\ell \colon z \mapsto (z^{l_1}, \dots, z^{l_k})$. This induces a commutative sequence of fibrations

$$\begin{array}{ccc}
S^1 & \xrightarrow{\Phi_\ell} & (S^1)^k \\
\downarrow & & \downarrow \\
W_{n,k} & = & W_{n,k} \\
\downarrow & & \downarrow \\
P_\ell W_{n,k} & \longrightarrow & F(1, \dots, 1, n-k) \\
\downarrow & & \downarrow \\
\mathbb{C}P^\infty & \xrightarrow{\phi_\ell} & (\mathbb{C}P^\infty)^k.
\end{array}$$

These fibre sequences extend one step further. Consider $G_k(\mathbb{C}^n)$, the Grassmann manifold of $k$–planes in $\mathbb{C}^n$, which is the quotient $U(n)/(U(k) \times U(n-k))$. One has a principal $U(k)$ bundle $W_{n,k} \to G_k(\mathbb{C}^n)$, and the map $(S^1)^k \to U(k)$ given by the inclusion of diagonal matrices forms a similar diagram of principal fibrations as above. Putting all this together one obtains a commutative diagram

$$\begin{array}{ccccc}
S^1 & \longrightarrow & (S^1)^k & \longrightarrow & U(k) \\
\downarrow & & \downarrow & & \downarrow \\
W_{n,k} & = & W_{n,k} & = & W_{n,k} \\
\downarrow & & \downarrow & & \downarrow \\
P_\ell W_{n,k} & \longrightarrow & F(1,\dots,1,n-k) & \longrightarrow & G_k(\mathbb{C}^n) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{C}P^\infty & \longrightarrow & (\mathbb{C}P^\infty)^k & \longrightarrow & BU(k).
\end{array}$$

In the diagram above, the bottom left square and the bottom right square are pullback squares of fibrations. Hence, the composite

$$\begin{array}{ccc}
P_\ell W_{n,k} & \longrightarrow & G_k(\mathbb{C}^n) \\
\downarrow & & \downarrow \\
\mathbb{C}P^\infty & \longrightarrow & BU(k)
\end{array} \tag{2.1}$$

is also a pullback. These fibrations induce Serre spectral sequences

$$E_2^{p,q} = H^p(BU(k)) \otimes H^q(W_{n,k}) \implies H^{p+q}(G_k(\mathbb{C}^n)), \tag{2.2}$$

$$E_2^{p,q} = H^p(\mathbb{C}P^\infty) \otimes H^q(W_{n,k}) \implies H^{p+q}(P_\ell W_{n,k}). \tag{2.3}$$

The pullback diagram (2.1) induces a map between the two spectral sequences that commutes with the differentials. Recall that the cohomology of $BU(k)$ is a polynomial algebra on the Chern classes $c_1, \dots, c_k$, where $c_i = c_i(\xi_k)$, $\xi_k$ being the universal $k$-plane bundle.

***Proposition 2.1*** *In the spectral sequence (2.2) the classes $y_j \in H^*(W_{n,k})$ are transgressive and support the differential $d(y_j) = -c'_j$ (the classes $c'_j$ satisfy the equation $(1 + c'_1 + \dots)(1 + c_1 + \dots + c_k) = 1$).*

**Proof** In the Serre spectral sequence for the fibration

$$U(n) \longrightarrow G_k(\mathbb{C}^n) \longrightarrow B\big(U(k) \times U(n-k)\big),$$

https://doi.org/10.4153/CMB-2016-098-8 Published online by Cambridge University Press

the differentials are given by $d(y_i) = c_i(\xi_k \oplus \xi_{n-k})$. This formula follows from [4]. We have the diagram of fibrations

$$
\begin{array}{ccc}
U(n) & \xrightarrow{q} & W_{n,k} \\
\downarrow & & \downarrow \\
G_k(\mathbb{C}^n) & = & G_k(\mathbb{C}^n) \\
\downarrow & & \downarrow \\
B(U(k) \times U(n-k)) & \longrightarrow & BU(k),
\end{array}
\tag{2.4}
$$

and hence a morphism of the associated Serre spectral sequences. Note that for $j \geq n - k + 1$, the classes $y_j \in H^{2j-1}(W_{n,k})$ pull back to $y_j \in H^{2j-1}(U(n))$. We try to read off the expression for $d(y_j)$ in the spectral sequence for the right column from the one on the left.

Denote by $E_*^{p,q}(l)$ the spectral sequence corresponding to the left vertical column in (2.4). The classes $y_1, \ldots, y_{n-k}$ are not in the image of $q^*$. Let $c_i = c_i(\xi_k)$ and $\widetilde{c}_i = c_i(\xi_{n-k})$. Note that $c_i = 0$ if $i > k$ and $\widetilde{c}_j = 0$ if $j > n - k$. The formula $d(y_j) = c_j(\xi_k \oplus \xi_{n-k})$ implies that in the page $E_{2(n-k)+1}(l)$, $\widetilde{c}_i$ is equivalent to $c'_i$ for $i \leq n - k$. Hence,

$$
\begin{aligned}
d_{2(n-k)+2}(y_{n-k+1}) = c_{n-k+1}(\xi_k \oplus \xi_{n-k}) &= \sum_{i+j=n-k+1} c_i \widetilde{c}_j \\
&= \sum_{\substack{i+j=n-k+1 \\ i \geq 1}} c_i c'_j + \widetilde{c}_{n-k+1} = \sum_{\substack{i+j=n-k+1 \\ i \geq 1}} c_i c'_j = -c'_{n-k+1}
\end{aligned}
$$

in $E_{2(n-k)+2}(l)$. The last equation above follows from $\sum_{i+j=d} c_i c'_j = 0$, which implies $\sum_{i+j=d, i \geq 1} c_i c'_j = -c'_d$.

We observe that the equation $d_{2j}(y_j) = -c'_j$ holds for all $j \geq n - k + 1$ in the page $E_{2j}(l)$. Proceeding by induction, we have in the page $E_{2j-1}(l)$, $\widetilde{c}_i = c'_i$ for $i \leq n - k$ and $c'_i = 0$ for $n - k + 1 \leq i \leq j - 1$. Then in the page $E_{2j}(l)$ we have the equation

$$
d(y_{2j-1}) = c_j(\xi_k \oplus \xi_{n-k}) = \sum_{p+q=j} c_p \widetilde{c}_q = \sum_{\substack{p+q=j \\ p \leq k \\ q \leq n-k}} c_p c'_q = \sum_{\substack{p+q=j \\ p \leq k}} c_p c'_q = -c'_j.
$$

Denote by $E_*^{p,q}(r)$ the spectral sequence for the right column of (2.4). For degree reasons, the differentials $d_j$ are 0 if $j < 2(n-k) + 2$. The morphism from the spectral sequence of the right column to the left column implies that the differentials on $y_i$ for $i > n - k$ are given by $d(y_i) = -c'_i$. ∎

Next we translate the Proposition 2.1 to obtain differentials in the spectral sequence (2.3). For a tuple $\ell = (l_1, \ldots, l_k)$ and integers $I = (i_1, \ldots, i_k)$, denote $|I| = \sum_j i_j$ and $\ell^I = \prod_j l_j^{i_j}$. We prove the following proposition.

***Proposition 2.2*** *In the spectral sequence (2.3), the classes $y_j$ (for $j > n - k$) are transgressive and the differentials are given by $d(y_j) = -\sum_{|I|=j} (-1)^j \ell^I x^j$.*

https://doi.org/10.4153/CMB-2016-098-8 Published online by Cambridge University Press

**Proof** In the diagram (2.1), the map $\phi_\ell\colon \mathbb{C}P^\infty \to BU(k)$ classifies the $k$-plane bundle $\xi^{l_1} \oplus \cdots \oplus \xi^{l_k}$. The Chern classes of this bundle are computed by

$$c(\oplus_j \xi^{l_j}) = \prod_j (1 + l_j x).$$

For the classes $c'_j$, define $c' = 1 + c'_1 + \cdots$ so that $cc' = 1$. This implies the pullback of $c'$ to $\mathbb{C}P^\infty$ is given by the equation

$$\phi_\ell^* c' = \prod_j (1 + l_j x)^{-1}.$$

Thus, $\phi_\ell^*(c'_j) = \sum_{|I|=j} (-1)^j \ell^I x^j$. The result follows. ■

Using the formulas above, we now compute the $\mathbb{Z}/p$ cohomology of $P_\ell W_{n,k}$.

***Theorem 2.3*** *For an odd prime $p$,*

$$H^*(P_\ell W_{n,k}; \mathbb{Z}/p) \cong (\mathbb{Z}/p)[x]/(x^N) \otimes \Lambda(y_{n-k+1}, \dots, y_{N-1}, y_{N+1}, \dots, y_n),$$

*where $N = \min\{r : r > n-k$ and $\sum_{|I|=r} \ell^I \not\equiv 0 \pmod p\}$.*

**Proof** We compute via the Serre spectral sequence (2.3) with $\mathbb{Z}/p$ coefficients whose differentials are computed in Proposition 2.2.

By the multiplicative structure, the first non-zero differential on a class in the vertical 0-line is forced to be a transgression. With $N$ defined as in the statement, note that the first non-zero transgression is given by $d_{2N}(y_N) = x^N$. Therefore, the page $E_{2N+1}^{*,*}$ is isomorphic to the algebra

$$(\mathbb{Z}/p)[x]/(x^N) \otimes \Lambda(y_{n-k+1}, \dots, y_{N-1}, y_{N+1}, \dots, y_n).$$

Since the classes $y_j$ are transgressive, there are no further differentials as $x^i = 0$ for $i > N$ in $E_{2N+1}^{*,*}$. Hence, $E_{2N+1} = E_\infty$. It follows that we must also have

$$H^*(P_\ell W_{n,k}; \mathbb{Z}/p) \cong (\mathbb{Z}/p)[x]/(x^N) \otimes \Lambda(y_{n-k+1}, \dots, y_{N-1}, y_{N+1}, \dots, y_n).$$

For the multiplicative structure, observe that the factor $(\mathbb{Z}/p)[x]/(x^N)$ is a subalgebra as it comes from the horizontal 0-line. Arbitrarily pick classes

$$y_j \in H^{2j-1}(P_\ell W_{n,k}; \mathbb{Z}/p)$$

(for $j > n-k$ and $j \neq N$), which pull back to $y_j \in H^{2j-1}(W_{n,k}; \mathbb{Z}/p)$ under the induced cohomology map of the quotient map $W_{n,k} \to P_\ell W_{n,k}$. These exist by the additive computation above. The classes $y_j$ are odd dimensional classes and hence square to 0. Thus, multiplication induces a ring map

$$(\mathbb{Z}/p)[x]/(x^N) \otimes \Lambda(y_{n-k+1}, \dots, y_{N-1}, y_{N+1}, \dots, y_n) \longrightarrow H^*(P_\ell W_{n,k}; \mathbb{Z}/p)$$

which is an additive isomorphism by the argument above. The result follows. ■

One can try to repeat the above argument for $p = 2$, but then the squares on the classes $y_j$ might not be zero. However, if $k = 2$, this case cannot arise, and we have the following result.

https://doi.org/10.4153/CMB-2016-098-8 Published online by Cambridge University Press

***Theorem 2.4*** *Let $\ell = (l_1, l_2)$ and suppose that 2 divides $\sum_{p+q=n-1} l_1^p l_2^q$. Then*

$$H^*(P_\ell W_{n,2}; \mathbb{Z}/2) \cong \mathbb{Z}/2[x, y_{n-1}]/(x^n, y_{n-1}^2).$$

*Otherwise,*

$$H^*(P_\ell W_{n,2}; \mathbb{Z}/2) \cong \mathbb{Z}/2[x, y_n]/(x^{n-1}, y_n^2).$$

**Proof** The proof for Theorem 2.3 can be repeated verbatim here. The only issue is with multiplicative extensions. Again choose representatives for $y_{n-1}, y_n$ in an arbitrary fashion. We examine the possible values for $y_j^2$. From dimension reasons no other class exists in the degree of $y_{n-1}^2$ and $y_n^2$ in either of the cases. The rest of the proof works as in Theorem 2.3. ∎

***Example 2.5*** Put $\ell = (1, \ldots, 1)$ so that we recover the complex projective Steifel manifold. In that case note that $\sum_{|I|=r} \ell^I$ is the number of ordered $k$-tuples of elements with sum $r$ that is $\binom{r+k-1}{k} = \binom{r+k-1}{r-1}$. Consider

$$N = \min\left\{ r : r > n-k \text{ and } \binom{r+k-1}{r-1} \not\equiv 0 \pmod{p} \right\}.$$

The first term in this set is $r = n - k + 1$ which is $\binom{n}{k}$. In view of the relation

$$\binom{r+k}{r} = \binom{r+k-1}{r-1} + \binom{r+k-1}{r},$$

if $\binom{r+k-1}{r-1} \equiv 0 \pmod{p}$, $\binom{r+k}{r} \equiv \binom{r+k-1}{r} \pmod{p}$. Therefore, one can rewrite the equation defining $N$ as $N = \min\{r : r > n-k \text{ and } \binom{n}{r} \not\equiv 0 \pmod{p}\}$. This matches the cohomology computation in [1, Theorem 1.1].

Refer to the commutative diagram (2.1). This is actually a homotopy pullback. Hence, one has an associated universal property for the manifold $P_\ell W_{n,k}$.

***Theorem 2.6*** *The space $P_\ell W_{n,k}$ classifies line bundles $L$ for which there exists an $(n-k)$-bundle $E$ such that $E \oplus_j L^{l_j}$ is a trivial bundle.*

**Proof** Since the diagram (2.1) is a homotopy pullback, $[X, P_\ell W_{n,k}]$ is equivalent to a map $X \to \mathbb{C}P^\infty$ and a map $X \to G_k(\mathbb{C}^n)$ such that the composites to $BU(k)$ are homotopic. Denote by $L$ the line bundle classified by the map to $\mathbb{C}P^\infty$ and by $E$ the pullback of the complementary canonical bundle $\xi_{n-k}$ over $G_k(\mathbb{C}^n)$. Then the maps are homotopic on composition to $BU(k)$ if and only if $\oplus_j L^{l_j} \oplus E = n\epsilon_{\mathbb{C}}$. The result follows. ∎

***Remark 2.7*** If $\ell = (1, \ldots, 1)$, the universal property classifies line bundles $L$ such that $kL \oplus E$ is a trivial bundle. We have the sequence of implications

$$kL \oplus E \cong n\epsilon_{\mathbb{C}} \iff L^* \otimes E \oplus k\epsilon_{\mathbb{C}} \cong nL^* \iff E^* \otimes L \oplus k\epsilon_{\mathbb{C}} \cong nL$$

Thus, the universal property is equivalent to having $k$ linearly independent sections to the bundle $nL$. This reduces to the universal property in [1, (5.2)].

## 3 Applications

In this section we consider applications of the cohomology computations in Section 2. We focus on the manifolds $P_\ell W_{n,2}$, but analogous computations can be done for $P_\ell W_{n,k}$ for $k \geq 3$. There are two kinds of applications we consider, the first being ranks of complementary bundles using Theorem 2.6 and the second being bounds on the number of linearly independent vector fields and immersion codimensions. We fix the notation

$$\phi_d(l_1, l_2) = \frac{l_1^{d+1} - l_2^{d+1}}{l_1 - l_2}$$

so that for $\ell = (l_1, l_2)$ with $l_1 \neq l_2$, $\sum_{|I|=d} \ell^I = \phi_d(l_1, l_2)$.

### 3.1 Ranks of Some Complementary Bundles

For a vector bundle $\xi$, call a bundle $\eta$ complementary if $\xi \oplus \eta$ is a trivial bundle. The universal property of $P_\ell W_{n,k}$ in Theorem 2.6 implies that the topology of the spaces $P_\ell W_{n,k}$ can be used to study the ranks of complementary bundles when $\xi$ is of the form $L^{l_1} \oplus \cdots \oplus L^{l_k}$. We concentrate on the case $k = 2$.

Suppose that $X$ is a manifold of dimension $2n$. Recall that a complex vector bundle $\xi$ over $X$ possesses a complementary bundle $\zeta$ of dimension $n$. Usually one tries to bound the dimension of $\zeta$ using Chern classes. Let $\eta$ be a complex line bundle over $X$ and let $\zeta$ be such that $\zeta \oplus \eta^{l_1} \oplus \eta^{l_2}$ is trivial. Suppose that $y = c_1(\eta)$. It follows that

$$c_i(\zeta) = y^i \sum_{p+q=i} (-1)^p l_1^p (-1)^q l_2^q = y^i (-1)^i \frac{l_1^{i+1} - l_2^{i+1}}{l_1 - l_2} = \phi_i(l_1, l_2) y^i.$$

If $c_n(\zeta) = \phi_n(l_1, l_2) y^n \neq 0$, then $\operatorname{rank}(\zeta) \geq n$. We ask the question: what happens if this element is 0? One can argue from the homotopy theory of classifying spaces that there exists a $\zeta$ of dimension $n-1$. One can also observe this using the spaces $P_\ell W_{n,k}$. Indeed, from Theorem 2.6, there exists a complementary $\zeta$ of dimension $n-1$ if and only if there is a lift in the diagram

$$\begin{array}{ccc} & & P_\ell W_{n+1,2} \\ & & \downarrow \\ X & \xrightarrow{\eta} & \mathbb{C}P^\infty \end{array}$$

for $\ell = (l_1, l_2)$. The fibre of the vertical map is $W_{n+1,2}$, and hence the obstructions to such a lift lies in $H^{k+1}(X; \pi_k W_{n+1,2})$. In this case the coefficient group is 0 unless $k \geq 2n-1$ and $\pi_{2n-1}(W_{n+1,2}) \cong \mathbb{Z}$. Therefore, the only possible obstruction to this lies in the group $H^{2n}(X;\mathbb{Z})$, and this can be explicitly computed as the $n^{th}$ Chern class. Next we consider an application where the spaces $P_\ell W_{n,2}$ give a better bound than the Chern classes. Let $L^d(m)$ denote the lens space $S^{2d+1}/(\mathbb{Z}/m)$. Consider the space $X = S^2 \times L^d(m)$ so that $H^2(X) = \mathbb{Z}\{e_2\} \oplus (\mathbb{Z}/m)\{u\}$ ($e_2$ is the pullback of the generator of $H^2 S^2$ and $u$ the pullback of the generator of $H^2 L^d(m)$). Consider the line bundle $\lambda$ given by the element $e_2 + u \in H^2(X)$. We consider the following question: if $\lambda^{l_1} \oplus \lambda^{l_2} \oplus \zeta$ is a trivial bundle, what are the possible restrictions on the rank of $\zeta$?

https://doi.org/10.4153/CMB-2016-098-8 Published online by Cambridge University Press

The dimension of $X$ equals $2d+3$; thus, we can choose $\zeta$ to have dimension $d+1$. The cohomology of $L^d(m)$ is given by

$$H^*(L^d(m);\mathbb{Z}) \cong \mathbb{Z}[u, v_{2d+1}]/(mu, u^{d+1}, v_{2d+1}^2, uv_{2d+1})$$

and by the Kunneth formula

$$H^*(X;\mathbb{Z}) \cong \mathbb{Z}[e_2, u, v_{2d+1}]/(e_2^2, mu, u^{d+1}, v_{2d+1}^2, uv_{2d+1}).$$

We ask whether one can choose $\zeta$ of rank $d$. The total Chern class of $\zeta$ is

$$c(\zeta) = \big(1 + l_1(e_2+u)\big)^{-1}(1 + l_2(e_2+u))^{-1}$$

This implies that

$$c_{d+1}(\zeta) = \phi_{d+1}(l_1, l_2)(e_2+u)^{d+1} \equiv \phi_{d+1}(l_1, l_2)(d+1)e_2u^d \pmod{m}.$$

Hence, $\dim(\zeta) \geq d+1$ if $m$ does not divide $\phi_{d+1}(l_1, l_2)(d+1)$.

We consider the case when $m$ divides $\phi_{d+1}(l_1, l_2)$ so that there is no obstruction to dimension of $\zeta$ being $d$ coming from the Chern class. Now we try to work out the obstruction theory. A choice of $\zeta$ is equivalent to a lift

$$\begin{array}{ccc} & & P_\ell W_{d+2,2} \\ & & \downarrow \\ X & \xrightarrow{\lambda} & \mathbb{C}P^\infty, \end{array}$$

with $\ell = (l_1, l_2)$.

The cohomology of $L^d(m)$ with $\mathbb{Z}/2$ coefficients (assuming that $m$ is even) is

$$H^*(L^d(m);\mathbb{Z}/2) \cong \mathbb{Z}[u, v]/(u^{d+1}, v^2 - \epsilon u)$$

with $\deg(u) = 2, \deg(v) = 1$, and $\epsilon \equiv \frac{m}{2} \pmod 2$. The Bockstein homomorphism $\beta\colon H^1(L^d(m);\mathbb{Z}/2) \to H^2(L^d(m);\mathbb{Z})$ is given by the formula $\beta(v) = \frac{m}{2}u$. Also, we have the formula

$$Sq^2(u^{d-1}v) = (d-1)u^d v.$$

Similarly, for the space $X$, we have

$$H^*(X;(\mathbb{Z}/2)) \cong \mathbb{Z}[e_2, u, v]/(e_2^2, u^{d+1}, v^2 - \epsilon u),$$
$$\beta(e_2u^{d-1}v) = \frac{m}{2}e_2u^d v,\ Sq^2(e_2u^{d-1}v) = (d-1)e_2u^d v,\ Sq^2(u^d v) = 0.$$

Next consider $P_\ell W_{d+2,2}$. We have $H^{2d+1}(P_\ell W_{d+2,2};\mathbb{Z}/2) = \mathbb{Z}/2$ generated by $y_{d+1}$, if $\phi_{d+1}(l_1, l_2) \equiv 0 \pmod 2$. In this case, the Bockstein

$$\begin{array}{ccc} H^{2d+1}(P_\ell W_{d+1,2};\mathbb{Z}/2) & = & \mathbb{Z}/2\{y_d\} \\ \downarrow{\scriptstyle\beta} & & \\ H^{2d+2}(P_\ell W_{d+1,2};\mathbb{Z}) & = & \mathbb{Z}/\phi(l_1, l_2)\{x^{d+1}\} \end{array}$$

is given by

$$\beta(y_{d+1}) = \tfrac{1}{2}\phi_{d+1}(l_1, l_2)x^{d+1}.$$

In this case, also note that $y_{d+2}$ is 0 in $H^{2d+3}(P_\ell W_{d+2,2};\mathbb{Z}/2)$ so that $Sq^2(y_{d+1}) = 0$. Using these computations we prove the following proposition.

**Proposition 3.1** *Suppose $d$ is even, $m$ is even, $m$ divides $\phi_{d+1}(l_1, l_2)$, and $\nu_2(m) = \nu_2(\phi_{d+1}(l_1, l_2))$, where $\nu_2(n)$ denotes the 2-adic valuation of $n$. Then* $\dim(\zeta) \geq d + 1$.

**Proof** Suppose $\dim(\zeta) = d$; then there exists $f\colon X \to P_\ell W_{d+2,2}$ such that $f^*(x) = c_1(\lambda) = e_2 + u$. We have

$$\beta(y_{d+1}) = \frac{1}{2}\phi_d(l_1, l_2)x^{d+1},$$

hence,

$$\beta(f^*(y_{d+1})) = \beta\Big(\frac{1}{2}\phi_{d+1}(l_1, l_2)x^{d+1}\Big) = \frac{\phi_{d+1}(l_1, l_2)}{2}(d+1)e_2u^d = \frac{m}{2}e_2u^d.$$

The last equality follows as $m$ divides $\phi_d(l_1, l_2)$, $\nu_2(m) = \nu_2(\phi_d(l_1, l_2))$ and $d+1$ is odd. This implies $f^*(y_{d+1}) = e_2u^{d-1}v + ku^dv$ for some $k$. Then

$$\begin{array}{ccc} H^{2d+1}(P_\ell W_{d+1,2};\mathbb{Z}/2) & \xrightarrow{f^*} & H^{2d+1}(X;\mathbb{Z}/2) \\ \downarrow{\scriptstyle Sq^2} & & \downarrow{\scriptstyle Sq^2} \\ H^{2d+3}(P_\ell W_{d+1,2};\mathbb{Z}/2) & \xrightarrow{f^*} & H^{2d+3}(X;\mathbb{Z}/2) \end{array}$$

implies

$$f^*Sq^2(y_{d+1}) = Sq^2f^*(y_{d+1}) = Sq^2(e_2u^{d-1}v + ku^dv) = e_2u^dv \neq 0$$

as $d$ is even. However, $f^*Sq^2(y_{d+1}) = f^*(0) = 0$, which leads to a contradiction. Hence, it follows that $\dim(\zeta) \geq d + 1$. ∎

## 3.2 Bounds for Span and Immersions in Euclidean Space

We compute the Pontrjagin classes for $P_\ell W_{n,2}$ and deduce some bounds for the span and immersion codimension. The dimension of the manifold $P_\ell W_{n,2}$ is $4n - 5$. Note the expression for the tangent bundle for $P_\ell W_{n,2}$ from [7, 2.2]:

$$\tau(P_\ell W_{n,2}) \cong r(\xi^{-l_1} \otimes_{\mathbb{C}} \xi^{l_2}) \oplus r(\xi^{-l_1} \otimes_{\mathbb{C}} \beta) \oplus r(\xi^{-l_2} \otimes_{\mathbb{C}} \beta) \oplus \epsilon_{\mathbb{R}}.$$

In this expression, $\xi$ is the complex line bundle associated with the principal $S^1$-bundle $W_{n,2} \to P_\ell(W_{n,2})$ and $\beta$ is the universal complex vector bundle satisfying $\xi^{l_1} \oplus \xi^{l_2} \oplus \beta \cong n\epsilon_{\mathbb{C}}$ in Theorem 2.6. The bundles $\xi^s$ are defined using the tensor product in the group of complex line bundles so that $\xi^{-n} \cong \bar{\xi}^n$. The operation $r$ is the realification functor carrying a complex bundle to its underlying real bundle. Eliminating the bundle $\beta$ from the equation above one has the following expression from [7, Lemma 2.1]:

$$\tau(P_\ell W_{n,2}) \oplus r(\xi^{-l_2} \otimes_{\mathbb{C}} \xi^{l_1}) \oplus 3\epsilon_{\mathbb{R}} \cong nr(\xi^{-l_1} \oplus \xi^{-l_2}).$$

Observe that the first Chern class of the line bundle $\xi$ equals the class $x$ defined in Section 2. It follows that the total Pontrjagin class of the tangent bundle is given by (modulo 2-torsion)

$$p\big(\tau(P_\ell W_{n,2})\big) = (1 - l_1^2x^2)^n(1 - l_2^2x^2)^n\big(1 - (l_2 - l_1)^2x^2\big)^{-1}. \qquad (3.1)$$

Thus, the Pontrjagin classes lie in the subalgebra of $H^*(P_\ell W_{n,2})$ generated by $x$. We have the following result on the span of these manifolds.

***Theorem 3.2*** *For any $\ell = (l_1, l_2)$ such that there is a prime $q$ dividing $n$ but not $l_2 - l_1$, the span of $P_\ell W_{n,2}$ is $\leq 4n - 5 - 2\lfloor \frac{n-2}{2} \rfloor$. In addition for $n$ odd, if $q$ divides $l_1^n - l_2^n$, then the span of $P_\ell W_{n,2}$ is $\leq 3n - 4$.*

**Proof** Recall that if the span of a vector bundle $\gamma$ is $k$, then the Pontrjagin classes $p_i(\gamma)$ are 0 for $i > \lfloor (\dim(\gamma) - k)/2 \rfloor$. In the spectral sequence for $H^*(P_\ell W_{n,2}; \mathbb{Z})$ of Section 2, the first differential onto a power of $x$ hits some multiple of the element $x^{n-1}$. Therefore, the powers $x^i$ are non-trivial for $i \leq n - 2$.

Suppose there is a prime $q$ that divides $n$ but not $l_2 - l_1$. Then the expression (3.1) modulo $q$ and $x^n$ is the same as $(1 - (l_2 - l_1)^2 x^2)^{-1}$, which has non-zero coefficient (modulo $q$) for every even power of $x$. Thus the coefficient of $x^{2\lfloor \frac{n-2}{2} \rfloor}$ is non-zero, implying the first part.

For $n$ odd, write $n - 1 = 2k$ and consider the possibility that the Pontrjagin class $p_k(\tau(P_\ell W_{n,2})) \not\equiv 0 \pmod q$. The expression $(1 - (l_2 - l_1)^2 x^2)^{-1}$ has a non-zero coefficient of $x^{n-1}$. Therefore, $p_k(\tau(P_\ell W_{n,2})$ is non-zero if the class $x^{n-1}$ is non-zero in $H^*(P_\ell W_{n,2}; \mathbb{Z}/q)$, which in turn is equivalent to the condition $\phi_{n-1}(l_1, l_2) \equiv 0 \pmod q$. Note that $\phi_{n-1}(l_1, l_2) = \frac{l_1^n - l_2^n}{l_1 - l_2}$. Hence, the result follows. ∎

***Remark 3.3*** Note that the second condition is easily satisfied (for example if $q - 1$ divides $n$ and $q$ does not divide any $l_i$). There can be other results similar to Theorem 3.2. For example, a similar argument demonstrates that if a prime $q$ divides $n - 1$ but not $l_1 - l_2$, $l_1$ or $l_1 + 3l_2$ the first conclusion holds. If in addition $q$ divides $l_1^n - l_2^n$ the second condition holds. One can make similar computations with $q$ dividing $n - 2$ and so on. Thus it is possible to write down many sets of divisibility relations for $l_1$ and $l_2$ which imply the first consequence, and in addition if the prime divides $l_1^n - l_2^n$ without dividing $l_1 - l_2$ then the second consequence also follows.

Next we consider the problem of immersing the manifold $P_\ell W_{n,2}$ in Euclidean space. If $P_\ell W_{n,2}$ is immersed in $\mathbb{R}^N$ for some $N$, then we have $\tau \oplus \nu \cong N\epsilon_{\mathbb{R}}$, where $\nu$ is the normal bundle. The total Pontrjagin class modulo elements of order 2, satisfies $p(\nu) = p(\tau(P_\ell W_{n,2}))^{-1}$. From (3.1), it follows that

$$p(\nu) = (1 - l_1^2 x^2)^{-n} (1 - l_2^2 x^2)^{-n} \big(1 - (l_2 - l_1)^2 x^2\big). \tag{3.2}$$

***Theorem 3.4*** *Suppose that there exists a prime $q$ dividing $n - 1$ and $l_2 - l_1$. Then the class $p_{\lfloor \frac{n-3}{2} \rfloor}(\nu)$ is non-zero. Hence the manifold $P_l W_{n,2}$ does not immerse in $\mathbb{R}^{4n-5+2\lfloor \frac{n-3}{2} \rfloor}$.*

**Proof** We compute $p(\nu)$ modulo $q$ and $x^n$ as in Theorem 3.2. Reducing (3.2) modulo $q$ and $x^n$, we get

$$p(\nu) \equiv (1 - l_1^2 x^2)^{-2} (\text{mod } q, x^n).$$

https://doi.org/10.4153/CMB-2016-098-8 Published online by Cambridge University Press

The coefficient of $x^{2\lfloor \frac{n-3}{2} \rfloor}$ in this expression is

$$\binom{-2}{2\lfloor \frac{n-3}{2} \rfloor} = \pm\Big(2\lfloor \frac{n-3}{2} \rfloor + 1\Big).$$

This equals $n-2$ or $n-3$, none of which are divisible by $q$ as $q$ divides $n-1$. ∎

**Acknowledgments** The authors would like to thank Shilpa Gondhali for certain useful discussions on this subject. The authors also wish to thank the referee for detailed comments that significantly improved the presentation of the paper.

## References

[1] L. Astey, S. Gitler, E. Micha, and G.Pastor, *Cohomology of complex projective Stiefel manifolds.* Canad. J. Math. **51**(1999), 897–914. http://dx.doi.org/10.4153/CJM-1999-039-2

[2] ———, *Parallelizability of complex projective Stiefel manifolds*. Proc. Amer. Math. Soc. **128**(1999), 1527–1530. http://dx.doi.org/10.1090/S0002-9939-99-05372-1

[3] P. Baum and W. Browder, *The cohomology of quotients of classical groups.* Topology **3**(1965), 305–336. http://dx.doi.org/10.1016/0040-9383(65)90001-7

[4] A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts.* Ann. of Math. **57**(1953), 115–207. http://dx.doi.org/10.2307/1969728

[5] S. Gitler, *The projective Stiefel manifolds. II. Applications.* Topology **7**(1968), 47–53. http://dx.doi.org/10.1016/0040-9383(86)90014-5

[6] S. Gitler and D. Handel, *The projective Stiefel manifolds. I.* Topology **7**(1968), 39–46. http://dx.doi.org/10.1016/0040-9383(86)90013-3

[7] S. Gondhali and B. Subhash, *Vector fields on right generalized complex projective Stiefel manifolds.* Proc. Roy. Soc. Edinburgh Sect. A **143**(2013), no. 6, 1237–1242. http://dx.doi.org/10.1017/S0308210512001035

[8] I. M. James, *The topology of Stiefel manifolds.* London Mathematical Society Lecture Note Series, 24, Cambridge University Press, Cambridge, 1976.

[9] C. Ruiz, *The cohomology of the complex projective Stiefel manifold.* Trans. Amer. Math. Soc. **146**(1969), 541–547. http://dx.doi.org/10.1090/S0002-9947-1969-0250345-0

*Department of Mathematics, Vivekananda University, Belur, Howrah 711202, West Bengal, India*
*e-mail*: samik.basu2@gmail.com

*Indian Institute of Science Education and Research Tirupati, Mangalam, Tirupati -517507, Andra Pradesh, India*
*e-mail*: subhash02@gmail.com

https://doi.org/10.4153/CMB-2016-098-8 Published online by Cambridge University Press