ON ASYMPTOTIC BEHAVIOURS OF TRIGONOMETRIC SERIES WITH δ-QUASI-MONOTONE COEFFICIENTS

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1. Let

$$f(x) = \sum_{k=1}^{\infty} a_k \cos kx,$$

$$g(x) = \sum_{k=1}^{\infty} a_k \sin kx.$$

The asymptotic behaviours of $f(x)$ and $g(x)$, as $x \to +0$, were first given by G. H. Hardy in (4), (5). In his papers $\{a_n\}$ is a monotone decreasing sequence. Further results on the asymptotic behaviours of $f(x)$ and $g(x)$, as $x \to +0$, for monotone coefficients have been given in (9) and (1). Recently, the results have been generalized to quasi-monotone coefficients.

This paper is concerned with asymptotic behaviours of $f(x)$ and $g(x)$ for δ-quasi-monotone coefficients.

In what follows, we shall denote by $L(x)$ a slowly varying function in the sense of Karamata (6), i.e.,

(*a*) $L(x)$ is positive and continuous for all $x>0$;

(*b*) $L(tx)/L(x) \to 1$, as $x \to \infty$ with every fixed $t>0$.

A sequence $\{a_n\}$ is called δ-quasi-monotonic (3), if

(*a*) $a_n > 0$ ultimately;

(*b*) $a_n \to 0$, as $n \to \infty$;

(*c*) $\Delta a_n = a_n - a_{n+1} \geqq -\delta_n$ for some positive sequence $\{\delta_n\}$.

A sequence $\{a_n\}$ of positive numbers is called quasi-monotonic if

$$a_n - a_{n+1} = \Delta a_n \geqq -\alpha n^{-1} a_n$$

for some $\alpha > 0$. We see that a quasi-monotonic sequence with $a_n \to 0$ is a δ-quasi-monotonic sequence when $\delta_n = \alpha n^{-1} a_n$.

By "$A(x) \simeq B(x)$, as $x \to a$" we mean that $A(x) = B(x)\{1+o(1)\}$, as $x \to a$. We shall make use of K to denote some positive constants which need not be the same from one occurrence to another. K's can depend on β.

The following theorems will be established in this paper.

Theorem 1. *Let* $0<\beta<1$ *and let* $\{a_n\}$ *be a* δ*-quasi-monotonic sequence with* $S = \sum_{k=1}^{\infty} \delta_k k^{\alpha} < \infty$ $(\beta < \alpha)$. *Then* $\{a_n\}$ *is of bounded variation and*

$$f(x) \simeq \tfrac{1}{2}\pi x^{\beta-1} L(x^{-1}) / \{\Gamma(\beta) \cos \tfrac{1}{2}\beta\pi\},$$

as $x \to +0$, *if and only if* $a_n \simeq n^{-\beta}L(n)$, *as* $n \to \infty$, *where* $L(n)$ *is a slowly varying function in the sense of Karamata.*

Theorem 2. *Let* $0<\beta<1$ *and let* $\{a_n\}$ *be a* δ*-quasi-monotonic sequence with* $S = \sum_{k=1}^{\infty} \delta_k k^{\alpha} < \infty$ $(\beta<\alpha)$. *Then* $\{a_n\}$ *is of bounded variation and*

$$g(x) \simeq \tfrac{1}{2}\pi x^{\beta-1}L(x^{-1})/\{\Gamma(\beta)\sin\tfrac{1}{2}\beta\pi\},$$

as $x \to +0$, *if and only if* $a_n \simeq n^{-\beta}L(n)$, *as* $n \to \infty$, *where* $L(n)$ *is a slowly varying function in the sense of Karamata.*

2. Preliminary Lemmas.

Lemma 1. *For any* $b>0$, *we have*

(*a*) $x^bL(x) \to \infty$ *and* $x^{-b}L(x) \to 0$, *as* $x \to \infty$;

(*b*) $\max_{0 \leqq \xi \leqq x} \{\xi^b L(\xi)\} \simeq x^b L(x)$,

$\max_{x \leqq \xi < \infty} \{\xi^{-b}L(\xi)\} \simeq x^{-b}L(x)$, *as* $x \to \infty$.

Lemma 1 is due to Karamata (8).

Lemma 2. *Let* $\{a_n\}$ *be* δ*-quasi-monotonic with* $\sum_{k=1}^{\infty} \delta_k k^b < \infty$ $(b>0)$. If $\sum_{=1}^{\infty} a_k k^{b-1}$ *converges, then* $\sum_{k=1}^{\infty} |\Delta a_k| k^b$ *converges.*

Lemma 2 is due to Boas (3).

Lemma 3. *Let* $0<\beta<1$ *and* $\beta<\alpha$. *Let* $\{a_n\}$ *be* δ*-quasi-monotonic with* $\sum_{k=1}^{\infty} \delta_k k^{\alpha} < \infty$. *If* $a_n \simeq n^{-\beta}L(n)$, *as* $n \to \infty$, *then*

(*a*) $\{a_n\}$ *is of bounded variation,*

(*b*) $\sum_{k=n}^{\infty} |\Delta a_k| < Kn^{-\beta}L(n)$, *as* $n \to \infty$.

Proof. Let $a_n = n^{-\beta}L(n)\bar{a}_n$. We see that $\bar{a}_n \to 1$ as $n \to \infty$ and $\bar{a}_n$ is bounded. Then we have

$$\sum_{k=1}^{\infty} a_k k^{-1+\frac{1}{2}\beta} = \sum_{k=1}^{\infty} k^{-1-\frac{1}{2}\beta}L(k)\bar{a}_k$$

$$\leqq \max_{1 \leqq k < \infty}\{\bar{a}_k\} \max_{1 \leqq \xi < \infty}\{\xi^{-\frac{1}{2}\beta}L(\xi)\} \sum_{k=1}^{\infty} k^{-1-\frac{1}{2}\beta} < K.$$

By Lemma 2 we have $\sum_{k=1}^{\infty} |\Delta a_k| < \infty$, i.e. $\{a_n\}$ is of bounded variation.

Next, putting $(\Delta a_k)^- = \max\{0, -\Delta a_k\}$, we have

$$\frac{n^\beta}{L(n)} \sum_{k=n}^{\infty} |\Delta a_k| = \frac{n^\beta}{L(n)} \left\{ \sum_{k=n}^{\infty} \Delta a_k + 2\sum_{k=n}^{\infty} (\Delta a_k)^- \right\} = S_1 + S_2,$$

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where

$$S_1 = \frac{n^\beta}{L(n)} \sum_{k=n}^{\infty} \Delta a_k = \frac{n^\beta}{L(n)} a_n < K, \text{ as } n \to \infty,$$

$$S_2 = 2\frac{n^\beta}{L(n)} \sum_{k=n}^{\infty} (\Delta a_k)^- \leqq 2\frac{n^\beta}{L(n)} \sum_{k=n}^{\infty} \delta_k \leqq K \sum_{k=n}^{\infty} \delta_k k^{\frac{1}{2}(\alpha+\beta)} < K, \text{ as } n \to \infty.$$

Then we have

$$\sum_{k=n}^{\infty} |\Delta a_k| < Kn^{-\beta} L(n), \text{ as } n \to \infty.$$

This completes the proof of Lemma 3.

Lemma 4. *Let* $0<\beta<1$ *and* $\beta<\alpha$. *Let* $\{a_n\}$ *be δ-quasi-monotonic with* $\sum_{k=1}^{\infty} \delta_k k^\alpha < \infty$. *If* $\sum_{k=1}^{n} a_k \simeq An^{1-\beta}L(n)$, *as* $n\to\infty$, *then* $a_n \simeq An^{-\beta}L(n)\{1-\beta\}$, *as* $n\to\infty$, *where A is some positive constant.*

Proof. Let $m = n+\eta n-\theta$, where m and n are positive integers, $0 \leqq \theta < 1$ and $\eta = \eta(n) > 0$. When $n\to\infty$ we have the asymptotic expression:

$$\begin{aligned} a_{n+1}+a_{n+2}+\ldots+a_m &= Am^{1-\beta}L(m) - An^{1-\beta}L(n) + o(n^{1-\beta}L(n)) \\ &= An^{1-\beta}L(n)\{(1+\eta)^{1-\beta} - 1 + o(1)\}. \end{aligned}$$

On the other hand, considering $\Delta a_n = a_n - a_{n+1} \geqq -\delta_n$, we have

$$a_k \leqq a_n + \sum_{\gamma=n}^{k-1} \delta_\gamma \leqq a_n + n^{-\alpha} \sum_{\gamma=n}^{\infty} \delta_\gamma \gamma^\alpha \leqq a_n + n^{-\alpha} S;$$

$$a_k \geqq a_m - \sum_{\gamma=k}^{m-1} \delta_\gamma \geqq a_m - n^{-\alpha} \sum_{\gamma=n}^{\infty} \delta_\gamma \gamma^\alpha \geqq a_m - n^{-\alpha} S,$$

where $n+1 \leqq k \leqq m$ and $S = \sum_{\gamma=1}^{\infty} \delta_\gamma \gamma^\alpha < \infty$.

Then

$$a_{n+1}+a_{n+2}+\ldots+a_m \leqq (m-n)\{a_n + n^{-\alpha}S\} \leqq \eta\{na_n + n^{1-\alpha}S\};$$

$$a_{n+1}+a_{n+2}+\ldots+a_m \geqq (m-n)\{a_m - n^{-\alpha}S\} = \eta(1-\theta/\eta n)\{na_m - n^{1-\alpha}S\}.$$

Put $\eta = \eta(n) = n^{-\frac{1}{2}}$. It follows that

$$\begin{aligned} \eta n a_n &\geqq An^{1-\beta}L(n)\left\{(1+\eta)^{1-\beta} - 1 + o(1) - \eta\frac{Sn^{-\alpha+\beta}}{AL(n)}\right\} \\ &= An^{1-\beta}L(n)\{(1+\eta)^{1-\beta} - 1 + o(1)\}; \end{aligned}$$

$$\begin{aligned} \eta\left(1-\frac{\theta}{\eta n}\right) n a_m &\leqq An^{1-\beta}L(n)\left\{(1+\eta)^{1-\beta} - 1 + o(1) + \eta\left(1-\frac{\theta}{\eta n}\right)\frac{Sn^{-\alpha+\beta}}{AL(n)}\right\} \\ &= An^{1-\beta}L(n)\{(1+\eta)^{1-\beta} - 1 + o(1)\}. \end{aligned}$$

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Then we have

$$\liminf_{n\to\infty}\frac{a_n}{n^{-\beta}L(n)}\geqq An^{\frac{1}{2}}\{(1+n^{-\frac{1}{2}})^{1-\beta}-1\}$$
$$=An^{\frac{1}{2}}\{1+(1-\beta)n^{-\frac{1}{2}}+O(n^{-1})-1\}\simeq A(1-\beta);$$

$$\limsup_{m\to\infty}\frac{a_m}{m^{-\beta}L(m)}\leqq\frac{An^{\frac{1}{2}}}{(1-\theta n^{-\frac{1}{2}})}\{(1+n^{-\frac{1}{2}})^{1-\beta}-1\}(1+n^{-\frac{1}{2}})^{\beta}\simeq A(1-\beta),$$

as $n\to\infty$. Thus $a_n\simeq An^{-\beta}L(n)(1-\beta)$ as $n\to\infty$.

This completes the proof of Lemma 4.

Lemma 5. *Let $c_1, c_2>0$. If $\int_{+0}^{\infty}y^k\,|f(y)|\,dy<\infty$ for $-c_1<k<c_2$, then*

$$\int_{+0}^{\infty}f(y)L\left(\frac{y}{1-r}\right)dy\simeq L\left(\frac{1}{1-r}\right)\int_{+0}^{\infty}f(y)dy,$$

as $r\to1-0$.

Lemma 5 is due to Aljančić, Bojanić and Tomić (2).

Lemma 6. *For $0<a$, $-1<b<1$, we have*

$$\int_0^a\frac{x^bL(x^{-1})dx}{\Delta_2(r,x)}=(1-r)^{b-1}L\left(\frac{1}{1-r}\right)\{C(b)+o(1)\},$$

as $r\to1-0$, where $\Delta_2(r,x)=(1-r)^2+x^2$ and $C(b)=\frac{1}{2}\pi/\sin\{\frac{1}{2}(b+1)\pi\}$.

Proof. Let $f(y)=y^{-b}/(1+y^2)$ $(y>0)$,

$$c=\min\{1-b,1+b\}.$$

We see that $-1<k+b<1$ where $|k|<c$. We have

$$\int_{+0}^{\infty}y^{-k}f(y)dy=\int_{+0}^{\infty}\frac{y^{-(b+k)}}{1+y^2}dy=\tfrac{1}{2}\pi/\sin\{\tfrac{1}{2}(b+k+1)\pi\}.$$

By Lemma 5 we have

$$\int_{+0}^{\infty}f(y)L\left(\frac{y}{1-r}\right)dy\simeq L\left(\frac{1}{1-r}\right)\int_{+0}^{\infty}f(y)dy,\text{ as }r\to1-0.\tag{2.1}$$

On the other hand,

$$\left.\begin{aligned}\left|\int_{+0}^{(1-r)/a}f(y)L\left(\frac{y}{1-r}\right)dy\right|&\leqq(1-r)^{\frac{1}{2}c}\int_{+0}^{(1-r)/a}y^{-\frac{1}{2}c}f(y)L\left(\frac{y}{1-r}\right)\left\{\frac{y}{1-r}\right\}^{\frac{1}{2}c}dy\\&\leqq(1-r)^{\frac{1}{2}c}\max_{0\leqq\xi\leqq1/a}\{L(\xi)\xi^{\frac{1}{2}c}\}\int_{+0}^{(1-r)/a}y^{-\frac{1}{2}c}f(y)dy=o\left(L\left(\frac{1}{1-r}\right)\right),\end{aligned}\right\}\tag{2.2}$$

as $r\to1-0$.

From (2.1) and (2.2) we obtain

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$$\int_{(1-r)/a}^{\infty} f(y)L\left(\frac{y}{1-r}\right)dy \simeq L\left(\frac{1}{1-r}\right)\int_{+0}^{\infty} f(y)dy, \text{ as } r\to 1-0. \tag{2.3}$$

Using (2.3) and putting $x = (1-r)/y$, we have

$$\begin{aligned}\int_0^a \frac{x^b L(x^{-1})dx}{(1-r)^2+x^2} &= (1-r)^{b-1}\int_{(1-r)/a}^{\infty} \frac{y^{-b}}{1+y^2} L\left(\frac{y}{1-r}\right)dy \\ &\simeq (1-r)^{b-1}L\left(\frac{1}{1-r}\right)\int_{+0}^{\infty}\{y^{-b}/(1+y^2)\}dy \\ &= (1-r)^{b-1}L\left(\frac{1}{1-r}\right)C(b), \text{ as } r\to 1-0.\end{aligned}$$

This proves Lemma 6.

Lemma 7. *Let* $b_k>0$ *for all positive integers* k *and let* $0<\beta<1$. *If*

$$\sum_{k=1}^{\infty} b_k r^k \simeq \Gamma(1-\beta)L\left(\frac{1}{1-r}\right)(1-r)^{\beta-1}$$

as $r\to 1-0$, *then* $\sum_{k=1}^{n} b_k \simeq n^{1-\beta}L(n)/(1-\beta)$, *as* $n\to\infty$.

Lemma 7 is due to Karamata (7).

Lemma 8. *For* $0<x \leqq \pi$ *and* $\frac{1}{2} \leqq r<1$, *let*

$$\begin{aligned}\Delta_1 &= \Delta_1(r, x) = 1+r^2-2r\cos x, \\ \Delta_2 &= \Delta_2(r, x) = (1-r)^2+x^2, \\ K_1(r, x) &= 1/\Delta_1-1/\Delta_2, \\ K_2(r, x) &= 4\sin^2 \tfrac{1}{2}x/\Delta_1-x^2/\Delta_2, \\ K_3(r, x) &= \sin x/\Delta_1-x/\Delta_2.\end{aligned}$$

We have

(*a*) $|K_1(r, x)| \leqq K(1-r+x^2)/\Delta_2$,

(*b*) $|K_2(r, x)| \leqq K$,

(*c*) $|K_3(r, x)| \leqq K\{(1-r)x+x^3\}/\Delta_2$.

Proof. Since

$$|4\sin^2 \tfrac{1}{2}x-x^2| = 2|\cos x-1+\tfrac{1}{2}x^2| = \tfrac{1}{3}\left|\int_0^x (x-t)^3\cos t\,dt\right| \leqq \tfrac{1}{12}x^4,$$

and similarly $|\sin x - x| \leqq \frac{1}{6}x^3$, we have

$$\begin{aligned}|K_1(r, x)| &= \frac{|x^2-4r\sin^2\frac{1}{2}x|}{\Delta_1\Delta_2} = \frac{|(1-r)x^2+r(x^2-4\sin^2\frac{1}{2}x)|}{\Delta_1\Delta_2} \\ &\leqq K(1-r+\tfrac{1}{12}rx^2)/\Delta_2 \leqq K(1-r+x^2)/\Delta_2, \\ |K_3(r, x)| &= |\sin x K_1(r, x)+(\sin x-x)/\Delta_2| \leqq x|K_1(r, x)|+\tfrac{1}{6}x^3/\Delta_2.\end{aligned}$$

And $|K_2(r, x)| \leqq K$ is trivial. This completes the proof of Lemma 8.

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3. Proof of Theorem 1.

We first prove the " only if " part, i.e. we assume that $\{a_n\}$ is of bounded variation and $f(x) \simeq \frac{1}{2}\pi x^{\beta-1}L(x^{-1})/\{\Gamma(\beta)\cos\frac{1}{2}\beta\pi\}$ as $x\to+0$. Since $\{a_n\}$ is of bounded variation, we have that $\sum_{k=1}^{\infty} a_k \cos kx$ converges uniformly outside any arbitrarily small neighbourhood of 0. Furthermore, by hypothesis

$$f(x) = x^{\beta-1}L(x^{-1})\{A(\beta)+o(1)\},$$

as $x\to+0$, where $A(\beta) = \frac{1}{2}\pi/\{\Gamma(\beta)\cos\frac{1}{2}\beta\pi\}$, whence we see that

$$f(x) = \sum_{k=1}^{\infty} a_k \cos kx$$

is integrable over $(0, \pi)$. Thus, the trigonometric series $\sum_{k=1}^{\infty} a_k \cos kx$ converges to the integrable function $f(x)$ in $(0, \pi)$. It follows that the a_n's are the Fourier cosine coefficients of $f(x)$ ((9), p. 326).
i.e.

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx.$$

Using the Poisson kernel

$$P(r, x) = \tfrac{1}{2} + \sum_{k=1}^{\infty} r^k \cos kx \quad (0<r<1),$$

i.e.

$$\sum_{k=1}^{\infty} r^k \cos kx = \{r(1-r)-2r\sin^2\tfrac{1}{2}x\}/\Delta_1(r, x),$$

where $\Delta_1(r, x) = 1+r^2-2r\cos x$, we have

$$\sum_{k=1}^{\infty} r^k a_k = \frac{2r(1-r)}{\pi}\int_0^{\pi}\frac{f(x)dx}{\Delta_1(r, x)} - \frac{r}{\pi}\int_0^{\pi}\frac{4\sin^2\frac{1}{2}xf(x)dx}{\Delta_1(r, x)}$$
$$= J_1(r, x)-J_2(r, x), \text{ say.}$$

Let $f(x) = x^{\beta-1}L(x^{-1})h(x)$. Then $h(x)\to A(\beta)$, as $x\to+0$. Hence $h(x)$ is bounded in $(0, \pi)$, say $h(x) \leqq M$.

Writing
$$K_1(r, x) = \frac{1}{\Delta_1(r, x)} - \frac{1}{\Delta_2(r, x)},$$
$$K_2(r, x) = \frac{4\sin^2\frac{1}{2}x}{\Delta_1(r, x)} - \frac{x^2}{\Delta_2(r, x)},$$

where $\Delta_2(r, x) = (1-r)^2+x^2$, we have

$$J_1(r, x) = \frac{2r}{\pi}(1-r)\int_0^{\pi}\frac{x^{\beta-1}L(x^{-1})h(x)dx}{\Delta_2(r, x)}$$
$$+\frac{2r}{\pi}(1-r)\int_0^{\pi} x^{\beta-1}L(x^{-1})h(x)K_1(r, x)dx$$
$$= J_{11}(r, x)+J_{12}(r, x);$$

$$J_2(r, x) = \frac{r}{\pi}\int_0^\pi \frac{x^{\beta+1}L(x^{-1})h(x)dx}{\Delta_2(r, x)} + \frac{r}{\pi}\int_0^\pi x^{\beta-1}L(x^{-1})h(x)K_2(r, x)dx$$
$$= J_{21}(r, x)+J_{22}(r, x), \text{ say}.$$

Let us consider J_{12}, J_{21}, J_{22}, J_{11} in greater detail.

From Lemma 8(a) we obtain

$$|J_{12}| = \left|\frac{2r}{\pi}(1-r)\int_0^\pi x^{\beta-1}L(x^{-1})h(x)K_1(r, x)dx\right|$$
$$\leqq (1-r)KM\int_0^\pi x^{\beta-1}L(x^{-1})\left\{\frac{(1-r)+x^2}{\Delta_2(r, x)}\right\}dx$$
$$= (1-r)KM(I_1+I_2), \text{ say},$$

where, by Lemma 6 with $a = \pi$, $b = \beta-1$,

$$I_1 = (1-r)\int_0^\pi \frac{x^{\beta-1}L(x^{-1})dx}{\Delta_2(r, x)} \simeq K(1-r)^{\beta-1}L\left(\frac{1}{1-r}\right),$$

and by Lemma 6 with $a = \pi$, $b = \beta$,

$$I_2 \leqq \pi\int_0^\pi \frac{x^{\beta}L(x^{-1})dx}{\Delta_2(r, x)} \simeq K(1-r)^{\beta-1}L\left(\frac{1}{1-r}\right).$$

Then $J_{12} = o\left((1-r)^{\beta-1}L\left(\frac{1}{1-r}\right)\right)$, as $r\to 1-0$.

By Lemma 6 with $a = \pi$, $b = \frac{1}{2}(1+\beta)$, we have

$$|J_{21}| = \left|\frac{r}{\pi}\int_0^\pi \frac{x^{\beta+1}L(x^{-1})h(x)dx}{\Delta_2(r, x)}\right|$$
$$\leqq \frac{r}{\pi}M\pi^{\frac{1}{2}(1+\beta)}\int_0^\pi \frac{x^{\frac{1}{2}(1+\beta)}L(x^{-1})dx}{\Delta_2(r, x)}$$
$$\leqq KM(1-r)^{\frac{1}{2}(1-\beta)}\left\{(1-r)^{\beta-1}L\left(\frac{1}{1-r}\right)\right\} \quad (0<\beta<1)$$
$$= o\left((1-r)^{\beta-1}L\left(\frac{1}{1-r}\right)\right),$$

as $r\to 1-0$.

It follows from Lemma 8(b) that

$$|J_{22}| = \left|\frac{r}{\pi}\int_0^\pi x^{\beta-1}L(x^{-1})h(x)K_2(r, x)dx\right|$$
$$\leqq KM\int_0^\pi x^{\beta-1}L(x^{-1})dx$$
$$\leqq KM \max_{0<\xi\leqq\pi}\{\xi^{\frac{1}{2}}L(\xi^{-1})\}\int_0^\pi x^{\frac{1}{2}\beta-1}dx \leqq KM.$$

Hence

$$J_{22} = o\left((1-r)^{\beta-1} L\left(\frac{1}{1-r}\right)\right), \text{ as } r \to 1-0.$$

We come now to estimate J_{11}. Since $h(x) \to A(\beta)$ when $x \to +0$, for any arbitrary given $\varepsilon > 0$, there exists $\delta > 0$, such that $| h(x) - A(\beta) | < \varepsilon$ for $0 < x < \delta$. It follows that

$$\left. \begin{aligned} & \left| J_{11} - 2r \frac{(1-r)}{\pi} \int_0^\delta \frac{x^{\beta-1} L(x^{-1}) A(\beta) dx}{\Delta_2(r, x)} \right| \\ & \leqq \frac{2r}{\pi}(1-r)\left\{\int_0^\delta \frac{x^{\beta-1} L(x^{-1}) \,|\, h(x) - A(\beta) \,|\, dx}{\Delta_2(r, x)} + \int_\delta^\pi \frac{x^{\beta-1} L(x^{-1}) \,|\, h(x) \,|\, dx}{\Delta_2(r, x)}\right\} \\ & \leqq \frac{2r}{\pi}(1-r)\left\{\varepsilon \int_0^\delta \frac{x^{\beta-1} L(x^{-1}) dx}{\Delta_2(r, x)} + M \int_\delta^\pi \frac{x^{\beta-1} L(x^{-1}) dx}{\Delta_2(r, x)}\right\} \\ & = \frac{2r}{\pi}(1-r)\{\varepsilon I_3 + M I_4\}, \text{ say,} \end{aligned} \right\} \quad (3.1)$$

where, by Lemma 6 with $a = \delta$, $b = \beta - 1$

$$I_3 = \int_0^\delta \frac{x^{\beta-1} L(x^{-1}) dx}{\Delta_2(r, x)} \simeq C(\beta-1)(1-r)^{\beta-2} L\left(\frac{1}{1-r}\right), \text{ as } r \to 1-0,$$

$$I_4 = \int_\delta^\pi \frac{x^{\beta-1} L(x^{-1}) dx}{(1-r)^2 + x^2} \leqq \int_\delta^\pi x^{\beta-3} L(x^{-1}) dx$$

$$\leqq K(\delta) \max_{0 < \xi \leqq \pi} \{\xi^{\frac{1}{2}\beta} L(\xi^{-1})\} = K(\varepsilon),$$

where $K(\varepsilon)$ is a constant which depends on ε and is independent of r. Since $\beta - 2 < 0$ we see that

$$I_3 \simeq C(\beta-1) L\left(\frac{1}{1-r}\right)(1-r)^{\beta-2} \to \infty, \text{ as } r \to 1-0.$$

Then for $\varepsilon > 0$ we have

$$\varepsilon I_3 + M I_4 \leqq \left\{\varepsilon + \frac{MK(\varepsilon)}{I_3}\right\} I_3 = \{\varepsilon + o(1)\} \int_0^\delta \frac{x^{\beta-1} L(x^{-1}) dx}{\Delta_2(r, x)}, \quad (3.2)$$

as $r \to 1-0$.

From (3.1) and (3.2) for arbitrarily small $\varepsilon > 0$, we have

$$J_{11} \leqq \frac{2r}{\pi}(1-r)\{A(\beta) + \varepsilon + o(1)\} \int_0^\delta \frac{x^{\beta-1} L(x^{-1}) dx}{\Delta_2(r, x)},$$

$$J_{11} \geqq \frac{2r}{\pi}(1-r)\{A(\beta) - \varepsilon + o(1)\} \int_0^\delta \frac{x^{\beta-1} L(x^{-1}) dx}{\Delta_2(r, x)},$$

as $r \to 1-0$. It follows from Lemma 6 that

$$J_{11} \simeq \frac{2}{\pi} A(\beta)C(\beta-1)(1-r)^{\beta-1}L\left(\frac{1}{1-r}\right)$$
$$= \Gamma(1-\beta)(1-r)^{\beta-1}L\left(\frac{1}{1-r}\right), \text{ as } r \to 1-0.$$

We therefore have

$$\sum_{k=1}^{\infty} r^k a_k = J_{11}+J_{12}-J_{21}-J_{22} = \{\Gamma(1-\beta)+o(1)\}(1-r)^{\beta-1}L\left(\frac{1}{1-r}\right),$$

as $r \to 1-0$. By Lemma 7 and Lemma 4 it follows that $a_n \simeq n^{-\beta}L(n)$, as $n \to \infty$.

We come now to prove the "if" part, i.e. we assume that $a_n \simeq n^{-\beta}L(n)$ as $n \to \infty$. By Lemma 3, we see that $\{a_n\}$ is of bounded variation.

Next, we set $0<\omega<1<\Omega<\infty$, and $[\omega/x] = p$, $[1/x] = q$, $[\Omega/x] = t$, where ω and Ω are some constants which will be defined later. Then we have

$$f(x) = \sum_{k=1}^{p} a_k \cos kx + \sum_{k=t+1}^{\infty} a_k \cos kx + \sum_{k=p+1}^{t} \{a_k - k^{-\beta}L(k)\} \cos kx$$
$$+ \sum_{k=p+1}^{t} \{L(k)-L(q)\}k^{-\beta} \cos kx - L(q)\sum_{k=1}^{p} k^{-\beta} \cos kx$$
$$-L(q)\sum_{k=t+1}^{\infty} k^{-\beta} \cos kx + L(q)\sum_{k=1}^{\infty} k^{-\beta} \cos kx$$
$$= \sum_{i=1}^{7} S_i, \text{ say.}$$

Here we have $S_7 \simeq A(\beta)L(x^{-1})x^{\beta-1}$, as $x \to +0$, where

$$A(\beta) = \tfrac{1}{2}\pi/\{\Gamma(\beta)\cos \tfrac{1}{2}\beta\pi\}$$

((9), p. 187). We shall now show that $S_i = o(x^{\beta-1}L(x^{-1}))$, as $x \to +0$, for $i = 1, 2, \ldots, 6$.

With a notation similar to that used in the proof of Lemma 3, we write $a_n = n^{-\beta}L(n)\bar{a}_n$. Then by Lemma 1 we have

$$|S_1| = \left|\sum_{k=1}^{p} a_k \cos kx\right| \leqq K\sum_{k=1}^{p} |a_k| = K\sum_{k=1}^{p} k^{-\beta}L(k)\,|\bar{a}_k|$$
$$< K \max_{1<\xi\leqq p} \{\xi^{\frac{1}{2}(1-\beta)}L(\xi)\} \sum_{k=1}^{p} k^{-\frac{1}{2}(1+\beta)} \leqq Kp^{\frac{1}{2}(1-\beta)}L(p)\int_1^p \xi^{-\frac{1}{2}(1+\beta)}d\xi$$
$$\leqq Kp^{1-\beta}L(p) \leqq K\omega^{1-\beta}L(x^{-1})x^{\beta-1}, \text{ as } x \to +0.$$

We are now in a position to define ω. For any arbitrarily small $\delta>0$, let $0<\omega = \omega(\delta)<1$ so that $K\omega^{1-\beta}/A(\beta)<\delta$.

Write

$$S_2 = \sum_{k=t+1}^{\infty} a_k \cos kx = \sum_{k=t+1}^{\infty} \Delta a_k D_k(x) - a_{t+1}D_{t+1}(x),$$

where

$$D_n(x) = \sum_{k=1}^{n} \cos kx = \sin \tfrac{1}{2}nx \cos \tfrac{1}{2}(n+1)x/\sin \tfrac{1}{2}x.$$

Then it follows from Lemma 3 that

$$|S_2| \leqq \left\{ \sum_{k=t+1}^{\infty} |\Delta a_k| + |a_{t+1}| \right\}/\sin \tfrac{1}{2}x \leqq Kx^{-1}\{(t+1)^{-\beta}L(t+1)\}$$
$$\leqq K\Omega^{-\beta}L(x^{-1})x^{\beta-1}, \text{ as } x \to +0.$$

Here, for $\delta > 0$ we define Ω to be a number $1 < \Omega = \Omega(\delta) < \infty$ so that

$$K\Omega^{-\beta}/A(\beta) < \delta.$$

Since $\bar{a}_n \to 1$ when $n \to \infty$, for any arbitrary given $\varepsilon > 0$, there exists p such that $|\bar{a}_n - 1| < \varepsilon$ for all $n > p$. Then by Lemma 1 we see that

$$|S_3| = \left| \sum_{k=p+1}^{t} (\bar{a}_k - 1)k^{-\beta}L(k) \cos kx \right| \leqq \varepsilon \max_{p < \xi \leqq t} \{\xi^{-\frac{1}{2}\beta}L(\xi)\} \sum_{k=p+1}^{t} k^{-\frac{1}{2}\beta}$$
$$\leqq \varepsilon KL(p)p^{-\frac{1}{2}\beta}\{t^{1-\frac{1}{2}\beta} - p^{1-\frac{1}{2}\beta}\} \leqq \varepsilon KL(x^{-1})x^{\beta-1}\Omega^{1-\frac{1}{2}\beta}\omega^{-\frac{1}{2}\beta},$$

as $x \to +0$. For ω and Ω defined above let $\varepsilon = \varepsilon(\delta)$ be small enough so that

$$\varepsilon K\Omega^{1-\frac{1}{2}\beta}\omega^{-\frac{1}{2}\beta}/A(\beta) < \delta.$$

It remains to consider S_4, S_5, S_6. Since these trigonometric sums are independent of $\{a_n\}$, we may follow the same arguments as shown in ((**1**), p. 112) to obtain

$$S_4, S_5, S_6 = o(x^{\beta-1}L(x^{-1})), \text{ as } x \to +0.$$

Hence

$$f(x) \simeq \tfrac{1}{2}\pi x^{\beta-1}L(x^{-1})/\{\Gamma(\beta) \cos \tfrac{1}{2}\beta\pi\},$$

as $x \to +0$. This completes the proof of Theorem 1.

4. Proof of Theorem 2.

We first prove the "only if" part, i.e. we assume that $\{a_n\}$ is of bounded variation and $g(x) \simeq \frac{1}{2}\pi x^{\beta-1}L(x^{-1})/\{\Gamma(\beta) \sin \frac{1}{2}\beta\pi\}$ as $x \to +0$. Following the same argument as in §3, we see that the a_n's are the Fourier sine coefficients of $g(x)$, i.e.

$$a_n = \frac{2}{\pi}\int_0^{\pi} g(x) \sin nx dx.$$

Next, let $g(x) = x^{\beta-1}L(x^{-1})h(x)$. Here $h(x)$ should not be confused with that in §3. We see that $h(x) \to B(\beta)$ as $x \to +0$, where $B(\beta) = \frac{1}{2}\pi/\{\Gamma(\beta) \sin \frac{1}{2}\beta\pi\}$ and $h(x)$ is bounded. Using the Poisson conjugate kernel

$$\sum_{k=1}^{\infty} r^k \sin kx = r \sin x/\Delta_1(r, x) \quad (0 < r < 1),$$

we have

$$\sum_{k=1}^{\infty} r^k a_k = \frac{2r}{\pi}\int_0^{\pi} \frac{x^{\beta}L(x^{-1})h(x)dx}{\Delta_2(r, x)} + \frac{2r}{\pi}\int_0^{\pi} x^{\beta-1}L(x^{-1})K_3(r, x)h(x)dx$$
$$= J_3(r, x) + J_4(r, x), \text{ say,}$$

where

$$K_3(r, x) = \frac{\sin x}{\Delta_1(r, x)} - \frac{x}{\Delta_2(r, x)}.$$

From Lemma 8(c) and Lemma 6 (cf. the discussion of J_{21} in §3) we obtain

$$| J_4(r, x) | \leqq KM \left\{ (1-r) \int_0^\pi \frac{x^\beta L(x^{-1}) dx}{\Delta_2(r, x)} + \int_0^\pi \frac{x^{\beta+2} L(x^{-1}) dx}{\Delta_2(r, x)} \right\}$$

$$\leqq KML\left(\frac{1}{1-r}\right) \{(1-r)^\beta + \pi^{\frac{1}{2}(3+\beta)}(1-r)^{\frac{1}{2}(\beta-1)}\}, \text{ as } r \to 1-0.$$

Then we have

$$J_4(r, x) = o\left((1-r)^{\beta-1} L\left(\frac{1}{1-r}\right)\right), \text{ as } r \to 1-0.$$

Since $h(x) \to B(\beta)$ as $x \to +0$, given $\varepsilon > 0$ we can find $\delta > 0$ such that

$$| h(x) - B(\beta) | < \varepsilon \text{ for } 0 < x < \delta.$$

We therefore have

$$\left. \begin{aligned} &\left| J_3(r, x) - \frac{2r}{\pi} \int_0^\delta \frac{x^\beta L(x^{-1}) B(\beta) dx}{\Delta_2(r, x)} \right| \\ &\quad = \left| \frac{2r}{\pi} \int_0^\delta \frac{x^\beta L(x^{-1}) \{h(x) - B(\beta)\} dx}{\Delta_2(r, x)} + \frac{2r}{\pi} \int_\delta^\pi \frac{x^\beta L(x^{-1}) h(x) dx}{\Delta_2(r, x)} \right| \\ &\quad \leqq \frac{2r}{\pi} \left\{ \varepsilon \int_0^\delta \frac{x^\beta L(x^{-1}) dx}{\Delta_2(r, x)} + M \int_\delta^\pi x^{\beta-2} L(x^{-1}) dx \right\} \\ &\quad \leqq \frac{2r}{\pi} \{\varepsilon I_5 + K(\varepsilon)\}, \end{aligned} \right\} \quad (4.1)$$

where

$$I_5 = \int_0^\delta \frac{x^\beta L(x^{-1})}{(1-r)^2 + x^2} dx.$$

By Lemma 6,

$$I_5 \simeq (1-r)^{\beta-1} L\left(\frac{1}{1-r}\right) C(\beta) \text{ as } r \to 1-0.$$

By arguments similar to that used in obtaining (3.2), we have

$$\{\varepsilon I_5 + M I_6\} \leqq \{\varepsilon + o(1)\} \int_0^\delta \frac{x^\beta L(x^{-1}) dx}{\Delta_2(r, x)}, \text{ as } r \to 1-0.$$

From (4.1), (4.2) and Lemma 6 we obtain

$$J_3(r, x) = \frac{2r}{\pi} \{B(\beta) + o(1)\} \int_0^\delta \frac{x^\beta L(x^{-1}) dx}{\Delta_2(r, x)}$$

$$\simeq \frac{2}{\pi} C(\beta) B(\beta) (1-r)^{\beta-1} L\left(\frac{1}{1-r}\right), \text{ as } r \to 1-0.$$

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Then we have

$$\sum_{k=1}^{\infty} r^k a_k = J_3(r, x) + J_4(r, x) = \{\Gamma(1-\beta)+o(1)\}(1-r)^{\beta-1} L\left(\frac{1}{1-r}\right),$$

as $r \to 1-0$. By Lemma 7 and Lemma 4 we have

$$a_n \simeq n^{-\beta} L(n), \text{ as } n \to \infty.$$

The "if" part of Theorem 2 follows by the same arguments as that of Theorem 1.

Finally it should be remarked that the range of β in Theorem 2 is $0<\beta<1$. I have been unable to establish the theorem for $0<\beta<2$ which is true for monotone and quasi-monotone coefficients. The main difficulty here is that the hypothesis in Lemma 4, "$\sum_{k=1}^{n} a_k \simeq An^{1-\beta}L(n)$, as $n \to \infty$," cannot be replaced by "$\sum_{k=1}^{n} ka_k \simeq An^{2-\beta}L(n)$, as $n \to \infty$."

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